18.706 HOMEWORK 10

DUE NOV. 18, 2020

Theorems

In this part you will see theorems that we studied in the class. Please supply a detailed proof of each.

Theorem 1. *Let R be a central simple algebra of finite dimension over a field k. Let S be a simple k-algebra and $\varphi_1, \varphi_2 : S \to R$ be two k-linear ring homomorphisms. Then there exists an invertible element $u \in R$ such that $\varphi_2(s) = u\varphi_1(s)u^{-1}$ for all $s \in S$.*

Exercises

Problem 1. Let k be a field with $\mathrm{char}(k) \neq 2$. For $a, b \in k^\times$ let $\left(\frac{a,b}{k}\right)$ denote the cyclic algebra of dimension 4 over k given by

$$\left(\frac{a,b}{k}\right) = k\langle x, y\rangle/(x^2 - a, y^2 - b, xy + yx).$$

(1) Show that every 4-dimensional central simple algebra over k is isomorphic to $\left(\frac{a,b}{k}\right)$ for some $a, b \in k^\times$.

(2) Show that $\left(\frac{a,b}{k}\right) \cong M_2(k)$ if and only if $u^2 - bv^2 = a$ has a solution $(u, v) \in k^2$. In particular, show that $\left(\frac{a,1-a}{k}\right)$ is isomorphic to $M_2(k)$.

(3) Let $k(\sqrt{c})$ be a quadratic extension of k. When does $k(\sqrt{c})$ embed into $\left(\frac{a,b}{k}\right)$?

(4) Show that $\left(\frac{a,b}{k}\right) \otimes_k \left(\frac{a,c}{k}\right) \cong \left(\frac{a,bc}{k}\right) \otimes M_2(k)$. Hence in the Brauer group $Br(k)$, the sum of the classes of $\left(\frac{a,b}{k}\right)$ and $\left(\frac{a,c}{k}\right)$ is the class of $\left(\frac{a,bc}{k}\right)$.

Hint: construct a module of $\left(\frac{a,b}{k}\right) \otimes_k \left(\frac{a,c}{k}\right)$ that is 8-dimensional over k, and identify its endomorphism algebra with $\left(\frac{a,bc}{k}\right)$.