Elliptic operators

§1 Differential operators on $\mathbb{R}^n$

Let U be an open subset of $\mathbb{R}^n$ and let D_k be the differential operator,

$$\frac{1}{\sqrt{-1}}\,\frac{\partial}{\partial x_k}\,.$$

For every multi-index, $\alpha = \alpha_1,\dots,\alpha_n$, we define

$$D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}\,.$$

A differential operator of order r:

$$P : \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U)\,,$$

is an operator of the form

$$Pu = \sum_{|\alpha|\le r} a_\alpha D^\alpha u\,, \quad a_\alpha \in \mathcal{C}^\infty(U)\,.$$

Here $|\alpha| = \alpha_1 + \cdots \alpha_n$.

The *symbol* of P is roughly speaking its "r^{th} order part". More explicitly it is the function on $U \times \mathbb{R}^n$ defined by

$$(x,\xi) \to \sum_{|\alpha|=r} a_\alpha(x)\xi^\alpha =: p(x,\xi)\,.$$

The following property of symbols will be used to define the notion of "symbol" for differential operators on manifolds. Let $f : U \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function.

Theorem 1.1. *The operator*

$$u \in \mathcal{C}^\infty(U) \to e^{-itf} P e^{itf} u$$

is a sum

$$\sum_{i=0}^{r} t^{r-i} P_i u \tag{1.1}$$

P_i being a differential operator of order i which doesn't depend on t. Moreover, P_0 is multiplication by the function

$$p_0(x) =: p(x,\xi)$$

with $\xi_i = \frac{\partial f}{\partial x_i}$, $i = 1,\dots n$.

Proof. It suffices to check this for the operators D^α. Consider first D_k:

$$e^{-itf} D_k e^{itf} u = D_k u + t\frac{\partial f}{\partial x_k}\,.$$

Next consider D^α

$$\begin{aligned} e^{-itf} D^\alpha e^{itf} u &= e^{-itf}(D_1^{\alpha_1}\cdots D_n^{\alpha_n}) e^{itf} u \\ &= (e^{-itf} D_1 e^{itf})^{\alpha_1} \cdots (e^{-itf} D_n e^{itf})^{\alpha_n} u \end{aligned}$$

which is by the above

$$\big(D_1 + t\frac{\partial f}{\partial x_1}\big)^{\alpha_1} \cdots \big(D_n + t\frac{\partial f}{\partial x_n}\big)^{\alpha_n}$$

and is clearly of the form (1.1). Moreover the t^r term of this operator is just multiplication by

$$\big(\frac{\partial}{\partial x_1} f\big)^{\alpha_1} \cdots \big(\frac{\partial f}{\partial x_n}\big)^{\alpha_n}\,. \tag{1.2}$$

$\square$

Corollary 1.2. *If P and Q are differential operators and $p(x,\xi)$ and $q(x,\xi)$ their symbols, the symbol of PQ is $p(x,\xi)\,q(x,\xi)$.*

Proof. Suppose P is of the order r and Q of the order s. Then

$$\begin{aligned} e^{-itf} PQ e^{itf} u &= \big(e^{-itf} P e^{itf}\big)\big(e^{-itf} Q e^{itf}\big) u \\ &= \big(p(x,\, df) t^r + \cdots\big)\big(q(x,\, df) t^s + \cdots\big) u \\ &= \big(p(x,\, df) q(x,\, df) t^{r+s} + \cdots\big) u\,. \end{aligned}$$

$\square$

Given a differential operator

$$P = \sum_{|\alpha|\le r} a_\alpha D^\alpha$$

we define its *transpose* to be the operator

$$u \in \mathcal{C}^\infty(U) \to \sum_{|\alpha|\le r} D^\alpha \overline{a}_\alpha u =: P^t u\,.$$

Theorem 1.3. *For $u, v \in \mathcal{C}_0^\infty(U)$*

$$\langle Pu, v\rangle =: \int Pu\overline{v}\, dx = \langle u, P^t v\rangle\,.$$

Proof. By integration by parts

$$\begin{aligned}\langle D_k u, v\rangle &= \int D_k u\overline{v}\,dx = \frac{1}{\sqrt{-1}}\int \frac{\partial}{\partial x_k} u\overline{v}\,dk \\ &= -\frac{1}{\sqrt{-1}}\int u\frac{\partial}{\partial x_k}\overline{v}\,dx = \int u\overline{D_k v}\,dx \\ &= \langle u,\, D_k v\rangle\,.\end{aligned}$$

Thus

$$\langle D^\alpha u, v\rangle = \langle u, D^\alpha v\rangle$$

and

$$\langle a_\alpha D^\alpha u, v\rangle = \langle D^\alpha u, \overline{a}_\alpha v\rangle = \langle u, D^\alpha \overline{a}_\alpha v\rangle, .$$

□

Exercises.

If $p(x,\xi)$ is the symbol of P, $\overline{p}(x,\xi)$ is the symbol of P^t.

Ellipticity.

P is elliptic if $p(x,\xi)\neq 0$ for all $x\in U$ and $\xi\in\mathbb{R}^n - 0$.

§2 Differential operators on manifolds.

Let U and V be open subsets of $\mathbb{R}^n$ and $\varphi : U\to V$ a diffeomorphism.
Claim. If P is a differential operator of order m on U the operator

$$u\in\mathcal{C}^\infty(V)\to(\varphi^{-1})^* P\varphi^* u$$

is a differential operator of order m on V.

Proof. $(\varphi^{-1})^* D^\alpha\varphi^* = \big((\varphi^{-1})^* D_1\varphi^*\big)^{\alpha_1}\cdots\big((\varphi^{-1})^* D_n\varphi^*\big)^{\alpha_n}$ so it suffices to check this for D_k and for D_k this follows from the chain rule

$$D_k\varphi^* f = \sum\frac{\partial\varphi_i}{\partial x_k}\varphi^* D_i f\,.$$

□

This invariance under coordinate changes means we can define differential operators on manifolds.

Definition 2.1. *Let $X = X^n$ be a real $\mathcal{C}^\infty$ manifold. An operator, $P : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$, is an m^{th} order differential operator if, for every coordinate patch, $(U, x_1, \dots, x_n)$ the restriction map*

$$u \in \mathcal{C}^\infty(X) \to Pu \upharpoonright U$$

is given by an m^{th} order differential operator, i.e., restricted to U,

$$Pu = \sum_{|\alpha| \le m} a_\alpha D^\alpha u\,, \quad a_\alpha \in \mathcal{C}^\infty(U)\,.$$

Remark. Note that this is a non-vacuous definition. More explicitly let $(U, x_1, \dots, x_n)$ and $(U', x'_1, \dots, x'_n)$ be coordinate patches. Then the map

$$u \to Pu \upharpoonright U \cap U'$$

is a differential operator of order m in the x-coordinates if and only if it's a differential operator in the x'-coordinates.

The symbol of a differential operator

Theorem 2.2. *Let $f : X \to \mathbb{R}$ be $\mathcal{C}^\infty$ function. Then the operator*

$$u \in \mathcal{C}^\infty(X) \to e^{-itf} P e^{-itf} u$$

can be written as a sum

$$\sum_{i=0}^{m} t^{m-i} P_i$$

P_i being a differential operator of order i which doesn't depend on t.

Proof. We have to check that for every coordinate patch $(U, x_1, \dots, x_n)$ the operator

$$u \in \mathcal{C}^\infty(X) \to e^{-itf} P e^{itf} \upharpoonright U$$

has this property. This, however, follows from Theorem 1.1.

$\square$

In particular, the operator, P_0, is a zero$^{\text{th}}$ order operator, i.e., multiplication by a $\mathcal{C}^\infty$ function, p_0.

Theorem 2.3. *There exists $\mathcal{C}^\infty$ function*

$$\sigma(P) : T^*X \to \mathbb{C}$$

not depending on f such that

$$p_0(x) = \sigma(P)(x, \xi) \tag{2.1}$$

with $\xi = df_x$.

Proof. It's clear that the function, $\sigma(P)$, is uniquely determined at the points, $\xi \in T_x^*$ by the property (2.1), so it suffices to prove the local existence of such a function on a neighborhood of x. Let $(U, x_1, \dots, x_n)$ be a coordinate patch centered at x and let $\xi_1, \dots, \xi_n$ be the cotangent coordinates on T^*U defined by

$$\xi \to \xi_1 \, dx_1 + \cdots + \xi_n \, dk_n \, .$$

Then if

$$P = \sum a_\alpha D^\alpha$$

on U the function, $\sigma(P)$, is given in these coordinates by $p(x, \xi) = \sum a_\alpha(x) \xi^\alpha$. (See (1.2).)

□

Composition and transposes

If P and Q are differential operators of degree r and s, PQ is a differential operator of degree $r + s$, and $\sigma(PQ) = \sigma(P)\sigma(Q)$.

Let $\mathcal{F}_X$ be the sigma field of Borel subsets of X. A *measure, dx, on* X is a measure on this sigma field. A measure, dx, is *smooth* if for every coordinate patch

$$(U, x_1, \dots, x_n) \, .$$

The restriction of dx to U is of the form

$$\varphi \, dx_1 \dots dx_n \tag{2.2}$$

φ being a non-negative $\mathcal{C}^\infty$ function and $dx_1 \dots dx_n$ being Lebesgue measure on U. dx is *non-vanishing* if the φ in (2.2) is strictly positive.

Assume dx is such a measure. Given u and $v \in \mathcal{C}_0^\infty(X)$ one defines the L^2 inner product

$$\langle u, v \rangle$$

of u and v to be the integral

$$\langle u, v \rangle = \int u \overline{v} \, dx \, .$$

Theorem 2.4. *If $P : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ is an m^{th} order differential operator there is a unique m^{th} order differential operator, P^t, having the property*

$$\langle Pu, v \rangle = \langle u, P^t v \rangle$$

for all $u, v \in \mathcal{C}_0^\infty(X)$.

Proof. Let's assume that the support of u is contained in a coordinate patch, $(U, x_1, \ldots, x_n)$. Suppose that on U

$$P = \sum a_\alpha D^\alpha$$

and

$$dx = \varphi dx_1 \ldots dx_n .$$

Then

$$\begin{aligned}
\langle Pu, v\rangle &= \sum_\alpha \int a_\alpha D^\alpha u \overline{v} \varphi dx_1 \ldots dx_n \\
&= \sum_\alpha \int a_\alpha \varphi D^\alpha u \overline{v} dx_1 \ldots dx_n \\
&= \sum \int u \overline{D^\alpha \overline{a}_\alpha \varphi v} dx_1 \ldots dx_n \\
&= \sum \int u \overline{\frac{1}{\varphi} D^\alpha \bar{a}_\alpha \varphi v} \varphi dx_1 \ldots dx_n \\
&= \langle u, P^t v\rangle
\end{aligned}$$

where

$$P^t v = \frac{1}{\varphi} \sum D^\alpha \overline{a}_\alpha \varphi v .$$

This proves the local existence *and local uniqueness* of P^t (and hence the *global* existence of P^t!).

□

Exercise.

$\sigma(P^t)(x, \xi) = \overline{\sigma(P)(x, \xi)}$.

Ellipticity.

P is elliptic if $\sigma(P)(x, \xi) \neq 0$ for all $x \in X$ and $\xi \in T_x^* - 0$.

The main goal of these notes will be to prove:

Theorem 2.5 (Fredholm theorem for elliptic operators.)**.** *If X is compact and*

$$P : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

is an elliptic differential operator, the kernel of P is finite dimensional and $u \in \mathcal{C}^\infty(X)$ is in the range of P if and only if

$$\langle u, v\rangle = 0$$

for all v in the kernel of P^t.

Remark. *Since P^t is also elliptic its kernel is finite dimensional.*

§3 Smoothing operators

Let X be an n-dimensional manifold equipped with a smooth non-vanishing measure, dx. Given $K \in \mathcal{C}^\infty(X \times X)$, one can define an operator

$$T_K : \mathcal{C}_0^\infty(X) \to \mathcal{C}^\infty(X)$$

by setting

$$T_K f(x) = \int K(x,y) f(y)\, dy\,. \tag{3.1}$$

Operators of this type are called *smoothing* operators. The definition (3.1) involves the choice of the measure, dx, however, it's easy to see that the notion of "smoothing operator" doesn't depend on this choice. Any other smooth measure will be of the form, $\varphi(x)\, dx$, where φ is an everywhere-positive $\mathcal{C}^\infty$ function, and if we replace dy by $\varphi(y)\, dy$ in (3.1) we get the smoothing operator, T_{K_1}, where $K_1(x,y) = K(x,y)\,\varphi(y)$.

A couple of elementary remarks about smoothing operators:

1. Let $L(x,y) = \overline{K(y,x)}$. Then T_L is the *transpose* of T_K. For f and g in $\mathcal{C}_0^\infty(X)$,

$$\begin{aligned} \langle T_K f, g\rangle &= \int \overline{g}(x) \left(\int K(x,y) f(y)\, dy \right) dx \\ &= \int f(y) \overline{(T_L g)(y)}\, dy = \langle f, T_L g\rangle\,. \end{aligned}$$

2. If X is compact, the composition of two smoothing operators is a smoothing operator. Explicitly:

$$T_{K_1} T_{K_2} = T_{K_3}$$

where

$$K_3(x,y) = \int K_1(x,z) K_2(z,y)\, dz\,.$$

We will now give a rough outline of how our proof of Theorem 2.5 will go. Let $I : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ be the identity operator. We will prove in the next few sections the following two results.

Theorem 3.1. *The elliptic operator, P is right-invertible modulo smoothing operators, i.e., there exists an operator, $Q : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ and a smoothing operator, T_K, such that*

$$PQ = I - T_K \tag{3.2}$$

and

Theorem 3.2. *The Fredholm theorem is true for the operator, $I - T_K$, i.e., the kernel of this operator is finite dimensional, and $f \in \mathcal{C}^\infty(X)$ is in the image of this operator if and only if it is orthogonal to kernel of the operator, $I - T_L$, where $L(x,y) = \overline{K(y,x)}$.*

Remark. In particular since T_K is the transpose of T_L, the kernel of $I - T_L$ is finite dimensional.

The proof of Theorem 3.2 is very easy, and in fact we'll leave it as a series of exercises. (See §9.) The proof of Theorem 3.1, however, is a lot harder and will involve the theory of pseudodifferential operators on the n-torus, T^n.

We will conclude this section by showing how to deduce Theorem 2.5 from Theorems 3.1 and 3.2. Let V be the kernel of $I - T_L$. By Theorem 3.2, V is a finite dimensional space, so every element, f, of $\mathcal{C}^\infty(X)$ can be written uniquely as a sum

$$f = g + h \tag{3.3}$$

where g is in V and h is orthogonal to V. Indeed, if $f_1, \ldots, f_m$ is an orthonormal basis of V with respect to the L^2 norm

$$g = \sum \langle f, f_i \rangle f_i$$

and $h = f - g$. Now let U be the orthocomplement of $V \cap \text{Image } P$ in V.

Proposition 3.3. *Every $f \in \mathcal{C}^\infty(M)$ can be written uniquely as a sum*

$$f = f_1 + f_2 \tag{3.4}$$

where $f_1 \in U$, $f_2 \in$ Image P and f_1 is orthogonal to f_2.

Proof. By Theorem 3.1

$$\text{Image } P \supset \text{Image}\,(I - T_K)\,. \tag{3.5}$$

Let g and h be the "g" and "h" in (3.3). Then since h is orthogonal to V, it is in Image $(I - T_K)$ by Theorem 3.2 and hence in Image P by (3.5). Now let $g = f_1 + g_2$ where f_1 is in U and g_2 is in the orthocomplement of U in V (i.e., in $V \cap \text{Image } P$). Then

$$f = f_1 + f_2$$

where $f_2 = g_2 + h$ is in Image P. Since f_1 is orthogonal to g_2 and h it is orthogonal to f_2.

□

Next we'll show that

$$U = \text{Ker } P^t\,. \tag{3.6}$$

Indeed $f \in U \Leftrightarrow f \perp \text{Image } P \Leftrightarrow \langle f, Pu \rangle = 0$ for all $u \Leftrightarrow \langle P^t f, u \rangle = 0$ for all $u \leftrightarrow P^t f = 0$.

This proves that all the assertions of the Theorem are true except for the finite dimensionality of Ker P. However, (3.6) tells us that Ker P^t is finite dimensional and so, with P and P^t interchanged, Ker P is finite dimensional.

§4 Elliptic systems

Let $\mathcal{C}^\infty(X, \mathbb{C}^k)$ be the space of $\mathcal{C}^\infty$ functions on X with values in $\mathbb{C}^k$. We will think of elements of this space as k-tuples of functions, $u = (u_1, \ldots, u_k)$, $u_i \in \mathcal{C}^\infty(X)$, and if u and v are in $\mathcal{C}^\infty(X, \mathbb{C}^k)$ and one of them is compactly supported we will define their inner product to be the integral

$$\langle u, v \rangle = \int_X (\sum u_i \bar{v}_i)\, dx \,.$$

Now let

$$[P_{i,j}]$$

be a $k \times \ell$ matrix whose i, j^{th} entry is a differential operator

$$P_{i,j} : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X) \,.$$

One gets from this matrix a differential operator

$$P : \mathcal{C}^\infty(X, \mathbb{C}^\ell) \to \mathcal{C}^\infty(X, \mathbb{C}^k) \tag{4.1}$$

mapping u to $v = Pu$ where

$$v_i = \sum P_{ij} u_j \,, \quad i = 1, \ldots, K \,. \tag{4.2}$$

Moreover, from the transpose matrix,

$$[P^t_{j,i}]$$

one gets an operator

$$P^t : \mathcal{C}^\infty(X, \mathbb{C}^k) \to \mathcal{C}^\infty(X, \mathbb{C}^\ell) \tag{4.3}$$

and it's easy to check that for $u \in \mathcal{C}^\infty_0(X, \mathbb{C}^\ell)$ and $v \in \mathcal{C}^\infty_0(X, \mathbb{C}^k)$

$$\langle Pu, v \rangle = \langle u, P^t v \rangle \,. \tag{4.4}$$

We'll say that P is of order r if the $P_{i,j}$'s are of order r and we'll define the *symbol* of P to be the matrix

$$[\sigma(P_{i,j}(x, \xi))]$$

at $\xi \in T^*_x$. If this symbol is invertible for all (x, ξ), $\xi \neq 0$, we'll say that P is *elliptic*. As we'll see in § 8, the Fredholm theorem for elliptic operators that we described in § 2 is valid as well for these more general elliptic operators.

Theorem 4.1. *If X is compact and the operator (4.1) is elliptic the kernel of this operator is finite dimensional, and u is in the range of P if and only iff $\langle u, v \rangle = 0$ for all v in the kernel of P^t.*

In its basic outline the proof of this is the same as the proof of Theorem 2.5 which we sketched in § 3. Let

$$K = [K_{i,j}] \quad 1 \leq i\,,\, j \leq k$$

be a $k \times k$ matrix of functions, $k_{i,j} \in \mathcal{C}^\infty(X \times X)$ and let

$$T_K : \mathcal{C}_0^\infty(X, \mathbb{C}^k) \to \mathcal{C}^\infty(X, \mathbb{C}^k)$$

be the operator mapping u to $T_K u = v$ where

$$v_i = \sum T_{K_{i,j}} u_j\,. \tag{4.5}$$

We will call operators of this sort smoothing. The transpose of this operator is the operator, T_L, where

$$L = [L_{ij}(x,y)] = [\bar{K}_{j,i}(y,x)]$$

so it, too is smoothing.

Just as in § 3 we will deduce Theorem 4.1 from the following two results.

Theorem 4.2. *If X is compact and the operator (4.1) is elliptic, then it's invertible module smoothing operators.*

and

Theorem 4.3. *The Fredholm theorem is true for the operator, $I - T_K$, where T_K is defined by (4.5).*

§5 Fourier analysis on the n-torus

In these notes the "n-torus" will be, by definition, the manifold: $T^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$. A $\mathcal{C}^\infty$ function, f, on T^n can be viewed as a $\mathcal{C}^\infty$ function on $\mathbb{R}^n$ which is *periodic* of period 2π: For all $k \in \mathbb{Z}^n$

$$f(x + 2\pi k) = f(x)\,. \tag{5.1}$$

Basic examples of such functions are the functions

$$e^{ikx}\,, \quad k \in \mathbb{Z}^n\,, \quad kx = k_1 x_1 + \cdots k_n x_n\,.$$

Let $\mathcal{P} = \mathcal{C}^\infty(T^n) = \mathcal{C}^\infty$ functions on $\mathbb{R}^n$ satisfying (5.1), and let $Q \subseteq \mathbb{R}^n$ be the open cube

$$0 < x_i < 2\pi\,. \quad i = 1, \ldots, n\,.$$

Given $f \in \mathcal{P}$ we'll define

$$\int_{T^n} f\,dx = \left(\frac{1}{2\pi}\right)^n \int_Q f\,dx$$

and given $f, g \in \mathcal{P}$ we'll define their L^2 inner product by

$$\langle f, g \rangle = \int_{T^n} f\overline{g}\, dx\,.$$

I'll leave you to check that

$$\langle e^{ikx}\,,\, e^{i\ell x} \rangle$$

is zero if $k \neq \ell$ and 1 if $k = \ell$. Given $f \in \mathcal{P}$ we'll define the k^{th} *Fourier coefficient* of f to be the L^2 inner product

$$c_k = c_k(f) = \langle f, e^{ikx} \rangle = \int_{T^n} f e^{-ikx}\, dx\,.$$

The *Fourier series* of f is the formal sum

$$\sum c_k e^{ikx}\,, \quad k \in \mathbb{Z}^n\,. \tag{5.2}$$

In this section I'll review (very quickly) standard facts about Fourier series.

It's clear that $f \in \mathcal{P} \Rightarrow D^\alpha f \in \mathcal{P}$ for all multi-indices, α.

Proposition 5.1. *If $g = D^\alpha f$*

$$c_k(g) = k^\alpha c_k(f)\,.$$

Proof.

$$\int_{T^n} D^\alpha f e^{-ikx}\, dx = \int_{T^n} f\overline{D^\alpha e^{ikx}}\, dx\,.$$

Now check

$$D^\alpha e^{ikx} = k^\alpha e^{ikx}\,.$$

□

Corollary 5.2. *For every integer $r > 0$ there exists a constant C_r such that*

$$|c_k(f)| \leq C_r(1 + |k|^2)^{-r/2}\,. \tag{5.3}$$

Proof. Clearly

$$|c_k(f)| \leq \frac{1}{(2\pi)^n} \int_Q |f|\, dx = C_0\,.$$

Moreover, by the result above, with $g = D^\alpha f$

$$|k^\alpha c_k(f)| = |c_k(g)| \leq C_\alpha$$

and from this it's easy to deduce an estimate of the form (5.3).

□

Proposition 5.3. *The Fourier series (5.2) converges and this sum is a $\mathcal{C}^\infty$ function.*

To prove this we'll need

Lemma 5.4. *If $m > n$ the sum*

$$\sum \left(\frac{1}{1+|k|^2} \right)^{m/2} , \quad k \in \mathbb{Z}^n , \tag{5.4}$$

converges.

Proof. By the "integral test" it suffices to show that the integral

$$\int_{\mathbb{R}^n} \left(\frac{1}{1+|x|^2} \right)^{m/2} dx$$

converges. However in polar coordinates this integral is equal to

$$\gamma_{n-1} \int_0^\infty \left(\frac{1}{1+|r|^2} \right)^{m/2} r^{n-1}\, dr$$

(γ_{n-1} being the volume of the unit $n-1$ sphere) and this converges if $m > n$. □

Combining this lemma with the estimate (5.3) one sees that (5.2) converges absolutely, i.e.,

$$\sum |c_k(f)|$$

converges, and hence (5.2) converges uniformly to a continuous limit. Moreover if we differentiate (5.2) term by term we get

$$D^\alpha \sum c_k e^{ikx} = \sum k^\alpha c_k e^{ikx}$$

and by the estimate (5.3) this converges absolutely and uniformly. Thus the sum (5.2) exists, and so do its derivatives of all orders.

Let's now prove the fundamental theorem in this subject, the identity

$$\sum c_k(f) e^{ikx} = f(x) \,. \tag{5.5}$$

Proof. Let $\mathcal{A} \subseteq \mathcal{P}$ be the algebra of trigonometric polynomials:

$$f \in \mathcal{A} \Leftrightarrow f(x) = \sum_{|k| \le m} a_k e^{ikx}$$

for some m. □

Claim. This is an algebra of continuous functions on T^n having the Stone–Weierstrass properties

1) Reality: If $f \in \mathcal{A}$, $\overline{f} \in \mathcal{A}$.

2) $1 \in \mathcal{A}$.

3) If x and y are points on T^n with $x \neq y$, there exists an $f \in \mathcal{A}$ with $f(x) \neq f(y)$.

Proof. Item 2 is obvious and item 1 follows from the fact that $\overline{e^{ikx}} = e^{-ikx}$. Finally to verify item 3 we note that the finite set, $\{e^{ix_1}, \ldots, e^{ix_n}\}$, already separates points. Indeed, the map

$$T^n \to (S^1)^n$$

mapping x to $e^{ix_1}, \ldots, e^{ix_n}$ is bijective.

Therefore by the Stone–Weierstrass theorem $\mathcal{A}$ is dense in $C^0(T^n)$. Now let $f \in \mathcal{P}$ and let g be the Fourier series (5.2). Is f equal to g? Let $h = f - g$. Then

$$\begin{aligned} \langle h, e^{ikx} \rangle &= \langle f, e^{ikx} \rangle - \langle g, e^{ikx} \rangle \\ &= c_k(f) - c_k(f) = 0 \end{aligned}$$

so $\langle h, e^{ikx} \rangle = 0$ for all e^{ikx}, hence $\langle h, \varphi \rangle = 0$ for all $\varphi \in \mathcal{A}$. Therefore since $\mathcal{A}$ is dense in $\mathcal{P}$, $\langle h, \varphi \rangle = 0$ for all $\varphi \in \mathcal{P}$. In particular, $\langle h, h \rangle = 0$, so $h = 0$.

□

I'll conclude this review of the Fourier analysis on the n-torus by making a few comments about the L^2 theory.

The space, $\mathcal{A}$, is dense in the space of continuous functions on T^n and this space is dense in the space of L^2 functions on T^n. Hence if $h \in L^2(T^n)$ and $\langle h, e^{ikx} \rangle = 0$ for all k the same argument as that I sketched above shows that $h = 0$. Thus

$$\{e^{ikx}\,,\ k \in \mathbb{Z}^n\}$$

is an orthonormal basis of $L^2(T^n)$. In particular, for every $f \in L^2(T^n)$ let

$$c_k(f) = \langle f, e^{ikx} \rangle\,.$$

Then the Fourier series of f

$$\sum c_k(f) e^{ikx}$$

converges in the L^2 sense to f and one has the Plancherel formula

$$\langle f, f \rangle = \sum |c_k(f)|^2\,, \quad k \in \mathbb{Z}^n\,.$$

§6 Pseudodifferential operators on T^n

In this section we will prove Theorem 2.5 for elliptic operators on T^n. Here's a road map to help you navigate this section. §6.1 is a succinct summary of the material in §4. Sections 6.2, 6.3 and 6.4 are a brief account of the theory of pseudodifferential operators on T^n and the symbolic calculus that's involved in this theory. In §6.5 and 6.6 we prove that an elliptic operator on T^n is right invertible modulo smoothing operators (and that its inverse is a pseudodifferential operator). Finally, in §6.7, we prove that pseudodifferential operators have a property called "pseudolocality" which makes them behave in some ways like differential operators (and which will enable us to extend the results of this section from T^n to arbitrary compact manifolds).

Some notation which will be useful below: for $a \in \mathbb{R}^n$ let

$$\langle a \rangle = (|a|^2 + 1)^{\frac{1}{2}} \,.$$

Thus

$$|a| \leq \langle a \rangle$$

and for $|a| \geq 1$

$$\langle a \rangle \leq 2|a| \,.$$

6.1 The Fourier inversion formula

Given $f \in \mathcal{C}^\infty(T^n)$, let $c_k(f) = \langle f, e^{ikx} \rangle$. Then:

1) $c_k(D^\alpha f) = k^\alpha c_k(f)$.
2) $|c_k(f)| \leq C_r \langle k \rangle^{-r}$ for all r.
3) $\sum c_k(f) e^{ikx} = f$.

Let S be the space of functions,

$$g : \mathbb{Z}^n \to \mathbb{C}$$

satisfying

$$|g(k)| \leq C_r \langle k \rangle^{-r}$$

for all r. Then the map

$$F : \mathcal{C}^\infty(T^n) \to S \,, \quad Ff(k) = c_k(f)$$

is bijective and its inverse is the map,

$$g \in S \to \sum g(k) e^{ikx} \,.$$

6.2 Symbols

A function $a : T^n \times \mathbb{R}^n \to \mathbb{C}$ is in $\mathcal{S}^m$ if, for all multi-indices, α and β,

$$|D_x^\alpha D_\xi^\beta a| \le C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|} \,. \tag{6.2.1}$$

Examples

1) $a(x,\xi) = \sum_{|\alpha| \le m} a_\alpha(x)\xi^\alpha$, $a_\alpha \in \mathcal{C}^\infty(T^n)$.

2) $\langle \xi \rangle^m$.

3) $a \in \mathcal{S}^\ell$ and $b \in \mathcal{S}^m \Rightarrow ab \in S^{\ell+m}$.

4) $a \in \mathcal{S}^m \Rightarrow D_x^\alpha D_\xi^\beta a \in \mathcal{S}^{m-|\beta|}$.

The asymptotic summation theorem

Given $b_i \in \mathcal{S}^{m-i}$, $i = 0, 1, \ldots$, there exists a $b \in \mathcal{S}^m$ such that

$$b - \sum_{j<i} b_j \in \mathcal{S}^{m-i}. \tag{6.2.2}$$

Proof. Step 1. Let $\ell = m + \epsilon$, $\epsilon > 0$. Then

$$|b_i(x,\xi)| < C_i \langle \xi \rangle^{m-i} = \frac{C_i \langle \xi \rangle^{\ell-i}}{\langle \xi \rangle^\epsilon} \,.$$

Thus, for some λ_i,

$$|b_i(x,\xi)| < \frac{1}{2^i} \langle \xi \rangle^{\ell-i}$$

for $|\xi| > \lambda_i$. We can assume that $\lambda_i \to +\infty$ as $i \to +\infty$. Let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be bounded between 0 and 1 and satisfy $\rho(t) = 0$ for $t < 1$ and $\rho(t) = 1$ for $t > 2$. Let

$$b = \sum \rho\left(\frac{|\xi|}{\lambda_i}\right) b_i(x,\xi) \,. \tag{6.2.3}$$

Then b is in $\mathcal{C}^\infty(T^n \times \mathbb{R}^n)$ since, on any compact subset, only a finite number of summands are non-zero. Moreover, $b - \sum_{j<i} b_j$ is equal to:

$$\sum_{j<i} \left(\rho\left(\frac{|\xi|}{\lambda_j}\right) - 1 \right) b_j + b_i + \sum_{j>i} \rho\left(\frac{|\xi|}{\lambda_j}\right) b_j \,.$$

The first summand is compactly supported, the second summand is in $\mathcal{S}^{m-1}$ and the third summand is bounded from above by

$$\sum_{k>i} \frac{1}{2^k} \langle \xi \rangle^{\ell-k}$$

which is less than $\langle\xi\rangle^{\ell-(i+1)}$ and hence, for $\epsilon < 1$, less than $\langle\xi\rangle^{m-i}$.

Step 2. For $|\alpha| + |\beta| \le N$ choose λ_i so that

$$|D_x^\alpha D_\xi^\beta b_i(x,\xi)| \le \frac{1}{2^i}\langle\xi\rangle^{\ell-i-|\beta|}$$

for $\lambda_i < |\xi|$. Then the same argument as above implies that

$$D_x^\alpha D_\xi^\beta (b - \sum_{j,i} b_j) \le C_N \langle\xi\rangle^{m-i-|\beta|} \tag{6.2.4}$$

for $|\alpha| + |\beta| \le N$.

Step 3. The sequence of λ_i's in step 2 depends on N. To indicate this dependence let's denote this sequence by $\lambda_{i,N}$, $i = 0, 1, \ldots$. We can, by induction, assume that for all i, $\lambda_{i,N} \le \lambda_{i,N+1}$. Now apply the Cantor diagonal process to this collection of sequences, i.e., let $\lambda_i = \lambda_{i,i}$. Then b has the property (6.2.4) for all N.

We will denote the fact that b has the property (6.2.2) by writing

$$b \sim \sum b_i\,. \tag{6.2.5}$$

The symbol, b, is not unique, however, if $b \sim \sum b_i$ and $b' \sim \sum b_i$, $b - b'$ is in the intersection, $\bigcap \mathcal{S}^\ell$, $-\infty < \ell < \infty$.

$\square$

6.3 Pseudodifferential operators

Given $a \in \mathcal{S}^m$ let

$$T_a^0 : S \to \mathcal{C}^\infty(T^n)$$

be the operator

$$T_a^0 g = \sum a(x,k) g(k) e^{ikx}\,.$$

Since

$$|D^\alpha a(x,k)e^{ikx}| \le C_\alpha \langle k\rangle^{m+\langle\alpha\rangle}$$

and

$$|g(k)| \le C_\alpha \langle k\rangle^{-(m+n+|\alpha|+1)}$$

this operator is well-defined, i.e., the right hand side is in $\mathcal{C}^\infty(T^n)$. Composing T_a^0 with F we get an operator

$$T_a : \mathcal{C}^\infty(T^n) \to \mathcal{C}^\infty(T^n)\,.$$

We call T_a the pseudodifferential operator with symbol a.

Note that

$$T_a e^{ikx} = a(x,k)e^{ikx}\,.$$

Also note that if

$$P = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha \tag{6.3.1}$$

and

$$p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha\,. \tag{6.3.2}$$

Then

$$P = T_p\,.$$

6.4 The composition formula

Let P be the differential operator (6.3.1). If a is in $\mathcal{S}^r$ we will show that PT_a is a pseudodifferential operator of order $m+r$. In fact we will show that

$$PT_a = T_{p\circ a} \tag{6.4.1}$$

where

$$p\circ a(x,\xi) = \sum_{|\beta|\le m} \frac{1}{\beta!}\partial_\xi^\beta p(x,\xi)D_x^\beta a(x,\xi) \tag{6.4.2}$$

and $p(x,\xi)$ is the function (6.3.2).

Proof. By definition

$$\begin{aligned} PT_a e^{ikx} &= Pa(x,k)e^{ikx} \\ &= e^{ikx}(e^{-ikx}Pe^{ikx})a(x,k)\,. \end{aligned}$$

Thus PT_a is the pseudodifferential operator with symbol

$$e^{-ix\xi}Pe^{ix\xi}a(x,\xi)\,. \tag{6.4.3}$$

However, by (6.3.1):

$$\begin{aligned} e^{-ix\xi}Pe^{ix\xi}u(x) &= \sum a_\alpha(x)e^{-ix\xi}D^\alpha e^{ix\xi}u(x) \\ &= \sum a_\alpha(x)(D+\xi)^\alpha u(x) \\ &= p(x,D+\xi)u(x)\,. \end{aligned}$$

Moreover,

$$p(x, \eta + \xi) = \sum \frac{1}{\beta!} \frac{\partial}{\partial \xi^\beta} p(x, \xi) \eta^\beta ,$$

so

$$p(x, D + \xi) u(x) = \sum \frac{1}{\beta!} \frac{\partial}{\partial \xi^\beta} p(x, \xi) D^\beta u(x)$$

and if we plug in $a(x, \xi)$ for $u(x)$ we get, by (6.4.3), the formula (6.4.2) for the symbol of PT_a.

□

6.5 The inversion formula

Suppose now that the operator (6.3.1) is elliptic. We will prove below the following inversion theorem.

Theorem 6.1. *There exists an $a \in \mathcal{S}^{-m}$ and an $r \in \bigcap S^\ell$, $-\infty < \ell < \infty$, such that*

$$PT_a = I - T_r .$$

Proof. Let

$$p_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha .$$

By ellipticity $p_m(x, \xi) \neq 0$ for $\xi \notin 0$. Let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a function satisfying $\rho(t) = 0$ for $t < 1$ and $\rho(t) = 1$ for $t > 2$. Then the function

$$a_0(x, \xi) = \rho(|\xi|) \frac{1}{p_m(x, \xi)} \tag{6.5.1}$$

is well-defined and belongs to S^{-m}. To prove the theorem we must prove that there exist symbols $a \in \mathcal{S}^{-m}$ and $r \in \bigcap \mathcal{S}^\ell$, $-\infty < \ell < \infty$, such that

$$p \circ q = 1 - r .$$

We will deduce this from the following two lemmas.

Lemma 6.2. *If $b \in \mathcal{S}^i$ then*

$$b - p \circ a_0 b$$

is in $\mathcal{S}^{i-1}$.

Proof. Let $q = p - p_m$. Then $q \in \mathcal{S}^{m-1}$ so $q \circ a_0 b$ is in $\mathcal{S}^{i-1}$ and by (**??**)

$$\begin{aligned} p \circ a_0 b &= p_m \circ a_0 b + q \circ a_0 b \\ &= p_m a_0 b + \cdots = b + \cdots \end{aligned}$$

where the dots are terms of order $i - 1$.

□

Lemma 6.3. *There exists a sequence of symbols $a_i \in \mathcal{S}^{-m-i}$, $i = 0, 1, \ldots$, and a sequence of symbols $r_i \in \mathcal{S}^{-i}$, $i = 0, \ldots$, such that a_0 is the symbol (6.5.1), $r_0 = 1$ and*

$$p \circ a_i = r_i - r_{i+1}$$

for all i.

Proof. Given $a_0, \ldots, a_{i-1}$ and $r_0, \ldots r_i$, let $a_i = r_i a_0$ and $r_{i+1} = r_i - p \circ a_i$. By Lemma 6.2, $r_{i+1} \in \mathcal{S}^{-i-1}$.

□

Now let $a \in \mathcal{S}^{-m}$ be the "asymptotic sum" of the a_i's

$$a \sim \sum a_i \,.$$

Then

$$p \circ a \sim \sum p \circ a_i = \sum_{i=0}^{\infty} r_i - r_{i+1} = r_0 = 1\,,$$

so $1 - p \circ a \sim 0$, i.e., $r = 1 - p \circ q$ is in $\bigcap \mathcal{S}^{\ell}$, $-\infty < \ell < \infty$.

□

6.6 The inversion formula for systems

The result above can easily be generalized to the case where P is an m^{th} order elliptic operator

$$P : \mathcal{C}^{\infty}(T^n, \mathbb{C}^k) \to \mathcal{C}^{\infty}(T^n, \mathbb{C}^k)\,.$$

As in (5.2) let P be defined by the matrix

$$[P_{i,j}]$$

where

$$P_{i,j} = \sum_{|\alpha| \leq m} a_{\alpha,i,j}(x) D^{\alpha}\,.$$

Let

$$p_{i,j}(x,\xi) = \sum_{|\alpha| \leq m} a_{\alpha,i,j}(x) \xi^{\alpha}$$

and

$$p_{m,i,j}(x,\xi) = \sum_{|\alpha| = m} a_{\alpha,i,j}(x) \xi^{\alpha}\,.$$

Since P is elliptic the matrix

$$P_m(x,\xi) = [p_{m,i,j}(x,\xi)]$$

is invertible for $\xi \neq 0$, and as in (6.5.1) we'll set

$$A_0 = \rho(|\xi|)P_m(x,\xi)^{-1}.$$

Letting

$$P(x,\xi) = [p_{ij}(x,\xi)]$$

we can, as in § 6.5, solve inductively the equations

$$P(x,\xi) \circ A_d(x,\xi) = R_d(x,\xi) - R_{d+1}(x,\xi)$$

where $R_0(x,\xi)$ is the identity $k \times k$ matrix, $R_d(x,\xi)$ has matrix entries in S^{-d}, $A_d(x,\xi)$ has the matrix entries in S^{-d-m} and the matrix entries of the product on the left are

$$\sum_\ell p_{i\ell}(x,\xi) \circ a_{d,\ell,j}(x,\xi).$$

Thus if

$$A \sim \sum A_d$$

one concludes that

$$P(x,\xi) \circ A(x,\xi) = I - R(x,\xi)$$

where $R(x,\xi)$ has matrix entries in $\bigcap S^d$, $-\infty < d < \infty$, and from this one gets the operator identity

(6.6.1) $$PT_A = I - T_R.$$

6.7 Smoothing properties of ΨDO's

Let $a \in \mathcal{S}^\ell$, $\ell < -m-n$. We will prove in this section that the sum

(6.7.1) $$K_a(x,y) = \sum a(x,k)e^{ik(x-y)}$$

is in $C^m(T^n \times T^n)$ and that T_a is the integral operator associated with K_a, i.e.,

$$T_a u(x) = \int K_a(x,y)u(y)\,dy.$$

Proof. For $|\alpha| + |\beta| \leq m$

$$D_x^\alpha D_y^\beta a(x,k)e^{ik(x-y)}$$

is bounded by $\langle k\rangle^{\ell+|\alpha|+|\beta|}$ and hence by $\langle k\rangle^{\ell+m}$. But $\ell + m < -n$, so the sum

$$\sum D_x^\alpha D_y^\beta a(x,k)e^{ik(x-y)}$$

converges absolutely. Now notice that

$$\int K_a(x,y)e^{iky}\,dy = a(x,k)e^{ikx} = T_\alpha e^{ikx}.$$

Hence T_a is the integral operators defined by K_a. Let

$$\mathcal{S}^{-\infty} = \bigcap \mathcal{S}^{\ell}\,, \quad -\infty < \ell < \infty\,. \tag{6.7.2}$$

If a is in $\mathcal{S}^{-\infty}$, then by (6.7.1), T_a is a smoothing operator.

□

6.8 Pseudolocality

We will prove in this section that if f and g are $\mathcal{C}^\infty$ functions on T^n with non-overlapping supports and a is in $\mathcal{S}^m$, then the operator

$$u \in \mathcal{C}^\infty(T^n) \to f T_a g u \tag{6.8.1}$$

is a smoothing operator. (This property of pseudodifferential operators is called *pseudolocality*.) We will first prove:

Lemma 6.4. *If $a(x,\xi)$ is in $\mathcal{S}^m$ and $w \in \mathbb{R}^n$, the function,*

$$a_w(x,\xi) = a(x,\xi+w) - a(x,\xi) \tag{6.8.2}$$

is in S^{m-1}.

Proof. Recall that $a \in \mathcal{S}^m$ if and only if

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}\,.$$

From this estimate is is clear that if a is in $\mathcal{S}^m$, $a(x,\xi+w)$ is in $\mathcal{S}^m$ and $\frac{\partial a}{\partial \xi_i}(x,\xi)$ is in $\mathcal{S}^{m-1}$, and hence that the integral

$$a_w(x,\xi) = \int_0^1 \sum_i \frac{\partial a}{\partial \xi_i}(x,\xi+tw)\,dt$$

in $\mathcal{S}^{m-1}$.

Now let ℓ be a large positive integer and let a be in $\mathcal{S}^m$, $m < -n-\ell$. Then

$$K_a(x,y) = \sum a(x,k) e^{ik(x-y)}$$

is in $C^\ell(T^n \times T^n)$, and T_a is the integral operator defined by K_a. Now notice that for $w \in \mathbb{Z}^n$

$$(e^{-i(x-y)w} - 1) K_a(x,y) = \sum a_w(x,k) e^{ik(x-y)}\,, \tag{6.8.3}$$

so by the lemma the left hand side of (**??**) is in $C^{\ell+1}(T^n \times T^n)$. More generally,

$$(e^{-i(x-y)w} - 1)^N K_a(x,y) \tag{6.8.4}$$

is in $C^{\ell+N}(T^n \times T^n)$. In particular, if $x \neq y$, then for some $1 \leq i \leq n$, $x_i - y_i \not\equiv 0$ mod $2\pi Z$, so if

$$w = (0, 0, \ldots, 1, 0, \ldots, 0)\,,$$

(a "1" in the i$^{\text{th}}$-slot), $e^{i(x-y)w} \neq 1$ and, by (6.8.4), $K_a(x, y)$ is $C^{\ell+N}$ in a neighborhood of (x, y). Since N can be arbitrarily large we conclude

Lemma 6.5. *$K_a(x, y)$ is a $\mathcal{C}^\infty$ function on the complement of the diagonal in $T^n \times T^n$.*

Thus if f and g are $\mathcal{C}^\infty$ functions with non-overlapping support, $fT_a g$ is the smoothing operator, T_K, where

$$K(x, y) = f(x)K_a(x, y)g(y)\,. \tag{6.8.5}$$

□

We have proved that T_a is pseudolocal if $a \in \mathcal{S}^m$, $m < -n - \ell$, ℓ a large positive integer. To get rid of this assumption let $\langle D\rangle^N$ be the operator with symbol $\langle\xi\rangle^N$. If N is an even positive integer

$$\langle D\rangle^N = (\sum D_i^2 + I)^{\frac{N}{2}}$$

is a differential operator and hence is a *local* operator: if f and g have non-overlapping supports, $f\langle D\rangle^N g$ is identically zero. Now let $a_N(x, \xi) = a(x, \xi)\langle\xi\rangle^{-N}$. Since $a_N \in \mathcal{S}^{m-N}$, T_{a_N} is pseudolocal for N large. But $T_a = T_{a_N}\langle D\rangle^N$, so T_a is the composition of an operator which is pseudolocal with an operator which is local, and therefore T_a itself is pseudolocal.

§7 Elliptic operators on open subsets of T^n

Let U be an open subset of T^n. We will denote by $\iota_U : U \to T^n$ the inclusion map and by $\iota_U^* : \mathcal{C}^\infty(T^n) \to \mathcal{C}^\infty(U)$ the restriction map: let V be an open subset of T^n containing $\overline{U}$ and

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha\,, \quad a_\alpha(x) \in \mathcal{C}^\infty(V)$$

an elliptic m^{th} order differential operator. Let

$$P^t = \sum_{|\alpha| \leq m} D^\alpha \overline{a}_\alpha(x)$$

be the transpose operator and

$$p_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha$$

the symbol of P. We will prove below the following localized version of the inversion formula of § **??**.

Theorem 7.1. *There exist symbols, $a \in \mathcal{S}^{-m}$ and $r \in \mathcal{S}^{-\infty}$ such that*

$$P\iota_U^* T_a = \iota_U^*(I - T_r)\,. \tag{7.1}$$

Proof. Let $\gamma \in \mathcal{C}_0^\infty(V)$ be a function which is bounded between 0 and 1 and is identically 1 in a neighborhood of $\overline{U}$. Let

$$Q = PP^t\gamma + (1-\gamma)(\sum D_i^2)^n\,.$$

This is a globally defined $2m^{\text{th}}$ order differential operator in T^n with symbol,

$$\gamma(x)|p_m(x,\xi)|^2 + (1-\gamma(x))|\xi|^{2m} \tag{7.2}$$

and since (7.2) is non-vanishing on $T^n \times (\mathbb{R}^n - 0)$, this operator is elliptic. Hence, by Theorem **??**, there exist symbols $b \in \mathcal{S}^{-2m}$ and $r \in \mathcal{S}^{-\infty}$ such that

$$QT_b = I - T_r\,.$$

Let $T_a = P^t\gamma T_b$. Then since $\gamma \equiv 1$ on a neighborhood of $\overline{U}$,

$$\begin{aligned}
\iota_U^*(I - T_r) &= \iota_U^* Q T_b \\
&= \iota_U^*(PP^t\gamma T_b + (1-\gamma)\sum D_i^2 T_b) \\
&= \iota_U^* PP^t\gamma T_b \\
&= P\iota_U^* P^t\gamma T_b = P\iota_U^* T_a\,.
\end{aligned}$$

§8 Elliptic operators on compact manifolds

Let X be a compact n dimensional manifold and

$$P : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

an elliptic m^{th} order differential operator. We will show in this section how to construct a *parametrix* for P: an operator

$$Q : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

such that $I - PQ$ is smoothing.

Let V_i, $i = 1,\ldots,N$ be a covering of X by coordinate patches and let U_i, $i = 1,\ldots,N$, $\overline{U}_i \subset V_i$ be an open covering which refines this covering. We can, without loss of generality, assume that V_i is an open subset of the hypercube

$$\{x \in \mathbb{R}^n \quad 0 < x_i < 2\pi \quad i = 1,\ldots,n\}$$

and hence an open subset of T^n. Let

$$\{\rho_i \in \mathcal{C}_0^\infty(U_i)\,, \quad i = 1,\ldots,N\}$$

be a partition of unity and let $\gamma_i \in \mathcal{C}_0^\infty(U_i)$ be a function which is identically one on a neighborhood of the support of ρ_i. By Theorem 6.1, there exist symbols $a_i \in \mathcal{S}^{-m}$ and $r_i \in \mathcal{S}^{-\infty}$ such that on T^n:

$$P\iota_{U_i}^* T_{a_i} = \iota_{U_i}^*(I - T_{r_i})\,. \tag{8.1}$$

Moreover, by pseudolocality $(1-\gamma_i)T_{a_i}\rho_i$ is smoothing, so

$$\gamma_i T_{a_i}\rho_i - \iota_{U_i}^* T_{a_i}\rho_i$$

and

$$P\gamma_i T_{a_i}\rho_i - P\iota_{U_i}^* T_{a_i}\rho_i$$

are smoothing. But by (7.1)

$$P\iota_{U_i}^* T_{a_i}\rho_i - \rho_i I$$

is smoothing. Hence

$$P\gamma_i T_{a_i}\rho_i - \rho_i I \tag{8.2}$$

is smoothing as an operator on T^n. However, $P\gamma_i T_{a_i}\rho_i$ and $\rho_i I$ are *globally defined* as operators on X and hence (7.2) is a globally defined smoothing operator. Now let $Q = \sum \gamma_i T_{a_i}\rho_i$ and note that by (8.2)

$$PQ - I$$

is a smoothing operator.

□

This concludes the proof of Theorem 3.1, and hence, modulo proving Theorem 3.2. This concludes the proof of our main result: Theorem 2.5. Also it's clear from the remarks in § 6.6 that the proof above goes through verbatim for elliptic operators of the form

$$P : \mathcal{C}^\infty(X, \mathbb{C}^k) \to \mathcal{C}^\infty(X, \mathbb{C}^k)\,.$$

The proof of Theorem 3.2 will be outlined, as a series of exercises, in the next section.

§9 The Fredholm theorem for smoothing operators

Let X be a compact n-dimensional manifold equipped with a smooth non-vanishing measure, dx. Given $K \in \mathcal{C}^\infty(X \times X)$ let

$$T_K : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

be the smoothing operator (3.1).

Exercise 1. Let V be the volume of X (i.e., the integral of the constant function, 1, over X). Show that if

$$\max |K(x,y)| < \frac{\epsilon}{V}\,, \quad 0 < \epsilon < 1$$

then $I - T_K$ is invertible and its inverse is of the form, $I - T_L$, $L \in \mathcal{C}^\infty(X \times X)$.
Hint 1. Let $K_i = K \circ \cdots \circ K$ (i products). Show that $\sup |K_i(x,y)| < C\epsilon^i$ and conclude that the series

$$\sum K_i(x,y) \tag{9.1}$$

converges uniformly.
Hint 2. Let U and V be coordinate patches on X. Show that on $U \times V$

$$D_x^\alpha D_y^\beta K_i(x,y) = K^\alpha \circ K_{i-2} \circ K^\beta(x,y)$$

where $K^\alpha(x,z) = D_x^\alpha K(x,z)$ and $K^\beta(z,y) = D_y^\beta K(z,y)$. Conclude that not only does (8.1) converge on $U \times V$ but so do its partial derivatives of *all* orders with respect to x and y.

Exercise 2. (finite rank operators.) T_K is a finite rank smoothing operator if K is of the form:

$$K(x,y) = \sum_{i=1}^{N} f_i(x) g_i(y)\,. \tag{9.2}$$

(a) Show that if T_K is a finite rank smoothing operator and T_L is any smoothing operator, $T_K T_L$ and $T_L T_K$ are finite rank smoothing operators.

(b) Show that if T_K is a finite rank smoothing operator, the operator, $I - T_K$, has finite dimensional kernel and co-kernel.

Hint. Show that if f is in the kernel of this operator, it is in the linear span of the f_i's and that f is in the image of this operator if

$$\int f(y) g_i(y)\, dy = 0\,, \quad i = 1, \ldots, N\,.$$

Exercise 3. Show that for every $K \in \mathcal{C}^\infty(X \times X)$ and every $\epsilon > 0$ there exists a function, $K_1 \in \mathcal{C}^\infty(X \times X)$ of the form (9.2) such that

$$\sup |K - K_1|(x,y) < \epsilon\,.$$

Hint. Let $\mathcal{A}$ be the set of all functions of the form (9.2). Show that $\mathcal{A}$ is a *subalgebra* of $C(X \times X)$ and that this subalgebra separates points. Now apply the Stone–Weierstrass theorem to conclude that $\mathcal{A}$ is dense in $C(X \times X)$.

Exercise 4. Prove that if T_K is a smoothing operator the operator

$$I - T_K : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

has finite dimensional kernel and co-kernel.
Hint. Show that $K = K_1 + K_2$ where K_1 is of the form (9.2) and K_2 satisfies the hypotheses of exercise 1. Let $I - T_L$ be the inverse of $I - T_{K_2}$. Show that the operators

$$(I - T_K) \circ (I - T_L)$$
$$(I - T_L) \circ (I - T_K)$$

are both of the form: identity minus a finite rank smoothing operator. Conclude that $I - T_K$ has finite dimensional kernel and co-kernel.

Exercise 5. Prove Theorem 3.2.
Hints: Find a finite dimensional vector subspace, V, of $\mathcal{C}^\infty(X)$ such that if $f \in \mathcal{C}^\infty(X)$ is orthogonal to V then f is in the image of $I - T_K$. (For example, let T_L be as in Exercise 4 and let $Q = I - T_L$. Then

$$(I - T_K)Q = I - T_{K_{f.r.}}$$

where

$$K_{f.r.} = \sum_{i=1}^{N} f_i(x)\bar{g}_i(y)\,.$$

Let $V = \mathrm{span}\{g_i\}$.)

Now let $h_1, \ldots, h_r$ be an orthogonal basis of V and for $f \in$ Image P let $f = f_1 + f_2$ where

$$f_1 = \sum_{i=1}^{r} \langle f, h_i \rangle h_i\,.$$

Note that f_2 is orthogonal to V, so it is in the image of P and hence $f_1 \in$ Image P. Finally let U be the orthocomplement of Image $P \cap V$ in V. Show that $U =$ Ker P^t.

Exercise 6. Prove Theorem 4.3.
Hint: Show that, with small modifications, the proof sketched above works for vector-valued smoothing operators

$$T_K : \mathcal{C}^\infty(X, \mathbb{C}^K) \to \mathcal{C}^\infty(X, \mathbb{C}^K).$$