Orbital Integrals and Dedekind Zeta Functions

Zhiwei Yun[*]

Department of Mathematics, Stanford University, 450 Serra Mall, Building 380,
Stanford, CA 94305
e-mail: zwyun@stanford.edu

Abstract.   Let F be a local non-archimedean field. We prove a formula relating orbital integrals in $\mathrm{GL}(n, F)$ (for the unit Hecke function) and the generating series counting ideals of a certain ring. Using this formula, we give an explicit estimate for such orbital integrals. We also derive an analogous formula for global fields, proving analytic properties of the Dedekind zeta function for orders in global fields.

Keywords.   Zeta functions, orbital integrals.

2010 Subject Classification: Primary 22E35; Secondary 11R54

(*Dedicated to Srinivasa Ramanujan's 125th Birthday.*)

1. Introduction

The goal of this note is two-fold. First, we give an explicit estimate of orbital integrals for the unit Hecke function on $\mathrm{GL}(n, F)$ where F is a local non-archimedean field. Second, we study analytic properties of the Dedekind zeta function of an order in a global field. Though at first glance the two goals seem to be unrelated to each other, there is a key formula on which both of them rely. The key formula relates the cardinality of the "compactified class group" of some order R in a local field to a local zeta function defined in terms of R.

1.1 *Dedekind zeta function for orders*

Let E be a number field and $R \subset \mathcal{O}_E$ be an order. Define the completed Dedekind zeta function of R to be

$$\Lambda_R(s) := D_R^{s/2} \Gamma_{E,\infty}(s) \sum_{M \subset R^\vee} (\#R^\vee/M)^{-s}. \qquad (1.1)$$

Here $R^\vee = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Z})$ viewed as an R-module. The sum is over all nonzero R-submodules $M \subset R^\vee$. The number D_R is the absolute discriminant of R (see §3.2) and $\Gamma_{E,\infty}(s)$ is the usual Gamma-factor attached to the archimedean places of E (see [11]). When $R = \mathcal{O}_E$, $R^\vee$ is an invertible R-module, and $\Lambda_{\mathcal{O}_E}(s)$ is the usual completed Dedekind zeta function of the number field E. One of the main results of this note says that $\Lambda_R(s)$ shares many analytic properties with the usual Dedekind zeta function.

Theorem 1.2. *Let $R \subset E$ be an order, then*

(1) *The function $\Lambda_R(s)$ admits a meromorphic continuation to all $s \in \mathbb{C}$. The only poles of $\Lambda_R(s)$ are simple poles at $s = 0, 1$ with residues*

$$-\mathrm{Res}_{s=0}\Lambda_R(s) = \mathrm{Res}_{s=1}\Lambda_R(s)$$

$$= \frac{2^{r_1}(2\pi)^{r_2}\#\mathrm{Cl}(R)\mathrm{Reg}_E}{\#\mathcal{O}_{E,\mathrm{tors}}^{\times}} \sum_{\{M\}\in\mathrm{Cl}(R)\backslash\overline{\mathrm{Cl}}(R)} \frac{\#(\mathcal{O}_E^{\times}/\mathrm{Aut}(M))}{\#\mathrm{Stab}_{\mathrm{Cl}(R)}([M])}.$$

$$(1.2)$$

Here, as usual, r_1, r_2 are the number of real and complex places of E, and Reg_E is the regulator of E. The finite set $\overline{\mathrm{Cl}}(R)$ is the set of $E^{\times}$-homothety classes of fractional R-ideals, on which the class group $\mathrm{Cl}(R)$ acts. For other notations, see §3.3.

(2) *The function $\Lambda_R(s)$ satisfies the functional equation*

$$\Lambda_R(s) = \Lambda_R(1-s).$$

In fact, more general zeta functions (with E replaced by a semisimple $\mathbb{Q}$-algebra, R an order in E and $R^{\vee}$ replaced by an arbitrary finitely generated R-submodule of E) were introduced by Solomon [10] and studied in depth in a series of papers by Bushnell and Reiner [2], [3] and [4]. However, the fact that the functional equation holds for our specific zeta function $\Lambda_R(s)$ does not seem to be found in the literature. The residue formula above is a special case of a result of Bushnell and Reiner [3].

Moreover, the meromorphic continuation and the same residue formula at $s = 0$ is valid for more general zeta functions, with $R^{\vee}$ replaced by any fractional R-ideal in (1.1).

We also have a function field analog of Theorem 1.2, see Theorem 3.7.

1.3 *Orbital integrals*

Another main result of this note concerns with estimates of orbital integrals. Let F be a local non-archimedean field with ring of integers $\mathcal{O}_F$ and residue field $k = \mathbb{F}_q$. Let $G = \mathrm{GL}(n)$, viewed as a split reductive group over $\mathcal{O}_F$. Let $\mathfrak{g} = \mathfrak{gl}(n)$ be its Lie algebra.

Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element. Let T_{γ} be the centralizer of γ in G, which is a maximal torus in G. Let dg be the Haar measure on $G(F)$ with $\mathrm{vol}(dg, G(\mathcal{O}_F)) = 1$. Let T_c be the maximal compact subgroup of $T_{\gamma}(F)$. Let dt be the Haar measure on T_{γ} with $\mathrm{vol}(dt, T_c) = 1$. The two measures dg and dt together induce a measure $d\mu = \frac{dg}{dt}$ on the coset space $T_{\gamma}(F)\backslash G(F)$ that is right invariant under $G(F)$. The *orbital integral* of γ is

$$O_{\gamma} = \int_{T_{\gamma}(F)\backslash G(F)} \mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)}(g^{-1}\gamma g)d\mu(g),$$

where $\mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)}$ is the characteristic function of $\mathfrak{g}(\mathcal{O}_F) \subset \mathfrak{g}(F)$.

1.4 The polynomials $M_{\delta,r}$ and $N_{\delta,r}$

For each pair of integers $\delta \geq 0, r \geq 1$, we introduce a monic polynomial of degree δ:

$$M_{\delta,r}(x) = \sum_{|\lambda| \leq \delta, m_1(\lambda) < r} x^{\delta - \ell(\lambda)} + \sum_{\delta - r \leq |\lambda| < \delta} x^{|\lambda| - \ell(\lambda)}. \tag{1.3}$$

Here the summations are over partitions λ (including the empty partition). For a partition $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots$, we use $|\lambda| = \sum_i \lambda_i$ to denote its size, $\ell(\lambda)$ for the number of (nonzero) parts and $m_1(\lambda)$ for the number of times one appears as a part of λ. For example, the first summation in (1.3) is over all partitions λ with size $\leq \delta$ in which 1 appears less than r times.

We list the first few values of the polynomial $M_{\delta,r}$:

$M_{0,r}(x) = 1$; $M_{1,1}(x) = x + 1$; $M_{1,r}(x) = x + 2$ for $r \geq 2$;

$M_{2,1}(x) = x^2 + x + 1$; $M_{2,2}(x) = x^2 + 2x + 2$; $M_{2,r}(x) = x^2 + 2x + 3$ for $r \geq 3$;

$M_{3,1}(x) = x^3 + 2x^2 + x + 1$; $M_{3,2}(x) = x^3 + 3x^2 + 2x + 2$;

$M_{3,3}(x) = x^3 + 3x^2 + 3x + 3$; $M_{3,r}(x) = x^3 + 3x^2 + 3x + 4$ for $r \geq 4$; ....

We also define a polynomial $N_{\delta,r} \in \mathbb{Z}_{\geq 0}[x]$, monic of degree δ:

$$N_{\delta,r}(x) = \begin{cases} x^\delta + x^{\delta-1} + \cdots + x^{\delta-r+1} + r & \text{if } r \leq \delta; \\ x^\delta + x^{\delta-1} + \cdots + x + \delta + 1 & \text{if } r > \delta. \end{cases}$$

Another main result of this note is

Theorem 1.5. *For a regular semisimple $\gamma \in \mathfrak{g}(F)$ whose characteristic polynomial $f(X)$ has coefficients in $\mathcal{O}_F$, we have*

$$q^{\rho(\gamma)} \prod_{i \in B(\gamma)} N_{\delta_i, r_i}(q^{d_i}) \leq O_\gamma \leq q^{\rho(\gamma)} \prod_{i \in B(\gamma)} M_{\delta_i, r_i}(q^{d_i}).$$

Here the products are over the set $B(\gamma)$ of irreducible factors $f_i(X)$ of $f(X)$, the integers δ_i, r_i, d_i and $\rho(\gamma)$ are certain invariants determined by the $f_i(X)$'s, see §4.1. Both the upper bound and the lower bound are monic polynomials of degree δ in q with positive integer coefficients.

We spell out the special case when γ is elliptic and that the field extension $F[\gamma]/F$ is totally ramified. In this case, we have

$$q^\delta + 1 \leq O_\gamma \leq \sum_{|\lambda| \leq \delta, \text{ with all parts } \geq 2} q^{\delta - \ell(\lambda)} + \sum_{|\lambda| = \delta - 1} q^{\delta - 1 - \ell(\lambda)}.$$

1.6 Potential applications

In the recent work of S-W. Shin and N. Templier [9], estimates of orbital integrals (for a general reductive group G and test function in the unramified Hecke algebra)

are used as an essential technical tool to prove their Sato-Tate theorem for families of automorphic representations, see [9, Proposition 7.1, Appendices A and B]. For their purposes, an upper bound in terms of $|D(\gamma)|_F^{-1/2}$ suffices. When $G = \mathrm{GL}_n$, $D(\gamma)$ is essentially the discriminant of the characteristic polynomial of γ, and $|D(\gamma)|_F^{-1/2}$ is comparable to $q^{\delta(\gamma)}$. Theorem 1.5 gives a much finer upper and lower bound for the orbital integrals in the case $G = \mathrm{GL}_n$ and the test function being the unit element in the Hecke algebra, and the leading terms $q^{\delta(\gamma)}$ of the upper and lower bounds are optimal. It would be interesting to generalize our result to other groups G and more general test functions in the Hecke algebra. We expect that such fine estimates for orbital integrals should have further applications in analytic number theory.

1.7 *Method of proof*

The proofs of both theorems are based on a local statement, Theorem 2.5, whose proof is a simple exercise following Tate's thesis [11]. To get the estimate in Theorem 1.5, one needs a formula for the number of ideals of a given colength in the two-dimensional ring $\mathcal{O}_F[[X]]$, which is worked out in Proposition 4.13.

2. Local study of zeta functions for orders

2.1 *The setup*

Let F be a local non-archimedean field with ring of integers $\mathcal{O}_F$ and residue field $k = \mathbb{F}_q$. Let $|\cdot| : F^\times \to q^{\mathbb{Z}}$ be the standard absolute value: $|\pi| = q^{-1}$ for any uniformizer π in F.

Let E be a finite-dimensional separable F-algebra. We extend the absolute value $|\cdot|$ to $E^\times$ by pre-composing with the norm $N_{E/F}$. Write E as a product of fields $E = \prod_{i \in B} E_i$ where B is some finite index set. Let $\widetilde{k}_i$ be the residue field of E_i and $n_i = [\widetilde{k}_i : k]$ for $i \in B$. The rings of integers of E (resp. E_i) is $\mathcal{O}_E$ (resp. $\mathcal{O}_{E_i}$).

An *order* in E is finitely generated $\mathcal{O}_F$-algebra $R \subset E$ with $R \otimes_{\mathcal{O}_F} F = E$. A *fractional R-ideal* is a finitely generated R-submodule $M \subset E$ such that $M \otimes_{\mathcal{O}_F} F = E$.

Let X_R be the set of fractional R-ideals, on which $E^\times$ acts via multiplication. An $E^\times$-orbit in X_R is called an *ideal class*, and we denote by $\overline{\mathrm{Cl}}(R) = E^\times \backslash X_R$ the set of ideal classes of R. The usual ideal class group is the quotient $\mathrm{Cl}(R) = E^\times \backslash X_R^\circ$ where X_R° is the set of invertible fractional ideals. We may view $\overline{\mathrm{Cl}}(R)$ as a "compactification" of the group $\mathrm{Cl}(R)$.

Choose a finite free $\mathbb{Z}$-module $\Lambda \subset E^\times$ complementary to $\mathcal{O}_E^\times$, for example $\Lambda = \prod_i \pi_i^{\mathbb{Z}}$ for uniformizers $\pi_i \in E_i$. Then Λ acts freely on X_R with finitely many orbits. The cardinality of the orbit set $\Lambda \backslash X_R$ is independent of the choice of Λ complementary to $\mathcal{O}_E$.

For any two fractional ideals M_1 and M_2, their relative ($\mathcal{O}_F$-)length $[M_1 : M_2]$ is defines as

$$[M_1 : M_2] := \mathrm{leng}_{\mathcal{O}_F}(M_1/M_1 \cap M_2) - \mathrm{leng}_{\mathcal{O}_F}(M_2/M_1 \cap M_2). \qquad (2.1)$$

The normalization of R is $\mathcal{O}_E$. The Serre invariant for R is $\delta_R = [\mathcal{O}_E : R]$, which we often abbreviate as δ.

2.2 *Duality*

Let $\mathfrak{d}_{E/F}$ be the different ideal of E/F (which is an ideal in $\mathcal{O}_E$). Choose an $\mathcal{O}_E$-generator of $c \in \mathfrak{d}_{E/F}$. On E we define a modified trace paring $(\cdot, \cdot) : E \times E \to F$ defined by

$$(x, y) = \mathrm{Tr}_{E/F}(c^{-1}xy). \qquad (2.2)$$

For any fractional R-ideal M, let $M^\vee = \{x \in E | (x, M) \subset \mathcal{O}_F\}$, which is also a fractional ideal of R.

The operation $M \mapsto M^\vee$ is an involution on the set of fractional R-ideals. The insertion of c^{-1} in the definition of the modified trace pairing ensures that $\mathcal{O}_E^\vee = \mathcal{O}_E$. For any two fractional ideals $M_1, M_2 \subset E$, we have

$$[M_1 : M_2] = [M_2^\vee : M_1^\vee].$$

We have $R \subset \mathcal{O}_E \subset R^\vee$ with $[\mathcal{O}_E : R] = [R^\vee : \mathcal{O}_E] = \delta$.

2.3 *Local zeta function*

For each $j \geq 0$, let $\mathrm{Quot}_{R^\vee}^j$ be the set of fractional R-ideals $M \subset R^\vee$ with $[R^\vee : M] = j$. Let

$$J_R(s) = \sum_{j \geq 0} \#\mathrm{Quot}_{R^\vee}^j \cdot q^{-js}. \qquad (2.3)$$

We define

$$V_R(s) := q^{\delta s} \prod_{i \in B} (1 - q^{-n_i s}), \qquad (2.4)$$

$$\widetilde{J}_R(s) := V_R(s) J_R(s). \qquad (2.5)$$

Remark 2.4. Let Hilb_R^j be the set of ideals $I \subset R$ such that $[R : I] = j$. If R is Gorenstein, i.e., $R^\vee$ is an invertible R-module, then $\mathrm{Quot}_{R^\vee}^j$ is naturally in bijection with Hilb_R^j. Therefore, in the Gorenstein case, $J_R(s)$ is the generating series made by counting points on the "Hilbert schemes of R".

Theorem 2.5. *We have*

(1) The function $\widetilde{J}_R(s)$ is of the form

$$\widetilde{J}_R(s) = q^{\delta s} P_R(q^{-s}) \qquad (2.6)$$

for some polynomial $P_R(t) \in 1 + t\mathbb{Z}[t]$ of degree $2\delta_R$.
(2) $\widetilde{J}_R(0) = \widetilde{J}_R(1) = \#(\Lambda \backslash X_R)$. Equivalently, $P_R(1) = q^\delta P_R(q^{-1}) = \#(\Lambda \backslash X_R)$.

(3) *There is a functional equation*

$$\widetilde{J}_R(s) = \widetilde{J}_R(1-s). \tag{2.7}$$

Or equivalently $P_R(t) = (qt^2)^\delta P_R(q^{-1}t^{-1})$.

Remark 2.6. Let M be a fractional R-ideal. We may define the function $J_R(s)$ using M instead of $R^\vee$, i.e., setting

$$J_R^M(s) := \sum_{j \geq 0} \#\mathrm{Quot}_M^j \cdot q^{-js},$$

and $\widetilde{J}_R^M(s) := V_R(s)J_R^M(s)$. Then $\widetilde{J}_R^M(s)$ is still of the form $q^{\delta s} P_R^M(q^{-s})$ for some polynomial $P_R^M(t)$ and

$$\widetilde{J}_R^M(0) = P_R^M(1) = \#(\Lambda \backslash X_R)$$

still holds. However, in general, $P_R^M(t)$ will not be of degree $2\delta_R$, and it will not satisfy the functional equation as in Theorem 2.5(3).

Before proving the theorem, we give some examples.

Example 2.7. Let $n \geq 2$, $E = F^n$ and $R \subset \mathcal{O}_F^n$ consist of $(x_1, \ldots, x_n)$ such that $x_1 \equiv x_2 \equiv \cdots \equiv x_n \mod \pi$. Then $\delta = n - 1$. The dual $R^\vee$ consists of triples $(y_1, \ldots, y_n) \in (\pi^{-1}\mathcal{O}_F)^n$ such that $y_1 + \cdots + y_n \in \mathcal{O}_F$. Note that R is not Gorenstein when $n \geq 3$. We have a formula for $P_R(t)$ by direct calculation:

$$P_R(t) = \sum_{r=0}^{n} \binom{n}{r}(1-t)^r t^{n-r} \sum_{c=0}^{n-1} t^c \left(\text{coeff. of } t^{n-c-1} \text{ in } \frac{(1-t)^{n-r}}{\prod_{j=0}^{c+\epsilon}(1-q^j t)} \right). \tag{2.8}$$

Here $\epsilon = 0$ if $r > 0$ and $\epsilon = 1$ if $r = 0$. It is an interesting combinatorial problem to show directly that this polynomial satisfies the functional equation $q^{n-1}t^{2n-2}P_R(q^{-1}t^{-1}) = P_R(t)$.

For a geometric description of $\Lambda \backslash X_R$ when F is a function field, see Remark 4.14.

Example 2.8. Continuing the above example and taking $n = 3$, a direct calculation shows

$$P_R(t) = 1 + (q-2)t + t^2 + (q^2 - 2q)t^3 + q^2 t^4.$$

We try with other definitions of the zeta function by replacing $R^\vee$ in the definition by other R-modules. For example, if we define

$$J_R^\sharp(s) := \sum_{j \geq 0} \#\mathrm{Quot}_{\mathcal{O}_F^3}^j \cdot q^{-js}.$$

Then the corresponding polynomial $P_R^\sharp(t)$ is

$$P_R^\sharp(t) = 1 + (q^2 + q - 2)t + (q^2 - 2q + 1)t^2.$$

which does not satisfy the functional equation. If we define

$$J_R^\flat(s) := \sum_{j \geq 0} \#\mathrm{Hilb}_R^j \cdot q^{-js}.$$

Then the corresponding polynomial $P_R^\flat(t)$ is

$$P_R^\flat(t) = 1 - 2t + (q^2 + q + 1)t^2 + (q^2 - 2q)t^3,$$

which again does not satisfy the functional equation. However, all these polynomials P_R, $P_R^\sharp$ and $P_R^\flat$ have the same special value $2q^2 - q$ at $t = 1$, which is the number of fractional R-ideals up to multiplication by the three uniformizers.

The rest of the section is devoted to the proof of Theorem 2.5. We first need some preparation.

2.9 *Contribution of an ideal class*

For a fractional R-ideal M, the ideal class containing it is the set $[M] = \{xM, x \in E^\times\}$. Let $c^j([M])$ be the cardinality of $\mathrm{Quot}_{R^\vee}^j \cap [M]$. The contribution of the ideal class $[M]$ to $J_R(s)$ is

$$J_R([M], s) := \sum_j c^j([M])q^{-js}.$$

The contribution of $[M]$ to $\widetilde{J}_R(s)$ is $\widetilde{J}_R([M], s) := V_R(s)J_R([M], s)$.

Fix a Haar measure dx on the additive group E such that $\mathrm{vol}(dx, \mathcal{O}_E) = 1$. So for any fractional ideal M we have $\mathrm{vol}(dx, M) = q^{[M:\mathcal{O}_E]}$. We also get a Haar measure $d^\times x = dx/|x|$ on the multiplicative group $E^\times$.

For a fractional R-ideal M, let $\mathrm{Aut}(M) = \{x \in E^\times | xM = M\}$. Then $\mathrm{Aut}(M)$ is a compact open subgroup of $E^\times$ and hence has a nonzero finite volume under the Haar measure $d^\times x$.

Lemma 2.10. *We have*

$$J_R([M], s) = q^{-[R^\vee:M]s}\mathrm{vol}(d^\times x, \mathrm{Aut}(M))^{-1} \int_{E^\times} \mathbf{1}_{M^\vee}(x)|x|^s d^\times x. \qquad (2.9)$$

Note the right side is independent of the choice of the representative M in the class $[M]$ and the choice of the measure $d^\times x$.

Proof. Let $M_j^\vee \subset M^\vee$ consist of elements $x \in M^\vee$ with $|x| = q^{-j}$. Then $M_j^\vee$ admits a free action of $\mathrm{Aut}(M)$ via multiplication. The right side of (2.9) breaks up into a sum

$$q^{-[R^\vee:M]s} \sum_{j \in \mathbb{Z}} \#(M_j^\vee/\mathrm{Aut}(M))q^{-js}.$$

The map $M_j^\vee \ni x \mapsto xM \in \mathrm{Quot}_{R^\vee}^{j+[R^\vee:M]}$ gives a bijection between $M_j^\vee/\mathrm{Aut}(M)$ and $\mathrm{Quot}_{R^\vee}^{j+[R^\vee:M]} \cap [M]$. Therefore the right side of (2.9) can be further written as

$$q^{-[R^\vee:M]s} \sum_{j \in \mathbb{Z}} \#(M_j^\vee/\mathrm{Aut}(M))q^{-js} = \sum_{j \in \mathbb{Z}} c_{j+[R^\vee:M]}([M])q^{-(j+[R^\vee:M])s} = J_R([M], s).$$

$\square$

Summing up the contributions from all ideal classes, we get

$$J_R(s) := \sum_{[M] \in \overline{\mathrm{Cl}}(R)} q^{-[R^\vee:M]s} \mathrm{vol}(d^\times x, \mathrm{Aut}(M))^{-1} \int_{E^\times} \mathbf{1}_{M^\vee}(x)|x|^s d^\times x. \quad (2.10)$$

2.11 *Proof of Theorem 2.5(1)*

In view of Lemma 4.5, we set

$$P_R(t) := \prod_{i \in B} (1 - t^{n_i}) \sum_{j \geq 0} \#\mathrm{Quot}^j_{R^\vee} \cdot t^j. \quad (2.11)$$

This is clearly in $1 + t\mathbb{Z}[[t]]$. To show this is a polynomial, it suffices to show that $\widetilde{J}_R([M], s) \in \mathbb{Q}[q^s, q^{-s}]$ for each ideal class $[M]$.

Using (2.10), we have

$$\widetilde{J}_R([M], s) = \frac{q^{(\delta - [R^\vee:M])s}}{\mathrm{vol}(d^\times x, \mathrm{Aut}(M))} \int_{E^\times} \mathbf{1}_{M^\vee}(x)|x|^s \prod_{i \in B} (1 - q^{-n_i s}) d^\times x.$$

Note that

$$\mathrm{vol}(d^\times x, \mathcal{O}_E^\times) \prod_{i \in B} (1 - q^{-n_i s})^{-1} = \int_{E^\times} \mathbf{1}_{\mathcal{O}_E}(x)|x|^s d^\times x. \quad (2.12)$$

Therefore

$$\widetilde{J}_R([M], s) = \frac{q^{(\delta - [R^\vee:M])s}}{\mathrm{vol}(d^\times x, \mathrm{Aut}(M))}$$

$$\cdot \left(\mathrm{vol}(d^\times x, \mathcal{O}_E^\times) + \int_{E^\times} (\mathbf{1}_{M^\vee}(x) - \mathbf{1}_{\mathcal{O}_E}(x))|x|^s \prod_{i \in B} (1 - q^{-n_i s}) d^\times x \right). \quad (2.13)$$

The integral above now only involves finitely many possible valuations of x, hence belongs to $\mathbb{Q}[q^s, q^{-s}]$, and therefore $\widetilde{J}_R([M], s) \in \mathbb{Q}[q^s, q^{-s}]$. This shows that $P(t)$ is a polynomial. The fact that $\deg P_R(t) = 2\delta$ will follow once we prove the functional equation below.

2.12 *Proof of Theorem 2.5(3)*

We will in fact prove the following refinement of the functional equation, which implies (2.7) by summing up the contributions from all ideal classes.

Lemma 2.13. *For each ideal class* $[M] \in \overline{\mathrm{Cl}}(R)$, *we have*

$$\widetilde{J}_R([M], s) = \widetilde{J}_R([M^\vee], 1 - s).$$

Proof. Using notation from Tate's thesis [11, §2.4], we write the local zeta function

$$\zeta(f, ||^s) := \int_{E^\times} f(x)|x|^s d^\times x,$$

for compactly supported locally constant functions $f : E \to \mathbb{C}$. Using Lemma 2.10 and (2.12), we can then write

$$\widetilde{J}_R([M], s) = q^{(\delta - [R^\vee : M])s} \frac{\text{vol}(d^\times x, \mathcal{O}_E^\times)}{\text{vol}(d^\times x, \text{Aut}(M))} \frac{\zeta(\mathbf{1}_{M^\vee}, ||^s)}{\zeta(\mathbf{1}_{\mathcal{O}_E}, ||^s)}. \qquad (2.14)$$

Fix an additive character $\psi : F \to \mathbb{C}^\times$ which is trivial on $\mathcal{O}_F$ but nontrivial on $\pi^{-1}\mathcal{O}_F$, we can define the Fourier transform $f \mapsto \widehat{f}$ for functions f on E:

$$\widehat{f}(y) = \int_E f(x)\psi((x, y))dx.$$

Here (x, y) is the modified trace pairing in (2.2). By [11, Theorem 2.4.1] (local functional equation), we have

$$\frac{\zeta(\mathbf{1}_{M^\vee}, ||^s)}{\zeta(\mathbf{1}_{\mathcal{O}_E}, ||^s)} = \frac{\zeta(\widehat{\mathbf{1}}_{M^\vee}, ||^{1-s})}{\zeta(\widehat{\mathbf{1}}_{\mathcal{O}_E}, ||^{1-s})}. \qquad (2.15)$$

A straightforward calculation shows that for any fractional R-ideal $M \subset E$, we have

$$\widehat{\mathbf{1}}_{M^\vee} = q^{[\mathcal{O}_E : M]}\mathbf{1}_M.$$

Since $\mathcal{O}_E$ is self-dual, we have $\widehat{\mathbf{1}}_{\mathcal{O}_E} = \mathbf{1}_{\mathcal{O}_E}$. Therefore (2.15) and (2.14) together imply

$$\widetilde{J}_R([M], s) = q^{(\delta - [R^\vee : M])s} \frac{\text{vol}(d^\times x, \mathcal{O}_E^\times)}{\text{vol}(d^\times x, \text{Aut}(M))} \frac{\zeta(\mathbf{1}_{M^\vee}, ||^s)}{\zeta(\mathbf{1}_{\mathcal{O}_E}, ||^s)}$$

$$= q^{(\delta - [R^\vee : M])s} \frac{\text{vol}(d^\times x, \mathcal{O}_E^\times)}{\text{vol}(d^\times x, \text{Aut}(M^\vee))} \frac{q^{[\mathcal{O}_E : M]}\zeta(\mathbf{1}_M, ||^{1-s})}{\zeta(\mathbf{1}_{\mathcal{O}_E}, ||^{1-s})}. \qquad (2.16)$$

The exponent of q on the right side is

$$(\delta - [R^\vee : M])s + [\mathcal{O}_E : M] = (\delta - [R^\vee : M] + [\mathcal{O}_E : M])s + [\mathcal{O}_E : M](1 - s)$$

$$= [\mathcal{O}_E : M](1 - s) = (\delta - [R^\vee : M^\vee])(1 - s).$$

Also $\text{Aut}(M) = \text{Aut}(M^\vee)$, therefore the right side of (2.16) is exactly $\widetilde{J}_R([M^\vee], 1 - s)$. $\qquad \square$

2.14 *Proof of Theorem 2.5(2)*

Plugging in $s = 0$ in (2.13) (which is now legitimate because it only involves finitely many powers of q^{-s}), and then sum over all ideal classes $[M]$, we get

$$\widetilde{J}_R(0) = \sum_{[M] \in \overline{\text{Cl}}(R)} \frac{\text{vol}(d^\times x, \mathcal{O}_E^\times)}{\text{vol}(d^\times x, \text{Aut}(M))} = \sum_{[M] \in \overline{\text{Cl}}(R)} \#(\mathcal{O}_E^\times / \text{Aut}(M)). \qquad (2.17)$$

Now we claim that the right side is $\#(\Lambda\backslash X_R)$. In fact, consider the quotient map $p : \Lambda\backslash X_R \to E^\times\backslash X_R = \overline{\mathrm{Cl}}(R)$. We have

$$\#p^{-1}([M]) = \#(E^\times/\Lambda\mathrm{Aut}(M)) = \#(\mathcal{O}_E^\times/\mathrm{Aut}(M)).$$

Therefore (2.17) implies $\widetilde{J}_R(0) = \#(\Lambda\backslash X_R)$. Using the functional equation (2.7), we get also $\widetilde{J}_R(1) = \#(\Lambda\backslash X_R)$.

This finishes the proof of Theorem 2.5.

3. Dedekind zeta function for orders

We would like to study the global field analog of the function $\widetilde{J}_R(s)$, i.e., the Dedekind zeta function of orders. We will treat number fields and function fields separately, and prove an analog of Theorem 2.5 in both situations.

3.1 *The number field setting*

Let E now be a number field and $R \subset E$ be an order (i.e., a finitely generated $\mathbb{Z}$-algebra whose fraction field is E). Let $|R|$ be the set of maximal ideals of R. For each $v \in |R|$, let R_v denote the v-adic completion of R with residue field k_v and ring of total fractions E_v. Let $q_v = \#k_v$.

We recall the completed Dedekind zeta function $\Lambda_R(s)$ defined in (1.1):

$$\Lambda_R(s) = D_R^{s/2}(\pi^{-s/2}\Gamma(s/2))^{r_1}((2\pi)^{1-s}\Gamma(s))^{r_2} \sum_{M\subset R^\vee} (\#R^\vee/M)^{-s}$$

where r_1 (resp. r_2) are the number of real (resp. complex) places of E, and the sum is over all nonzero R-submodules $M \subset R^\vee$. The absolute discriminant D_R is defined below.

3.2 *Discriminant*

The trace pairing $(x, y) \mapsto \mathrm{Tr}_{E/\mathbb{Q}}(xy)$ allows to identify E and the $\mathbb{Q}$-linear dual of E. In particular, the dual $R^\vee = \mathrm{Hom}_\mathbb{Z}(R, \mathbb{Z})$ can also be viewed as a fractional ideal in E which contains R. We define the absolute discriminant D_R to be $\#(R^\vee/R)$. We have

$$D_R = \#(\mathcal{O}_E/R)^2|\Delta_{E/\mathbb{Q}}|. \tag{3.1}$$

3.3 *Global ideal classes*

Let $\mathbb{A}_{E,f}^\times$ be the finite part of the idèle group of E. The class group $\mathrm{Cl}(R)$ of R is the group

$$\mathrm{Cl}(R) = E^\times\backslash\mathbb{A}_{E,f}^\times/\prod_{v\in|R|} R_v^\times.$$

For $v \in |R|$, let X_v be the set of fractional R_v-ideals. The orbit space

$$\overline{\mathrm{Cl}}(R) := E^{\times} \backslash \prod{}'_{v \in |R|} X_v$$

classifies fractional R-ideals up to multiplication by $E^{\times}$, or, equivalently, isomorphism classes of torsion-free R-modules of rank one. Here the restricted product $\prod'$ requires that almost all factors be equal to the standard lattice R_v. For each fractional ideal $M \subset E$, we define $\mathrm{Aut}(M) := \{x \in E^{\times} | xM = M\}$, which only depends on the ideal class $[M]$.

The class group $\mathrm{Cl}(R)$ acts on $\overline{\mathrm{Cl}}(R)$ and $\overline{\mathrm{Cl}}(R)$ can be viewed as a "compactification" of $\mathrm{Cl}(R)$. The orbit space $\mathrm{Cl}(R) \backslash \overline{\mathrm{Cl}}(R)$ is the set of *genera* of R, i.e., the equivalence classes of fractional R-ideals under local isomorphism. For a fractional R-ideal M, we denote the $\mathrm{Cl}(R)$-orbit containing $[M]$ by $\{M\} \subset \overline{\mathrm{Cl}}(R)$. Let $\mathrm{Stab}_{\mathrm{Cl}(R)}([M])$ be the stabilizer of $[M]$ under $\mathrm{Cl}(R)$, which only depends on the $\mathrm{Cl}(R)$-orbit of $[M]$. Now we have explained all notations in the statement of Theorem 1.2.

3.4 *Proof of Theorem 1.2*

For each $v \in |R|$, we apply the discussion in §2 to the order $R_v \subset E_v$. We can then define the functions $J_v(s)$, $\widetilde{J}_v(s)$ and the polynomial $P_v(t)$. We also have the set X_v of fractional R_v-ideals and choose a free abelian group $\Lambda_v \subset E_v^{\times}$ complementary to $\mathcal{O}_{E_v}^{\times}$, as in 2.1.

Both $\Lambda_R(s)$ and $\Lambda_E(s)$ (the usual completed Dedekind zeta function for the number field E) admit Euler products over $v \in |R|$ (which is a partition of finite places of E). We can express the ratio $\Lambda_R(s)/\Lambda_E(s)$ as a product of ratios of local zeta functions

$$\frac{\Lambda_R(s)}{\Lambda_E(s)} = (\#\mathcal{O}_E/R)^s \prod_{v \in |R|} J_v(q_v^{-s}) \prod_{u \in |\mathcal{O}_E|, u \mapsto v} (1 - q_u^{-s})$$

$$= \prod_{v \in |R|} q_v^{\delta_v s} P_v(q_v^{-s}) = \prod_{v \in |R|} \widetilde{J}_v(s). \tag{3.2}$$

Here we used (3.1) to compute the ratio of the discriminant terms. By Theorem 2.5(1), this ratio is an entire function in s. Since $\Lambda_E(s)$ has a meromorphic continuation to $\mathbb{C}$ with simple poles at $s = 0, 1$, $\Lambda_R(s)$ also has a meromorphic continuation to $\mathbb{C}$ with at most simple poles at $s = 0, 1$.

The functional equation in Theorem 1.2(2) follows from (3.2) and the local functional equation (2.7).

Let us compute the residue of $\Lambda_R(s)$ at $s = 1$. The residue at $s = 0$ follows from the functional equation. Using (3.2) and Theorem 2.5(2), we have

$$\mathrm{Res}_{s=1} \Lambda_R(s) = \mathrm{Res}_{s=1} \Lambda_E(s) \cdot \prod_{v \in |R|} \#(\Lambda_v \backslash X_v). \tag{3.3}$$

The residue formula for $\Lambda_E(s)$ in [11, Main Theorem 4.4.1] says (note the factor $|\Delta_{E/\mathbb{Q}}|^{1/2}$ disappears because of our definition of $\Lambda_E(s)$)

$$\operatorname{Res}_{s=1}\Lambda_E(s) = \frac{2^{r_1}(2\pi)^{r_2}\#\operatorname{Cl}(\mathcal{O}_E)\operatorname{Reg}_E}{\#\mathcal{O}_{E,\mathrm{tors}}^{\times}}.$$

Multiplying the above two formulae, we see that in order to prove (1.2), we only need to show that

$$\#\operatorname{Cl}(\mathcal{O}_E) \prod_{v\in|R|} \#(\Lambda_v\backslash X_v) = \#\operatorname{Cl}(R) \sum_{\{M\}\in\operatorname{Cl}(R)\backslash\overline{\operatorname{Cl}}(R)} \frac{\#(\mathcal{O}_E^{\times}/\operatorname{Aut}(M))}{\#\operatorname{Stab}_{\operatorname{Cl}(R)}([M])}. \qquad (3.4)$$

We first break the left side into a sum over the genera. Note that $\operatorname{Cl}(R)\backslash\overline{\operatorname{Cl}}(R) = \prod(E_v^{\times}\backslash X_v)$ and we have a natural map $c : \prod(\Lambda_v\backslash X_v) \to \prod(E_v^{\times}\backslash X_v) = \operatorname{Cl}(R)\backslash\overline{\operatorname{Cl}}(R)$. For each genus $\{M\}$, we may represent it by $(M_v)_{v\in|R|}$ with $M_v \in X_v$. The preimage $c^{-1}(\{M\})$ is identified with $\prod(E_v^{\times}/\Lambda_v\operatorname{Stab}(M_v)) = \prod(\mathcal{O}_{E_v}^{\times}/\operatorname{Stab}(M_v))$, where $\operatorname{Stab}(M_v)$ is the stabilizer of $M_v \in X_v$ under the action of $E_v^{\times}$. Hence we have

$$\prod_{v\in|R|} \#(\Lambda_v\backslash X_v) = \sum_{\{M\}\in\operatorname{Cl}(R)\backslash\overline{\operatorname{Cl}}(R)} \#\prod(\mathcal{O}_{E_v}^{\times}/\operatorname{Stab}(M_v)).$$

On the other hand, the term $\#\operatorname{Cl}(R)\#\operatorname{Stab}_{\operatorname{Cl}(R)}([M])^{-1}$ on the right side of (3.4) is the cardinality of $\{M\} = E^{\times}\backslash\prod'(E_v^{\times}/\operatorname{Stab}(M_v))$. Therefore, to prove (3.4), we only need to show for each genus $\{M\}$ that

$$\frac{\#\prod(\mathcal{O}_{E_v}^{\times}/\operatorname{Stab}_v)}{\#(\mathcal{O}_E^{\times}/\operatorname{Aut}(M))} = \frac{\#(E^{\times}\backslash\prod'(E_v^{\times}/\operatorname{Stab}_v))}{\#\operatorname{Cl}(\mathcal{O}_E)}. \qquad (3.5)$$

Here we have abbreviated $\operatorname{Stab}(M_v)$ to Stab_v. Indeed, (3.5) follows from the exact sequence

$$1 \to \mathcal{O}_E^{\times}/\operatorname{Aut}(M) \to \prod(\mathcal{O}_{E_v}^{\times}/\operatorname{Stab}_v) \to E^{\times}\backslash\prod{}'(E_v^{\times}/\operatorname{Stab}_v) \to \operatorname{Cl}(\mathcal{O}_E) \to 1,$$

where the last homomorphism is the natural projection $E^{\times}\backslash\prod'(E_v^{\times}/\operatorname{Stab}_v) \to E^{\times}\backslash\prod'(E_v^{\times}/\mathcal{O}_{E_v}^{\times}) = \operatorname{Cl}(\mathcal{O}_E)$. This finishes the proof of Theorem 1.2.

3.5 *The function field setting*

In the function field case, we will use geometric language. Let C be a projective curve over a finite field $k = \mathbb{F}_q$ which is geometrically integral (i.e., $C\otimes_k\overline{k}$ is irreducible and reduced). Let $E = k(C)$ be the function field of C. Let $|C|$ be the set of closed points of C. For each $v \in |C|$, let R_v be the completed local ring at v with ring of fractions E_v and residue field $k_v = \mathbb{F}_{q_v}$. We again use X_v to denote the set of fractional R_v-ideals.

The arithmetic genus of C is defined as $g_a = g_a(C) = \dim_k \mathrm{H}^1(C, \mathcal{O}_C)$. Let $\nu : \widetilde{C} \to C$ be the normalization, then the Serre invariant of C is $\delta_C = \dim_k(\mathcal{O}_{\widetilde{C}}/\mathcal{O}_C)$.

The quantity q^{g_a-1} is the analog of $D_R^{1/2}$ for an order R in a number field, and the relation $g_a(C) = g(\widetilde{C}) + \delta_C$ is the analog of (3.1).

Let $\omega = \omega_C$ be the dualizing sheaf of C. This is a coherent sheaf which is generically a line bundle. Let Quot_ω^j be the Quot-scheme of length j quotients of ω. Define the completed Dedekind zeta function for C to be

$$\Lambda_C(s) = q^{(g_a-1)s} \sum_{j \geq 0} \#\mathrm{Quot}_\omega^j(k) q^{-js}.$$

When C is a smooth projective curve, we have

$$\Lambda_C(s) = q^{(g_a-1)s} Z(C/\mathbb{F}_q, s) := q^{(g_a-1)s} \exp\left(\sum_{i \geq 1} \#C(\mathbb{F}_{q^i}) \frac{q^{-is}}{i}\right),$$

which is essentially the usual zeta function of C introduced by E. Artin. Hence we view $\Lambda_C(s)$ as a generalization of the zeta function for a possibly singular curve.

3.6 *The compactified Jacobian*

In [1], Altman and Kleiman defined the compactified Picard scheme $\overline{\mathrm{Pic}}_C$ of C classifying torsion-free coherent sheaves on C of generic rank one. The compactified Jacobian $\overline{\mathrm{Jac}}_C$ is the component of $\overline{\mathrm{Pic}}_C$ consisting of coherent sheaves with the same degree as the structure sheaf $\mathcal{O}_C$. We have a natural bijection

$$\overline{\mathrm{Jac}}_C(k) = E^\times \backslash \left(\prod_{v \in |C|}' X_v\right)^1. \tag{3.6}$$

Here, $(-)^1$ again means the norm one part. The norm is given by the product of local norms $X_v \to q_v^{\mathbb{Z}}$ sending a fractional R_v-ideal M to $q_v^{[M:R_v]}$. To see the bijection, consider a coherent sheaf $\mathcal{F} \in \overline{\mathrm{Jac}}_C(k)$ together with a trivialization of it at the generic point $\iota : \mathcal{F}|_{\mathrm{Spec}E} \cong E$. Then this datum determines a fractional R_v-ideal $\mathcal{F} \otimes R_v \subset \mathcal{F} \otimes E_v \xrightarrow{\iota_v} E_v$, hence a point $(\mathcal{F} \otimes R_v)_{v \in |C|} \in \prod X_v$. Since $\mathcal{F}$ has the same degree as $\mathcal{O}_C$, we have $(\mathcal{F} \otimes R_v) \in (\prod_v X_v)^1$. Changing the trivialization ι amounts to translating $(\mathcal{F} \otimes R_v)$ by an element in $E^\times$, hence the bijection (3.6).

The following theorem is a global analog of Theorem 2.5 for function fields.

Theorem 3.7. *We have*

(1) The function $\Lambda_C(s)$ is of the form

$$\Lambda_C(s) = q^{(g_a-1)s} \frac{P_C(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})},$$

where $P_C(t) \in 1 + t\mathbb{Z}[t]$ is a polynomial of degree $2g_a$ with special values

$$P_C(1) = q^{g_a} P_C(q^{-1}) = \#\overline{\mathrm{Jac}}_C(k).$$

(2) *The function* $\Lambda_C(s)$ *satisfies the functional equation*

$$\Lambda_C(s) = \Lambda_C(1-s).$$

Equivalently, $P_C(t) = (qt^2)^{g_a} P_C(q^{-1}t^{-1}).$

The proof is similar to that of Theorem 1.2 using the local results in Theorem 2.5. We omit the proof here. In [8], in the case C has only planar singularities, we gave a cohomological interpretation of the coefficients of $P_C(t)$, which does not seem to have a number field analog.

4. Orbital integrals and local zeta functions

Throughout this section, F is a local non-archimedean field. We continue to use the notation set up in §1.3.

4.1 *Invariants attached to* γ

Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element as in §1.3. Let $f(X) \in F[X]$ be the characteristic polynomial of γ. Note that $O_\gamma \neq 0$ only if $f(X) \in \mathcal{O}_F[X]$.

Let $R = \mathcal{O}_F[X]/(f(X))$. This is an $\mathcal{O}_F$-algebra with ring of fractions $E = F[X]/(f(X))$. The pair $(R \subset E)$ is considered in §2.1, and we shall use the notation from there. In particular, E is a product of fields $\prod_{i \in B(\gamma)} E_i$ where $B(\gamma)$ is an index set in bijection with the irreducible factors $f_i(X)$ of $f(X)$. We have the Serre invariant $\delta = \delta_R$.

Set $R_i = \mathcal{O}_F[X]/(f_i(X))$ then $E_i = F[X]/(f_i(X))$ is the field of fractions of R_i. Let $\widetilde{k}_i$ and k_i be the residue fields of $\mathcal{O}_{E_i}$ and R_i respectively. Summarizing

$$
\begin{array}{ccccc}
\mathcal{O}_F & \longrightarrow & R_i & \longrightarrow & \mathcal{O}_{E_i} \\
\downarrow & & \downarrow & & \downarrow \\
k & \xrightarrow{d_i} & k_i & \xrightarrow{r_i} & \widetilde{k}_i
\end{array}
$$

where the integers d_i and r_i stand for the degrees of the respective field extensions, and we let $n_i = d_i r_i = [\widetilde{k}_i : k]$.

The normalization $\mathcal{O}_E = \prod_i \mathcal{O}_{E_i}$ of R is also the maximal compact subgroup T_c of $T_\gamma(F)$. We also define the Serre invariant of R_i to be $\delta_i = \text{leng}_{R_i}(\mathcal{O}_{E_i}/R_i)$. Let

$$\rho(\gamma) := \delta - \sum_{i \in B(\gamma)} d_i \delta_i$$

Then we have

$$\rho(\gamma) = \sum_{\{i,j\} \subset B(\gamma), i \neq j} \text{val}_F(\text{Res}(f_i, f_j)),$$

where $\text{Res}(\cdot, \cdot)$ is the resultant of two polynomials.

4.2 *Orbital integrals and lattices*

Let $X_\gamma = \{g \in G(F)/G(\mathcal{O}_F)|g^{-1}\gamma g \in \mathfrak{g}(\mathcal{O}_F)\}$. Then $T_\gamma(F)$ acts on X_γ by left translation. Choose a free abelian group $\Lambda_\gamma \in E^\times = T_\gamma(F)$ complementary to the maximal compact T_c. For example, we may fix a uniformizer π_i for each E_i, and let $\Lambda_\gamma = \prod_i \pi_i^{\mathbb{Z}}$. Then Λ_γ acts freely on X_γ. We have

$$O_\gamma = \#(\Lambda_\gamma \backslash X_\gamma). \tag{4.1}$$

Let $V = F^n$ be an n-dimensional vector space over F. We identify $\mathfrak{g}(F)$ with $\mathrm{End}_F(V)$, and thus view $\gamma \in \mathfrak{g}(F)$ as an F-linear endomorphism $\gamma : V \to V$. The action of γ on V extends to an action of $E = F[X]/(f(X))$ on V, with X acting as γ. Since γ is regular semisimple, V is a free E-module of rank one.

A *lattice* in V is an $\mathcal{O}_F$-submodule M of V of rank n. For example $L_0 = \mathcal{O}_F^n \subset F^n$ is a lattice. For two $\mathcal{O}_F$-lattices $L_1, L_2 \subset V$, we define their relative $\mathcal{O}_F$-length using formula (2.1).

The map $G(F)/G(\mathcal{O}_F) \ni g \mapsto gL_0$ gives a bijection between $G(F)/G(\mathcal{O}_F)$ and the set of lattices in V. The subset $X_\gamma \subset G(F)/G(\mathcal{O}_F)$ corresponds bijectively to lattices L which are stable under γ (i.e., $\gamma L \subset L$), or equivalently, stable under $R \subset E$.

4.3 *Another orbital integral*

Let V^* be the dual vector space of V. It also contains a standard lattice L_0^* spanned by the standard dual basis over $\mathcal{O}_F$. The algebra E acts on V^* on the right (using the adjoint of its action on V), under which V^* becomes a rank one free E-module (since V is). Let $e^* \in V^*$ be a basis of this free E-module of rank one. The R-translation of e^* inside of V^* gives another lattice e^*R.

Inspired by Jacquet [6], we consider the following orbital integral with a complex parameter s:

$$J_\gamma(s) := q^{[e^*R:L_0^*]s} \int_{G(F)} \mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)}(g^{-1}\gamma g)\mathbf{1}_{L_0^*}(e^*g)|\det(g)|^s dg.$$

Note that *a priori* $J_\gamma(s)$ depends on the choice of $e^* \in V^*$.

4.4 *Relation with local zeta function*

We may relate $J_\gamma(s)$ to the local zeta function $J_R(s)$ defined in §2.3. Using the vector e^* we may identify V^* with E: $E \ni x \mapsto e^*x \in V^*$. Transporting the modified trace pairing (2.2) to V^* using this identification, we can also identify V with E, such that under these identifications, the usual pairing between V^* and V becomes the modified trace pairing on E itself.

Lemma 4.5. *We have*

$$J_\gamma(s) = J_R(s), \tag{4.2}$$

where $J_R(s)$ is defined in (2.3). In particular, $J_\gamma(s)$ is independent of the choice of $e^* \in V^*$.

Proof. For each $g \in G(F)/G(\mathcal{O}_F)$, the condition that $g^{-1}\gamma g \in \mathfrak{g}(\mathcal{O}_F)$ and $e^*g \in L_0^*$ is the same as saying that the $\mathcal{O}_F$-lattice $L = gL_0$ is stable under γ and that $\langle e^*R, gL_0 \rangle \subset \mathcal{O}_F$ (here $\langle \cdot, \cdot \rangle$ means the pairing between V^* and V). Under the identification of $V \cong E \cong V^*$ given above, this is the same as saying that gL_0 is a fractional R-ideal contained in $R^\vee$. Moreover $[R^\vee : gL_0] = [R^\vee : L_0] + [L_0 : gL_0] = -[e^*R : L_0^*] + \mathrm{val}_F(\det(g))$. Therefore

$$J_\gamma(s) = q^{[e^*R:L_0^*]s} \sum_{j \in \mathbb{Z}} \#\mathrm{Quot}_{R^\vee}^j \cdot q^{(-j-[e^*R:L_0^*])s} = \sum_{j \geq 0} \#\mathrm{Quot}_{R^\vee}^j \cdot q^{-js}. \qquad \square$$

Now we define $\widetilde{J}_\gamma(s)$ using the same formula (2.5), using the invariants $\delta = \delta_R$ and $n_i = [\widetilde{k}_i : k]$.

Theorem 2.5 then implies

Corollary 4.6 (of Theorem 2.5). *We have*

(1) *The function $\widetilde{J}_\gamma(s)$ is of the form*

$$\widetilde{J}_\gamma(s) = q^{\delta s} P_\gamma(q^{-s})$$

for some polynomial $P_\gamma(t) \in 1 + t\mathbb{Z}[t]$ of degree 2δ.
(2) $\widetilde{J}_\gamma(0) = \widetilde{J}_\gamma(1) = O_\gamma$.
(3) *There is a functional equation*

$$\widetilde{J}_\gamma(s) = \widetilde{J}_\gamma(1 - s).$$

Or equivalently $P_\gamma(t) = (qt^2)^\delta P_\gamma(q^{-1}t^{-1})$.

Corollary 4.7. *Suppose γ is elliptic (i.e., $f(X)$ is irreducible) and the residue field of R is k, then*

$$O_\gamma = \sum_{j=0}^{\delta-r-1}(q^{\delta-j}-q^{\delta-r-j})\#\mathrm{Hilb}_R^j + q^r\#\mathrm{Hilb}_R^{\delta-r} + \sum_{j=\delta-r+1}^{\delta-1}(q^{\delta-j}+1)\#\mathrm{Hilb}_R^j + \#\mathrm{Hilb}_R^\delta.$$

(4.3)

Proof. As noticed in Remark 2.4, in the situation R is Gorenstein, which it is now, one can replace $\mathrm{Quot}_{R^\vee}^j$ by Hilb_R^j. Direct calculation using the definition of $P_\gamma(t)$ gives

$$P_\gamma(t) = \sum_{j=0}^{2\delta}(\#\mathrm{Hilb}_R^j - \#\mathrm{Hilb}_R^{j-r})t^j.$$

The functional equation $P_\gamma(t) = (qt^2)^\delta P_\gamma(q^{-1}t^{-1})$ in Corollary 4.6(3) implies that for $j = 0, \ldots, \delta$, we have

$$q^{\delta-j}(\#\mathrm{Hilb}_R^j - \#\mathrm{Hilb}_R^{j-r}) = \#\mathrm{Hilb}_R^{2\delta-j} - \#\mathrm{Hilb}_R^{2\delta-j-r}.$$

Therefore,

$$P_\gamma(1) = \#\mathrm{Hilb}_R^\delta - \#\mathrm{Hilb}_R^{\delta-r} + \sum_{j=0}^{\delta-1}(\#\mathrm{Hilb}_R^j - \#\mathrm{Hilb}_R^{j-r})(1 + q^{\delta-j}). \qquad (4.4)$$

Switching the order of summation, the right side above transforms to the right side of (4.3). By Corollary 4.6(2), $O_\gamma = P_\gamma(1)$, therefore (4.3) holds. $\qquad \square$

The rest of the note is devoted to the proof of Theorem 1.5.

4.8 *Reduction of Levi subgroups*

Let $L = \prod_{i\in B} L_i \subset \mathrm{GL}(n) = G$ be a Levi subgroup (so $L_i \cong \mathrm{GL}(n_i)$ with $\sum_{i\in B} n_i = n$). Let $\gamma = (\gamma_i) \in L(F)$ with $\gamma_i \in L_i$. One can similarly define $X_\gamma^L = \prod_i X_{\gamma_i}^{L_i}$, on which $T_\gamma(F) = \prod_i T_{\gamma_i}(F)$ acts. We may fix a choice of $\Lambda_i \subset T_{\gamma_i}(F)$ (complementary to the maximal compact in $T_{\gamma_i}(F)$) and let $\Lambda_\gamma = \prod_i \Lambda_i \subset T_\gamma(F)$. It is clear that $\Lambda_\gamma \backslash X_\gamma^L$ is a product of $\Lambda_i \backslash X_{\gamma_i}^{L_i}$. Hence by the analog of (4.1) for L and L_i, we have

$$O_\gamma^L = \prod_{i\in B} O_{\gamma_i}^{L_i}.$$

Choosing a parabolic subgroup $P \subset G$ with Levi subgroup L and unipotent radical N_P, we have a Cartan decomposition $G(F) = P(F)G(\mathcal{O}_F) = N_P(F)L(F)G(\mathcal{O}_F)$. The assignment $g = nlG(\mathcal{O}_F) \in G(F)/G(\mathcal{O}_F) \mapsto l \in L(F)/L(\mathcal{O}_F)$ gives a well-defined map $p : G(F)/G(\mathcal{O}_F) \to L(F)/L(\mathcal{O}_F)$ and restricts to a map $p_\gamma : X_\gamma \to X_\gamma^L$.

Let $\rho_G^L(\gamma) = \sum_{\{i,j\}\subset B, i\neq j} \mathrm{val}_F(\mathrm{Res}(f_i, f_j))$, where f_i is the characteristic polynomial of γ_i.

Lemma 4.9. *The fibers of the map $p_\gamma : X_\gamma \to X_\gamma^L$ all have cardinality $q^{\rho_G^L(\gamma)}$.*

Proof. The choice of the parabolic P is the same as an ordering of the set $B(\gamma)$, which thus allows us to identify $B(\gamma)$ with the set $\{1, 2, \ldots, b\}$ for some integer $b \geq 1$. This parabolic P also gives a flag of the vector space $V = F^n$:

$$V_{\leq 1} \subset V_{\leq 2} \subset \cdots V_{\leq b} = V$$

such that $\mathrm{Gr}_i V := V_{\leq i}/V_{\leq i-1}$ is a free E_i-module of rank one. A point $x \in X_\gamma^L$ is the same as a choice of lattices $U_i \subset \mathrm{Gr}_i V$, one for each i, such that U_i is stable under γ_i. The fiber of p_γ over $x = (U_i)$ is in bijection with the set of lattices $U \subset V$, stable under γ, such that $U_{\leq i}/U_{\leq i-1} = U_i$ (where $U_{\leq i} = U \cap V_{\leq i}$) for each $i = 1, 2, \ldots, b$.

By induction on $b = \#B$, one reduces to the case where $B = \{1, 2\}$. For each $\mathcal{O}_F$-linear map $\phi : U_2 \to V_1/U_1$ which satisfies $\phi\gamma_2 = \gamma_1\phi$, we get a lattice $U \subset V$ spanned by U_1 and $\{u_2 + \phi(u_2) \mod U_1 | u_2 \in U_2\}$. It is easy to see that such ϕ's are in bijection with the fiber $p_\gamma^{-1}(U_1, U_2)$. Therefore we only need to compute the

number of such ϕ's. However, such a ϕ is the same as an element $\psi \in U_2^\vee \otimes_{\mathcal{O}_F}$ $(V_1/U_1) = (V_2^\vee \otimes_F V_1)/(U_2^\vee \otimes_{\mathcal{O}_F} U_1)$ which is killed by the operator $\epsilon = \gamma_2^\vee \otimes$ $\mathrm{id} - \mathrm{id} \otimes \gamma_1 \in \mathrm{End}(V_1^\vee \otimes V_2)$, i.e., $\psi \in \epsilon^{-1}(U_2^\vee \otimes_{\mathcal{O}_F} U_1)/(U_2^\vee \otimes_{\mathcal{O}_F} U_1)$. The definition of the resultant implies that $\rho_G^L(\gamma) = \mathrm{val}_F \det(\epsilon)$, which is also the length of $\epsilon^{-1}(U_2^\vee \otimes_{\mathcal{O}_F} U_1)/(U_2^\vee \otimes_{\mathcal{O}_F} U_1)$ as an $\mathcal{O}_F$-module. Therefore, the number of ψ's, which is the same as the cardinality of $p_\gamma^{-1}(U_1, U_2)$, is $q^{\rho_G^L(\gamma)}$. $\square$

When F is a local function field, one can appeal to (an positive characteristic analogue of) a geometric result of Kazhdan and Lusztig [7, §5, Proposition 1] for affine Springer fibers.

Applying Lemma 4.9 to the Levi $L \subset G$ which corresponds to the factorization of $f(X)$ into irreducible polynomials, we get

Corollary 4.10. *We have*

$$O_\gamma = q^{\rho(\gamma)} \prod_{i \in B(\gamma)} O_{\gamma_i}^{L_i}.$$

Moreover, each $O_{\gamma_i}^{L_i}$ may be computed by changing F to $F_i = F \otimes_k k_i$ and work with $\mathrm{GL}(n_i/d_i, F_i)$ (to which γ_i belongs after conjugation) instead of $\mathrm{GL}(n_i, F)$. Therefore we have reduced the proof of Theorem 1.5 to the case γ is elliptic and the residue field of R is the same as that of $\mathcal{O}_F$. From now on we will restrict ourselves to this situation. We denote the residue field of E by $\widetilde{k}$ and let $r = [\widetilde{k} : k]$.

4.11 *The lower bound*

We have a very coarse lower bound

$$\#\mathrm{Hilb}_R^i \geq 1.$$

Therefore, when $r \leq \delta$, by (4.3), we have

$$O_\gamma \geq \sum_{i=0}^{\delta-r-1} (q^{\delta-i} - q^{\delta-r-i}) + q^r + \sum_{i=\delta-r+1}^{\delta-1} (q^{\delta-i} + 1) + 1 = q^{\delta-r+1}(q^{r-1} + \cdots + 1) + r.$$

When $r > \delta$, a similar calculation shows that $O_\gamma \geq q^\delta + \cdots + q + \delta + 1$. This shows the lower bound $O_\gamma \geq N_{\delta,r}(q)$.

4.12 *The upper bound*

To obtain an upper bound for O_γ, we only need to give an upper bound for $\#\mathrm{Hilb}_R^i$ for each $0 \leq i \leq \delta$ because all coefficients in (4.3) are positive. Since R is a quotient of a formal power series ring $\mathcal{O}_F[[X]]$, we have the naive estimate

$$\#\mathrm{Hilb}_R^j \leq \#\mathrm{Hilb}_{\mathcal{O}_F[[X]]}^j.$$

Proposition 4.13. *We have*

$$\#\mathrm{Hilb}^j_{\mathcal{O}_F[[X]]} = \sum_{|\lambda|=j} q^{j-\ell(\lambda)}.$$

Here the sum is over partitions λ of size j, and $\ell(\lambda)$ is the number of parts of λ.

Proof. The leading term of an element $f(X) \in \mathcal{O}_F[[X]]$ is the monomial appearing in $f(X)$ with lowest degree (together with its coefficient). For each ideal $I \subset \mathcal{O}_F[[X]]$, let $I^\natural$ be the ideal generated by the leading terms of elements in I.

Each $I^\natural$ is of the form $I_\mu = (\pi^{\mu_0}, \pi^{\mu_1} X, \pi^{\mu_2} X^2, \dots)$ for a sequence $\mu_0 \geq \mu_1 \geq \cdots$. We view this sequence as a partition $\mu = (\mu_i)_{i \geq 0}$. The size of the partition is $|\mu| = \sum \mu_i$, which is the length of $\mathcal{O}_F[[X]]/I_\mu$.

We first show that $I^\natural$ has the same length as I, provided that the latter is finite. In fact, let $I_m = (I, X^m) \subset \mathcal{O}_F[[X]]$ (with the convention that $I_0 = \mathcal{O}_F[[X]]$), then we have

$$\mathrm{leng}(\mathcal{O}_F[[X]]/I) = \sum_{m \geq 0} \mathrm{leng}(I_m/I_{m+1}). \tag{4.5}$$

We also have isomorphisms $\mathcal{O}_F/\pi^{\mu_m}\mathcal{O}_F \xrightarrow{\sim} I^\natural_m/I^\natural_{m+1}$ and $\mathcal{O}_F/\pi^{\mu_m}\mathcal{O}_F \xrightarrow{\sim} I_m/I_{m+1}$ both given by $a \mapsto aX^m$. Therefore (4.5) applied to both $I^\natural$ and I confirms that $I^\natural$ and I have the same colength.

Now we count how many I satisfies $I^\natural = I_\mu$ for a fixed partition μ. Suppose $I \cap (X^{m+1})$ is determined and we would like to determine $I \cap (X^m)$. Clearly, $I \cap (X^m)$ is generated by $I \cap (X^{m+1})$ and an element $f_m(X) \in \mathcal{O}_F[[X]]$ of the form

$$f_m(X) = \pi^{\mu_m} X^m + \sum_{i > m} a_i X^i$$

where $a_i \in \mathcal{O}_F/\pi^{\mu_i}\mathcal{O}_F$. There are constraints on the coefficients a_i. Since $I^\natural = I_\mu$, there is an element $f_{m+1}(X) \in I \cap (X^{m+1})$ of the form $f_{m+1}(X) \equiv \pi^{\mu_{m+1}} X^{m+1}$ mod X^{m+2}. Then $X f_m(X) - \pi^{\mu_m - \mu_{m+1}} f_{m+1}(X) \in I \cap (X^{m+2})$. Comparing the coefficient of X^{m+2}, we find that a_{m+1} is already determined modulo $\pi^{\mu_{m+2}}$. Hence the number of choices for a_{m+1} is $q^{\mu_{m+1} - \mu_{m+2}}$. Continuing like this, once $a_{m+1}, \dots, a_{i-1}$ is determined, the number of choices for a_i is $q^{\mu_i - \mu_{i+1}}$. Therefore, the number of choices for $f_m(X)$, hence $I \cap (X^m)$ is

$$\prod_{i > m} q^{\mu_i - \mu_{i+1}} = q^{\mu_{m+1}}.$$

Continuing this argument inductively for m, we see that the total number of I with the same $I^\natural = I_\mu$ is

$$\prod_{m \geq 0} q^{\mu_{m+1}} = q^{\mu_1 + \mu_2 + \cdots} = q^{|\mu| - \mu_0}.$$

If we take the transposition λ of μ, then $|\mu| - \mu_0 = |\lambda| - \ell(\lambda)$. Summing over all possible partitions λ, we get the desired formula. $\square$

When F is a local function field, Proposition 4.13 can also be deduced from the cellular decomposition of the punctual Hilbert scheme for $\mathbb{A}^2$, see [5, Theorem 1.1(iv)].

We can now finish the proof of the upper bound in Theorem 1.5. Since O_γ is less than or equal to the quantity on the right side of (4.3) if Hilb_R^j is replaced by $\mathrm{Hilb}_{\mathcal{O}_F[[X]]}^j$, we get from (4.4) that

$$O_\gamma \leq \#\mathrm{Hilb}_{\mathcal{O}_F[[X]]}^\delta - \#\mathrm{Hilb}_{\mathcal{O}_F[[X]]}^{\delta-r} + \sum_{j=0}^{\delta-1} (\#\mathrm{Hilb}_{\mathcal{O}_F[[X]]}^j - \#\mathrm{Hilb}_{\mathcal{O}_F[[X]]}^{j-r})(1 + q^{\delta-j})$$

$$= \sum_{j=0}^{\delta} q^{\delta-j}(\#\mathrm{Hilb}_{\mathcal{O}_F[[X]]}^j - \#\mathrm{Hilb}_{\mathcal{O}_F[[X]]}^{j-r}) + \sum_{j=\delta-r}^{\delta-1} \#\mathrm{Hilb}_{\mathcal{O}_F[[X]]}^j. \tag{4.6}$$

Using Proposition 4.13 we have

$$\#\mathrm{Hilb}_{\mathcal{O}_F[[X]]}^j - \#\mathrm{Hilb}_{\mathcal{O}_F[[X]]}^{j-r} = \sum_{|\lambda|=j} q^{|\lambda|-\ell(\lambda)} - \sum_{|\lambda|=j-r} q^{|\lambda|-\ell(\lambda)}. \tag{4.7}$$

For a partition λ, we assign a new partition μ by adding r 1's to λ, so that $m_1(\mu) \geq r$ and $|\mu| - \ell(\mu) = |\lambda| - \ell(\lambda)$. This sets up a bijection between $\{\lambda | |\lambda| = j - r\}$ and $\{\mu | |\mu| = j, m_1(\mu) \geq r\}$ preserving the function $\lambda \mapsto |\lambda| - \ell(\lambda)$. We can then rewrite (4.7) as

$$\#\mathrm{Hilb}_{\mathcal{O}_F[[X]]}^j - \#\mathrm{Hilb}_{\mathcal{O}_F[[X]]}^{j-r} = \sum_{|\lambda|=j, m_1(j)<r} q^{j-\ell(\lambda)}.$$

Plugging this into (4.6) and use Proposition 4.13 again we get the desired upper bound $O_\gamma \leq M_{\delta,r}(q)$.

Remark 4.14. In Example 2.7, the order R is not generated by one element over $\mathcal{O}_F$ when $n \geq 3$, so our estimate in Theorem 1.5 does not apply. By (2.8) we have

$$P_R(1) = \sum_{c=0}^{n-1} \text{coeff. of } t^{n-c-1} \text{ in } \frac{(1-t)^n}{(1-t)(1-qt)\cdots(1-q^{c+1}t)}.$$

It is easy to see that the leading term is $q^{n^2/4}$ for n even and $2q^{(n^2-1)/4}$ for n odd, which has much bigger degree than $\delta = n - 1$.

Now suppose F is a local function field. We give a geometric description of X_R in this case. For each subset $I \subset \{1, 2, \ldots, n\}$ and a number $0 \leq m \leq \#I - 1$ ($m = 0$ if $I = \varnothing$), we call a *cell of type* (I, m) *with center* $\lambda \in \mathbb{Z}^n$ the subset $C(I, m, \lambda) \subset \mathbb{Z}^n$ consisting of all n-tuples $(\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ such that

- $\mu_i = \lambda_i$ if $i \notin I$;
- $\mu_i = \lambda_i$ or $\mu_i = \lambda_i + 1$ if $i \in I$;
- $\sum_i \mu_i = \sum_i \lambda_i + m$.

This is a finite subset of $\mathbb{Z}^n$ of cardinality $\binom{\#I}{m}$. Let $\mathcal{C}$ be the set of all such triples (I, m, λ). We introduce a partial ordering on $\mathcal{C}$ by declaring $(I, m, \lambda) \leq (I', m', \lambda')$ if and only if $C(I, m, \lambda) \subset C(I', m', \lambda')$. Equivalently, $(I, m, \lambda) \leq (I', m', \lambda')$ if and only if

- $I \subset I', m \leq m'$;

- $\lambda - \lambda'$ consists of 0's and 1's, the 1's only occur in coordinates $I' - I$ and there are exactly $m' - m$ of them.

For each (I, m, λ), we assign to it the variety $\mathrm{Gr}(k^I, m)$, the Grassmannian of m-planes in the vector space k^I. For $(I, m, \lambda) \leq (I', m', \lambda')$, we have a natural embedding $\mathrm{Gr}(k^I, m) \hookrightarrow \mathrm{Gr}(k^{I'}, m')$ given by $V \mapsto V \oplus k^J$ where $J \subset I' - I$ is the subset of i such that $\lambda_i = \lambda'_i + 1$. Then, on the level of the reduced (ind-)schemes, we have

$$X_R = \varinjlim_{(I, m, \lambda) \in \mathcal{C}} \mathrm{Gr}(k^I, m).$$

The action of $\Lambda = \mathbb{Z}^n$ on X_R is induced from its action on the index set $\mathcal{C}$ by $\mu \cdot (I, m, \lambda) = (I, m, \lambda + \mu)$. In particular, there are $n - 1$ irreducible components of $\Lambda \backslash X_R$, isomorphic to $\mathrm{Gr}(k^n, m)$ where $1 \leq m \leq n - 1$.

We may visualize $\Lambda \backslash X_R$ in the following way. Draw a cube $[0, 1]^{n-1}$ and subdivide it into $n - 1$ polytopes using the slices $S_i = \{(x_1, \ldots, x_{n-1}) \in [0, 1]^{n-1} \mid \sum_j x_j = i\}$, for $i = 1, \ldots, n - 1$. Fill in the polytope between S_{i-1} and S_i by the Grassmannian $\mathrm{Gr}(k^n, i)$. Then glue each pair of opposite faces of the cube together.

For example, when $n = 3$, we may draw $\Lambda \backslash X_R$ as
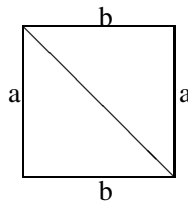
Here, each triangle stands for a copy of $\mathbb{P}^2$, and each segment stands for $\mathbb{P}^1$. The segments with the same labels are identified. Therefore, $\Lambda \backslash X_R$ can be stratified into a union of two copies of $\mathbb{G}_m^2$, three copies of $\mathbb{G}_m$ and a point, totaling $2(q - 1)^2 + 3(q - 1) + 1 = 2q^2 - q$ points, which coincides with $P_R(1)$.

Acknowledgement

References

[1] A. Altman and S. Kleiman, Compactifying the Picard scheme, *I. Adv. in Math.*, **35** (1980), no. 1, 50–112. *II. Amer. J. Math.*, **101** (1979) no. 1, 10–41.

[2] C. J. Bushnell and I. Reiner, Zeta functions of arithmetic orders and Solomon's conjectures, *Math. Z.*, **173** (1980) no. 2, 135–161.

[3] C. J. Bushnell and I. Reiner, L-functions of arithmetic orders and asymptotic distribution of ideals, *J. Reine Angew. Math.*, **327** (1981) 156–183.

[4] C. J. Bushnell and I. Reiner, Functional equations for L-functions of arithmetic orders, *J. Reine Angew. Math.*, **329** (1981) 88–124.

[5] G. Ellingsrud and S. Strømme, On the homology of the Hilbert scheme of points in the plane, *Invent. Math.*, **87** (1987) no. 2, 343–352.

[6] H. Jacquet, A conjectural fundamental lemma. Preprint.

[7] D. Kazhdan and G. Lusztig, Fixed point varieties on affine flag manifolds, *Israel J. Math.*, **62** (1988) no. 2, 129–168.

[8] D. Maulik and Z. Yun, Macdonald formula for curves with planar singularities, to appear in Crelle's Journal.

[9] S.-W. Shin, and N. Templier, Sato-Tate theorem for families and low-lying zeros of automorphic *L*-functions. Preprint. aXiv:1208.1945.

[10] L. Solomon, Zeta functions and integral representation theory, *Advances in Math.*, **26** (1977) no. 3, 306–326.

[11] J. Tate, Fourier analysis in number fields, and Hecke's zeta-functions. Thesis (Ph.D.) Princeton University (1950). In *Algebraic Number Theory* (Proc. Instructional Conf., Brighton, 1965) 305–347, Thompson, Washington, D.C.