On the Approximation of Laplacian Eigenvalues in Graph Disaggregation

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Graph disaggregation is a technique used to address the high cost of computation for power law graphs on parallel processors. The few high-degree vertices are broken into multiple small-degree vertices, in order to allow for more efficient computation in parallel. In particular, we consider computations involving the graph Laplacian, which has significant applications, including diffusion mapping and graph partitioning, among others. We prove results regarding the spectral approximation of the Laplacian of the original graph by the Laplacian of the disaggregated graph. In addition, we construct an alternate disaggregation operator whose eigenvalues interlace those of the original Laplacian. Using this alternate operator, we construct a uniform preconditioner for the original graph Laplacian.

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1. Introduction

A variety of real-world graphs, including web networks [1], social networks [15], and bioinformatics networks [10], exhibit a degree power law. Namely, the fraction of nodes of degree k, denoted by $P(k)$, follows a power distribution of the form $P(k) \sim k^{-\gamma}$, where γ is typically in the range $2 < \gamma < 3$. Networks of this variety are often referred to as scale-free networks. The pairing of a few high-degree vertices with many low-degree vertices on large scale-free networks makes computations such as Laplacian matrix-vector products and solving linear and eigenvalue equations challenging. The computation of the minimal nontrivial eigenpair can become prohibitively expensive. This eigenpair has many important applications, such as diffusion mapping and graph partitioning [2, 13, 16, 17].

Breaking the few high degree nodes into multiple smaller degree nodes is a way to address this issue, especially when large-scale parallel computers are available. This technique, called graph disaggregation, was introduced by Kuhlemann and

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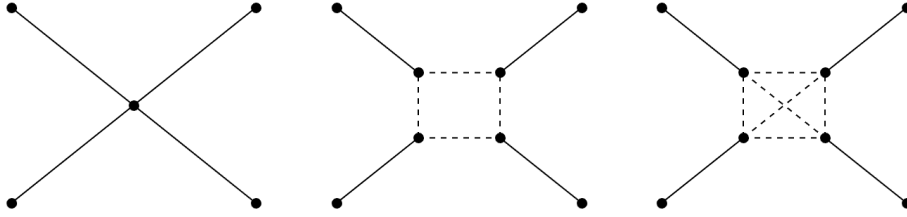*Corresponding author. Email: urschel@mit.edu

Figure 1.1.   Example of disaggregation: original graph (left); disaggregate using cycle (middle); disaggregate using clique (right).

Vassilevski [6, 11]. In this process, each of the high-degree vertices of the network is replaced by a graph, such as a cycle or a clique, where each incident edge of the original node now connects to a node of the cycle or clique (see Figure 1.1).

Independently, Lee, Peng, and Spielman investigated the concept of graph disaggregation, referred to as vertex splitting, in the setting of combinatorial spectral sparsifiers [12]. They proved results for graphs disaggregated from complete graphs and expanders, and used the Schur complement of the disaggregated Laplacian with respect to the disaggregated vertices to approximate the original Laplacian. The basic motivating assumption in such constructions is that the spectral structure of the graph Laplacian induced by the disaggregated graph approximates the spectral structure of the original graph well.

In [6, 11] Kuhlemann and Vassilevski took a numerical approach. We extend, expand upon, and prove precise and rigorous theoretical results regarding this technique. First, we look at the case of a single disaggregated vertex and establish bounds on the error in spectral approximation with respect to the Laplacians of the original and disaggregated graph, as well as results related to the Cheeger constant. We investigate a conjecture made in [11] and give strong theoretical evidence that it does not hold in general. Then, we treat the more general case of disaggregation of multiple vertices and prove analogous results. Finally, we construct an alternative disaggregation operator whose eigenvalues interlace with those of the original graph Laplacian, and, hence, provide excellent approximation to the spectrum of the latter. We then use this new disaggregation operator to construct a uniform preconditioner for the graph Laplacian of the original graph. We prove that the preconditioned graph Laplacian can be made arbitrarily close to the identity operator if we require that the weights of the internal disaggregated edges are sufficiently large.

2.   Single Vertex Disaggregation

Let us restrict ourselves to the class of weighted, connected, undirected graphs with no self-loops, denoted by $\mathcal{G}$. Consider a graph $\mathsf{G} = (V, E, \omega) \in \mathcal{G}$, $|V| = n$. Let $e = (i, j)$ denote an edge that connects vertices i and j, and $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the standard ℓ^2-inner product and the corresponding induced norm. The associated graph Laplacian $L \in \mathbb{R}^{n \times n}$ is given by

$$\langle L\boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{e=(i,j) \in E} \omega_e (u_i - u_j)(v_i - v_j), \quad \omega_e = (-l_{ij}),$$

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where we denote the (i,j)-th element of L by l_{ij}. Without loss of generality, let us disaggregate the last vertex v_n of the graph G. Then, the Laplacian can be written in the following block form

$$L = \begin{pmatrix} L_0 & -\boldsymbol{l}_n \\ -\boldsymbol{l}_n^T & \delta_n \end{pmatrix},$$

where δ_i is the weighted degree of vertex v_i defined as $\delta_i = \sum_{e=(i,j)\in E} \omega_e$. Because the graph is connected, the associated Laplacian L has eigenvalues

$$0 = \lambda_1(L) < \lambda_2(L) \leq \cdots \leq \lambda_n(L)$$

and corresponding eigenvectors

$$\mathbf{1}_n = \boldsymbol{\varphi}^{(1)}(L), \boldsymbol{\varphi}^{(2)}(L), \cdots, \boldsymbol{\varphi}^{(n)}(L), \qquad \text{where } \mathbf{1}_n = (\underbrace{1, \cdots, 1}_{n})^T.$$

The eigenpair $(\lambda_2(L), \boldsymbol{\varphi}^{(2)}(L))$ has special significance, and therefore $\lambda_2(L)$ is referred to as the *algebraic connectivity*, denoted $a(\mathsf{G})$, and $\boldsymbol{\varphi}^{(2)}(L)$ is referred to as the *Fiedler vector*.

We can also write a given nontrivial eigenpair $(\lambda(L), \boldsymbol{\varphi}(L))$, $\lambda(L) \neq 0$, $\|\boldsymbol{\varphi}(L)\| = 1$, in block notation, namely

$$\boldsymbol{\varphi}(L) = \begin{pmatrix} \boldsymbol{\varphi}_0 \\ \varphi_n \end{pmatrix}.$$

We have the relations

$$\begin{aligned} \langle \boldsymbol{\varphi}_0, \mathbf{1}_{n_0} \rangle + \varphi_n &= 0, \\ L_0 \boldsymbol{\varphi}_0 - \varphi_n \boldsymbol{l}_n &= \lambda(L) \boldsymbol{\varphi}_0, \\ \delta_n \varphi_n - \boldsymbol{l}_n^T \boldsymbol{\varphi}_0 &= \lambda(L) \varphi_n, \end{aligned}$$

where $n_0 = n - 1$. Suppose that the vertex v_n is disaggregated into d vertices, with an unspecified connected structure between the disaggregated elements. We will denote this graph by G_D. This induces a disaggregated graph Laplacian $L_D \in \mathbb{R}^{N \times N}$, $N = n_0 + d$, with eigenvalues $0 = \lambda_1(L_D) < \lambda_2(L_D) \leq \cdots \leq \lambda_N(L_D)$ and corresponding eigenvectors $\mathbf{1}_N = \boldsymbol{\varphi}^{(1)}(L_D), \boldsymbol{\varphi}^{(2)}(L_D), \cdots, \boldsymbol{\varphi}^{(N)}(L_D)$. We can write L_D in block form

$$L_D = \begin{pmatrix} L_0 & -L_{0n} \\ -L_{0n}^T & L_n \end{pmatrix}.$$

We have the relations

$$\begin{aligned} \delta_n &= \boldsymbol{l}_n^T \mathbf{1}_{n_0}, \\ L_{0n}^T \mathbf{1}_{n_0} &= L_n \mathbf{1}_d, \\ \boldsymbol{l}_n &= L_0 \mathbf{1}_{n_0} = L_{0n} \mathbf{1}_d. \end{aligned}$$

Let us introduce the prolongation matrix $P : \mathbb{R}^n \to \mathbb{R}^N$,

$$P = \begin{pmatrix} I_{n_0 \times n_0} & 0 \\ 0 & \mathbf{1}_d \end{pmatrix}. \tag{2.1}$$

The following result is immediate.

LEMMA 1   *Let L and L_D be the graph Laplacian of the original graph $\mathsf{G} \in \mathcal{G}$ and the disaggregated graph $\mathsf{G}_D \in \mathcal{G}$, respectively. If P is defined as (2.1), then we have*

$$L = P^T L_D P.$$

We aim to show that the algebraic connectivity of L_D is bounded away from the algebraic connectivity of the original graph L. To do so, suppose we have an eigenpair (λ, φ) of the Laplacian of the original graph G. We prolongate the eigenvector φ to the disaggregated graph G_D and obtain an approximate eigenvector by the procedure

$$\widetilde{\varphi} = P\varphi - s\mathbf{1}_N = \begin{pmatrix} \varphi_0 \\ \varphi_n \mathbf{1}_d \end{pmatrix} - s\mathbf{1}_N, \quad \text{where } s = \frac{d-1}{N}\varphi_n. \tag{2.2}$$

This gives $\langle \widetilde{\varphi}, \mathbf{1}_N \rangle = 0$.

We consider $\widetilde{\varphi}$ to be an approximation of φ on the non-trivial eigenspace of the disaggregated operator L_D. We have the following relation between the eigenvalue λ of L and the Rayleigh quotient of $\widetilde{\varphi}$ with respect to L_D.

LEMMA 2   *Let (λ, φ), $\|\varphi\| = 1$, be an eigenpair of the graph Laplacian L associated with a graph $\mathsf{G} \in \mathcal{G}$, and $\widetilde{\varphi}$ be defined by (2.2). We have*

$$\mathrm{RQ}(\widetilde{\varphi}) := \frac{\langle L_D \widetilde{\varphi}, \widetilde{\varphi} \rangle}{\langle \widetilde{\varphi}, \widetilde{\varphi} \rangle} = \frac{\lambda}{1 + \frac{(d-1)n}{N}\varphi_n^2}.$$

Proof. We have

$$\begin{aligned}
\langle \widetilde{\varphi}, \widetilde{\varphi} \rangle &= \langle P\varphi - s\mathbf{1}_N, P\varphi - s\mathbf{1}_N \rangle = \langle P\varphi, P\varphi \rangle - 2s\langle P\varphi, \mathbf{1}_N \rangle + s^2 \langle \mathbf{1}_N, \mathbf{1}_N \rangle \\
&= \langle \varphi_0, \varphi_0 \rangle + \varphi_n^2 \langle \mathbf{1}_d, \mathbf{1}_d \rangle - 2s \left(\langle \varphi_0, \mathbf{1}_{n_0} \rangle + \varphi_n \langle \mathbf{1}_d, \mathbf{1}_d \rangle \right) + s^2 N \\
&= \langle \varphi_0, \varphi_0 \rangle + \varphi_n^2 + (d-1)\varphi_n^2 - 2s(d-1)\varphi_n + s^2 N \\
&= 1 + \left[(d-1) - 2\frac{(d-1)^2}{N} + \frac{(d-1)^2}{N} \right] \varphi_n^2 \\
&= 1 + \frac{(d-1)n}{N}\varphi_n^2
\end{aligned}$$

and

$$\begin{aligned}
\langle L_D \widetilde{\varphi}, \widetilde{\varphi} \rangle &= \langle L_D(P\varphi - s\mathbf{1}_N), P\varphi - s\mathbf{1}_N \rangle \\
&= \langle L_D P\varphi, P\varphi \rangle - 2s\langle L_D \mathbf{1}_N, P\varphi \rangle + s^2 \langle L_D \mathbf{1}_N, \mathbf{1}_N \rangle \\
&= \langle P^T L_D P\varphi, \varphi \rangle = \langle L\varphi, \varphi \rangle = \lambda.
\end{aligned}$$

This completes the proof. □

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The following result quickly follows by applying Lemma 2 to the Fielder vector.

THEOREM 1   *Let $\boldsymbol{\varphi} = (\boldsymbol{\varphi}_0, \varphi_n)^T$, $\|\boldsymbol{\varphi}\| = 1$, be the Fiedler vector of the graph Laplacian L associated with a graph $\mathsf{G} \in \mathcal{G}$. Let L_D be the graph Laplacian corresponding to the disaggregated graph $\mathsf{G}_D \in \mathcal{G}$ resulting from disaggregating one vertex into $d > 1$ vertices. We have*

$$\frac{a(\mathsf{G})}{a(\mathsf{G}_D)} \geq 1 + \frac{(d-1)n}{N}\varphi_n^2.$$

Proof. Noting that $\widetilde{\boldsymbol{\varphi}}$ is orthogonal to $\mathbf{1}_N$, we have

$$a(\mathsf{G}_D) \leq \frac{\langle L_D\widetilde{\boldsymbol{\varphi}}, \widetilde{\boldsymbol{\varphi}}\rangle}{\langle \widetilde{\boldsymbol{\varphi}}, \widetilde{\boldsymbol{\varphi}}\rangle} = \frac{\lambda}{1 + \frac{(d-1)n}{N}\varphi_n^2} = \frac{a(\mathsf{G})}{1 + \frac{(d-1)n}{N}\varphi_n^2},$$

which completes the proof. $\square$

If the characteristic value of the disaggregated vertex is non-zero, then the algebraic connectivity of the disaggregated graph stays bounded away from that of the original graph, independent of the structure of L_n. Therefore, as the weight on the internal edges approaches infinity, the approximation stays bounded away.

In [11], the authors made the following conjecture.

CONJECTURE 1   *Under certain conditions the Laplacian eigenvalues of the graph Laplacian of the disaggregated graph approximate the eigenvalues of the graph Laplacian of the original graph, provided that the weight on the internal edges of the disaggregation is chosen to be large enough.*

Theorem 1 directly implies that Conjecture 1 is false when the characteristic value of the disaggregated vertex is non-zero. Though there exist examples of graphs with characteristic values equal to zero for some vertices [18], this often implies some sort of symmetry of the graph. Theoretical results regarding the non-existence of vertices with zero characteristic value in random graphs do not exist, though it is well know that almost all finite graphs are asymmetric [8].

We also have the following result, providing an estimate of how close the approximation $\widetilde{\boldsymbol{\varphi}}$ is to the invariant subspace with respect to L_D.

LEMMA 3   *Let L and L_D be the graph Laplacian of the original graph $\mathsf{G} \in \mathcal{G}$ and the disaggregated graph $\mathsf{G}_D \in \mathcal{G}$, respectively, and $\widetilde{\boldsymbol{\varphi}}$ be defined by (2.2). We have*

$$\|L_D\widetilde{\boldsymbol{\varphi}} - \mathrm{RQ}(\widetilde{\boldsymbol{\varphi}})\widetilde{\boldsymbol{\varphi}}\| \leq \left(\|L_{0n}^T(\mathbf{1}_{n_0} - \boldsymbol{\varphi}_0/\varphi_n)\| + \frac{\sqrt{dn(d+n)}}{N}\lambda + \frac{dn}{N}\lambda|\varphi_n|\right)|\varphi_n|.$$

Proof. We recall that

$$\|\widetilde{\boldsymbol{\varphi}}\| = \left(1 + \frac{(d-1)n}{N}\varphi_n^2\right)^{1/2}$$

and

$$|\lambda - \mathrm{RQ}(\widetilde{\boldsymbol{\varphi}})| = \frac{\frac{(d-1)n}{N}\varphi_n^2}{1 + \frac{(d-1)n}{N}\varphi_n^2}\lambda.$$

We also have

$$L_D\widetilde{\boldsymbol{\varphi}} = L_D\left(P\boldsymbol{\varphi} - s\mathbf{1}_N\right) = L_D P\boldsymbol{\varphi} = \begin{pmatrix} L_0\boldsymbol{\varphi}_0 - \varphi_n L_{0n}\mathbf{1}_d \\ \varphi_n L_n \mathbf{1}_d - L_{0n}^T\boldsymbol{\varphi}_0 \end{pmatrix} = \begin{pmatrix} L_0\boldsymbol{\varphi}_0 - \varphi_n \boldsymbol{l}_n \\ L_{0n}^T(\varphi_n\mathbf{1}_{n_0} - \boldsymbol{\varphi}_0) \end{pmatrix}$$

$$= \lambda P\boldsymbol{\varphi} + \begin{pmatrix} \mathbf{0} \\ L_{0n}^T(\varphi_n\mathbf{1}_{n_0} - \boldsymbol{\varphi}_0) - \lambda\varphi_n\mathbf{1}_d \end{pmatrix} = \lambda\widetilde{\boldsymbol{\varphi}} + s\lambda\mathbf{1}_N + \begin{pmatrix} \mathbf{0} \\ L_{0n}^T(\varphi_n\mathbf{1}_{n_0} - \boldsymbol{\varphi}_0) - \lambda\varphi_n\mathbf{1}_d \end{pmatrix}$$

$$= \lambda\widetilde{\boldsymbol{\varphi}} + \frac{\lambda\varphi_n}{N}\left[(d-1)\begin{pmatrix}\mathbf{1}_0 \\ \mathbf{0}\end{pmatrix} - n\begin{pmatrix}\mathbf{0} \\ \mathbf{1}_d\end{pmatrix}\right] + \begin{pmatrix}\mathbf{0} \\ L_{0n}^T(\varphi_n\mathbf{1}_{n_0} - \boldsymbol{\varphi}_0)\end{pmatrix},$$

giving

$$\|L_D\widetilde{\boldsymbol{\varphi}} - \mathrm{RQ}(\widetilde{\boldsymbol{\varphi}})\widetilde{\boldsymbol{\varphi}}\| \le \|L_D\widetilde{\boldsymbol{\varphi}} - \lambda\widetilde{\boldsymbol{\varphi}}\| + |\lambda - \mathrm{RQ}(\widetilde{\boldsymbol{\varphi}})|\|\widetilde{\boldsymbol{\varphi}}\|$$

$$\le \|L_{0n}^T(\varphi_n\mathbf{1}_{n_0} - \boldsymbol{\varphi}_0)\| + \frac{\sqrt{(d-1)^2 n_0 + dn^2}}{N}|\varphi_n|\lambda + \frac{\frac{(d-1)n}{N}}{\sqrt{1 + \frac{(d-1)n}{N}\varphi_n^2}}\varphi_n^2\lambda$$

$$\le \left(\|L_{0n}^T(\mathbf{1}_{n_0} - \boldsymbol{\varphi}_0/\varphi_n)\| + \frac{\sqrt{dn(d+n)}}{N}\lambda + \frac{dn}{N}\lambda|\varphi_n|\right)|\varphi_n|.$$

$$\square$$

In many applications, we are only concerned with minimal Laplacian eigenpairs. For minimal eigenvalues of scale-free graphs, we have $\lambda = O(1)$ and $\varphi_n = O(N^{-1/2})$. In this way, often the largest source of error comes from the term $\|L_{0n}^T(\varphi_n\mathbf{1}_{n_0} - \boldsymbol{\varphi}_0)\|$. Heuristically, the error of this term is typically best controlled when d is relatively small and each new disaggregate is connected to roughly the same number of exterior vertices.

Next, we consider how disaggregation affects the eigenvalues of the random walk Laplacian. This gives us insight into how the Cheeger constant changes after disaggregating a vertex. Suppose we have an eigenpair $(\nu, \boldsymbol{\phi})$ of the random walk graph Laplacian $D^{-1}L$, where D is the degree matrix of G, namely

$$D = \mathrm{diag}(\delta_1, \delta_2, \cdots, \delta_n).$$

By similarity, ν is also an eigenvalue of the normalized Laplacian $D^{-1/2}LD^{-1/2}$. In addition, $(\nu, \boldsymbol{\phi})$ is also an eigenpair of the generalized eigenvalue problem $L\boldsymbol{\phi} = \nu D\boldsymbol{\phi}$ and the eigenvalues ν can be characterized by the Courant-Fischer minmax values of the generalized Rayleigh quotient, i.e.

$$\frac{\langle L\boldsymbol{x}, \boldsymbol{x}\rangle}{\langle D\boldsymbol{x}, \boldsymbol{x}\rangle}, \quad 0 \ne \boldsymbol{x} \in \mathbb{R}^n.$$

Since D is symmetric positive definite, we can define the inner product $\langle\boldsymbol{x}, \boldsymbol{y}\rangle_D := \langle D\boldsymbol{x}, \boldsymbol{y}\rangle$ and it induces a norm $\|\boldsymbol{x}\|_D := \sqrt{\langle\boldsymbol{x}, \boldsymbol{x}\rangle_D}$. Noting that $D^{-1}L$ is symmetric

with respect to $\langle \cdot, \cdot \rangle_D$, we naturally normalize the eigenvectors ϕ such that $\|\phi\|_D = 1$ and they are orthogonal with each other with respect to $\langle \cdot, \cdot \rangle_D$. Let D_D be the weighted degree matrix of G_D. In order to obtain an approximate eigenvector of the matrix $D_D^{-1} L_D$, we prolongate ϕ as follows

$$\widetilde{\phi} = P\phi - s\mathbf{1}_N, \quad \text{where } s = \frac{\langle P\phi, \mathbf{1}_N \rangle_{D_D}}{\langle \mathbf{1}_N, \mathbf{1}_N \rangle_{D_D}}, \quad \langle \boldsymbol{x}, \boldsymbol{y} \rangle_{D_D} := \langle D_D \boldsymbol{x}, \boldsymbol{y} \rangle. \qquad (2.3)$$

Here s is chosen such that $\langle \widetilde{\phi}, \mathbf{1}_N \rangle_{D_D} = 0$. We may write D_D in the following way

$$\begin{aligned} D_D &= \mathrm{diag}(\delta_1^D, \cdots, \delta_{n_0}^D, \delta_n^D, \delta_{n+1}^D, \cdots, \delta_N^D) \\ &= \mathrm{diag}(\delta_1^D, \cdots, \delta_{n_0}^D, \delta_n, \delta_{n+1}, \cdots, \delta_N) + \mathrm{diag}(0, \cdots, 0, d_n^{ex}, d_{n+1}^{ex}, \cdots, d_N^{ex}) \\ &=: D_D^1 + D_D^{ex}, \end{aligned}$$

where δ_i and δ_i^D are the weighted degrees of vertex v_i on the original and disaggregated graph, respectively. Moreover, $d_i^{ex} = \delta_i^D - \delta_i$, $i = n, n+1, \cdots, N$. We may also rewrite the shift s as

$$s = \frac{\langle P\phi, \mathbf{1}_N \rangle_{D_D}}{\langle \mathbf{1}_N, \mathbf{1}_N \rangle_{D_D}} = \frac{\sum_{i=n}^N d_i^{ex} \phi_n}{\sum_{i=1}^N \delta_i^D}.$$

Let $\omega_{\mathrm{total}}(\mathsf{H})$ denote the total weights of a graph H, and let G_a be the disaggregated local subgraph. Similarly, we consider $\widetilde{\phi}$ as an approximation of the eigenvectors of the matrix $D_D^{-1} L_D$. In fact, we have the following lemma.

THEOREM 2   *Let (ν, ϕ), $\|\phi\|_D = 1$ be an eigenpair of $D^{-1}L$ associated with a simply connected graph G and $\widetilde{\phi}$ be defined by (2.3). We have*

$$\frac{\langle L_D \widetilde{\phi}, \widetilde{\phi} \rangle}{\langle D_D \widetilde{\phi}, \widetilde{\phi} \rangle} = \frac{\nu}{1 + \frac{2\omega_{total}(\mathsf{G})\,\omega_{total}(\mathsf{G}_a)}{\omega_{total}(\mathsf{G}_D)} \phi_n^2}, \qquad (2.4)$$

and, therefore,

$$\nu_2^D = \alpha \nu_2, \quad \alpha = \left(1 + \frac{2\omega_{total}(\mathsf{G})\,\omega_{total}(\mathsf{G}_a)}{\omega_{total}(\mathsf{G}_D)} \phi_n^2\right)^{-1} \leq 1. \qquad (2.5)$$

Proof. We have

$$\langle L_D \widetilde{\phi}, \widetilde{\phi} \rangle = \langle L_D(P\phi - s\mathbf{1}_N), P\phi - s\mathbf{1}_N \rangle = \langle L_D P\phi, P\phi \rangle = \langle L\phi, \phi \rangle = \nu$$

and

$$\langle D_D \widetilde{\phi}, \widetilde{\phi} \rangle = \langle D_D P\phi, P\phi \rangle - s\langle D_D \mathbf{1}_N, P\phi \rangle$$

$$= \langle D_D^1 P\phi, P\phi \rangle + \langle D_D^{ex} P\phi, P\phi \rangle - s\langle D_D \mathbf{1}_N, P\phi \rangle$$

$$= \langle D\phi, \phi \rangle + \sum_{i=n}^{N} d_i^{ex} \phi_n^2 - \frac{\left(\sum_{i=n}^{N} d_i^{ex} \phi_n \right)^2}{\sum_i^N \delta_i^D}$$

$$= 1 + \frac{\left(\sum_i^N \delta_i^D - \sum_{i=n}^N d_i^{ex} \right) \sum_{i=n}^N d_i^{ex}}{\sum_i^N \delta_i^D} \phi_n^2$$

$$= 1 + \frac{\left(\sum_{i=1}^n \delta_i \right) \left(\sum_{i=n}^N d_i^{ex} \right)}{\sum_{i=1}^N \delta_i^D} \phi_n^2.$$

Noting that $\sum_{i=1}^N \delta_i = 2\omega_{\text{total}}(\mathsf{G})$, $\sum_{i=1}^N \delta_i^D = 2\omega_{\text{total}}(\mathsf{G}_D)$, and $\sum_{i=n}^N d_i^{ex} = 2\omega_{\text{total}}(\mathsf{G}_a)$, we obtain (2.4). Moreover, (2.5) follows directly from (2.4). □

The Cheeger constant of a weighted graph is defined as follows [4, 9]

$$h(\mathsf{G}) = \min_{\emptyset \neq U \subset V} \frac{|E(U, \bar{U})|}{\min(\text{vol}(U), \text{vol}(\bar{U}))},$$

where

$$\bar{U} = V \backslash U, \quad \text{vol}(U) = \sum_{i \in U} \delta_i, \quad |E(U, \bar{U})| = \sum_{\substack{e=(i,j) \in E \\ i \in U, j \in \bar{U}}} \omega_e.$$

Due to the Cheeger inequality [4, 9], the Cheeger constant $h(\mathsf{G})$ and ν_2 are related as follows

$$1 - \sqrt{1 - h(\mathsf{G})^2} \leq \nu_2 \leq 2h(\mathsf{G}). \tag{2.6}$$

THEOREM 3 *For the Cheeger constant of the original graph $\mathsf{G} \in \mathcal{G}$ and the disaggregated graph $\mathsf{G}_D \in \mathcal{G}$, we have*

$$h(\mathsf{G}_D) \leq \sqrt{1 - (1 - 2\alpha h(\mathsf{G}))^2},$$

where α is defined by (2.5). If $h(\mathsf{G}) \geq \frac{4\alpha}{4\alpha^2 + 1}$, then $h(\mathsf{G}_D) \leq h(\mathsf{G})$.

Proof. Based on (2.6) and (2.5), we have

$$h(\mathsf{G}_D) \leq \sqrt{1 - (1 - \nu_2^D)^2} = \sqrt{1 - (1 - \alpha\nu_2)^2} \leq \sqrt{1 - (1 - 2\alpha h(\mathsf{G}))^2}.$$

Basic algebra shows that $h(\mathsf{G}_D) \leq h(\mathsf{G})$ if $h(\mathsf{G}) \geq \frac{4\alpha}{4\alpha^2 + 1}$. □

3.   Graph Disaggregation

We now move on to the more general case of multiple disaggregated vertices. Without loss of generality, for a graph Laplacian $L \in \mathbb{R}^{n \times n}$, suppose we are disaggregating the first m vertices. This gives us the disaggregated Laplacian $L_D \in \mathbb{R}^{N \times N}$. Here, N is the number of vertices in the disaggregated graph, given by

$$N = n - m + \sum_{k=1}^{m} d_k = n - m + n_d, \quad n_d = \sum_{k=1}^{m} d_k.$$

Note that we have m groups of vertices associated with the disaggregation, which we can also number consecutively

$$\{1, \ldots, N\} = \{\underbrace{1, \ldots, d_1}_{d_1}, \ldots \underbrace{d_1 + 1, \ldots, d_1 + d_2}_{d_2}, \ldots, n_d + 1, \ldots, N\}. \tag{3.1}$$

Similar to the case of a single disaggregated vertex, we can establish a relationship between L and L_D through a prolongation matrix $P : \mathbb{R}^n \to \mathbb{R}^N$, given by

$$P = \begin{pmatrix} P_m & 0 \\ 0 & I_{n_0 \times n_0} \end{pmatrix}, \quad \text{where} \quad n_0 = (n - m). \tag{3.2}$$

Here, $P_m \in \mathbb{R}^{n_d \times m}$, and

$$P_m = \begin{pmatrix} \mathbf{1}_{d_1} & 0 & \ldots & 0 \\ 0 & \mathbf{1}_{d_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \mathbf{1}_{d_m} \end{pmatrix}.$$

Note that $P_m^T P_m = \mathrm{diag}(d_1, \ldots d_m)$. We have the following lemma, which can be easily verified by simple algebraic calculation.

LEMMA 4   *Let L and L_D be the graph Laplacian of the original graph $\mathsf{G} \in \mathcal{G}$ and the disaggregated graph $\mathsf{G}_D \in \mathcal{G}$, respectively. If P is defined as (3.2), then we have*

$$L = P^T L_D P. \tag{3.3}$$

If we look at the disaggregated graph Laplacian L_D directly, we can obtain a similar bound on the algebraic connectivity as shown in Theorem 1. Let $(\lambda, \boldsymbol{\varphi})$ be an eigenpair of L. We can define an approximated eigenvector of L_D by prolongating $\boldsymbol{\varphi}$ as follows

$$\widetilde{\boldsymbol{\varphi}} = P\boldsymbol{\varphi} - s\mathbf{1}_N, \quad \text{where} \quad s = \frac{1}{N} \sum_{i=1}^{m} (d_i - 1)\varphi_i. \tag{3.4}$$

It is easy to check that $\langle \widetilde{\boldsymbol{\varphi}}, \mathbf{1}_N \rangle = 0$. Now we have the following lemma about the Rayleigh quotient of $\widetilde{\boldsymbol{\varphi}}$ with respect to L_D.

LEMMA 5 *Let (λ, φ) be an eigenpair of the graph Laplacian L associated with a simply connected graph G and $\widetilde{\varphi}$ be defined by (3.4). We have*

$$\mathrm{RQ}(\widetilde{\varphi}) := \frac{\langle L_D \widetilde{\varphi}, \widetilde{\varphi} \rangle}{\langle \widetilde{\varphi}, \widetilde{\varphi} \rangle} \leq \frac{\lambda}{1 + \frac{1}{N} \sum_{i=1}^m (d_i - 1)(n + n_d - md_i)\varphi_i^2}.$$

Moreover, if for $d_i^{\max} = \max_i d_i$, we have $n + n_d - md_i^{\max} > 0$, then $RQ(\widetilde{\varphi}) < \lambda$.

Proof. We note that

$$\langle L_D \widetilde{\varphi}, \widetilde{\varphi} \rangle = \langle L_D(P\varphi - s\mathbf{1}_N), P\varphi - s\mathbf{1}_N \rangle = \langle L_D P\varphi, P\varphi \rangle - 2s\langle L_D \mathbf{1}_N, P\varphi \rangle + s^2 \langle L_D \mathbf{1}_N, \mathbf{1}_N \rangle$$
$$= \langle P^T L_D P\varphi, \varphi \rangle = \langle L\varphi, \varphi \rangle = \lambda.$$

Denoting φ by $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_m, \boldsymbol{\varphi}_0)^T$, we have

$$\langle \widetilde{\varphi}, \widetilde{\varphi} \rangle = \langle P\varphi - s\mathbf{1}_N, P\varphi - s\mathbf{1}_N \rangle = \langle P\varphi, P\varphi \rangle - 2s\langle P\varphi, \mathbf{1}_N \rangle + s^2 \langle \mathbf{1}_N, \mathbf{1}_N \rangle$$
$$= \langle \boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0 \rangle + \sum_{i=1}^m \varphi_i^2 \langle \mathbf{1}_{d_i}, \mathbf{1}_{d_i} \rangle - 2s \left(\langle \boldsymbol{\varphi}_0, \mathbf{1}_{n_0} \rangle + \sum_{i=1}^m \varphi_i \langle \mathbf{1}_{d_i}, \mathbf{1}_{d_i} \rangle \right) + s^2 N$$
$$= \langle \boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0 \rangle + \sum_{i=1}^m \varphi_i^2 + \sum_{i=1}^m (d_i - 1)\varphi_i^2 - 2s \sum_{i=1}^m (d_i - 1)\varphi_i + s^2 N$$
$$= 1 + \sum_{i=1}^m (d_i - 1)\varphi_i^2 - \frac{1}{N} \left(\sum_{i=1}^m (d_i - 1)\varphi_i \right)^2$$
$$\geq 1 + \sum_{i=1}^m (d_i - 1)\varphi_i^2 - \frac{m}{N} \sum_{i=1}^m (d_i - 1)^2 \varphi_i^2$$
$$= 1 + \sum_{i=1}^m (d_i - 1)\frac{N - m(d_i - 1)}{N}\varphi_i^2$$
$$= 1 + \frac{1}{N} \sum_{i=1}^m (d_i - 1)(n + n_d - md_i)\varphi_i^2.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

From the Rayleigh quotient and applying the above lemma to the Fielder vector, we have the following theorem concerning the algebraic connectivity.

THEOREM 4 *Let φ be the Fiedler vector of the graph Laplacian L associated with a graph $\mathsf{G} \in \mathcal{G}$ and L_D be the graph Laplacian corresponding to the disaggregated graph $\mathsf{G}_D \in \mathcal{G}$. Suppose we have disaggregated m vertices and each of those vertices are disaggregated into $d_i > 1$ vertices, $i = 1, 2, \cdots, m$. We have*

$$\frac{a(\mathsf{G})}{a(\mathsf{G}_D)} \geq 1 + \frac{1}{N} \sum_{i=1}^m (d_i - 1)(n + n_d - md_i)\varphi_i^2.$$

Proof. The proof is similar to the proof of Theorem 1 and uses Lemma 5. $\qquad \square$

It is possible to perform a more careful estimate, using the fact that disaggregating m vertices at once is equivalent to disaggregating m vertices one by one. Denote $n_0 = n$ and $n_i = n_{i-1} - 1 + d_i$, $i = 1, 2, \cdots, m$. Note that $n_i = n - i + \sum_{k=1}^{i} d_k$, $i = 1, 2, \cdots, m$ and $N = n_m$. Recursively applying Theorem 1, we have the following result.

THEOREM 5 *Let φ be the Fiedler vector of the graph Laplacian L associated with a graph $\mathsf{G} \in \mathcal{G}$ and L_D be the graph Laplacian corresponding to the disaggregated graph $\mathsf{G}_D \in \mathcal{G}$. Suppose we disaggregated m vertices and each of those are disaggregated into $d_i > 1$ vertices, $i = 1, 2, \cdots, m$. We have*

$$\frac{a(\mathsf{G})}{a(\mathsf{G}_D)} \geq \prod \left(1 + \frac{(d_i - 1)n_{i-1}}{n_i} \varphi_i^2 \right).$$

Proof. Let G_D^i be the resulting graph after disaggregating the i-th vertex in the graph G_D^{i-1} and note that $\mathsf{G}_D = \mathsf{G}_D^m$. We have

$$\frac{a(\mathsf{G})}{a(\mathsf{G}_D)} = \frac{a(\mathsf{G})}{a(\mathsf{G}_D^1)} \times \frac{a(\mathsf{G}_D^1)}{a(\mathsf{G}_D^2)} \times \cdots \times \frac{a(\mathsf{G}_D^{m-1})}{a(\mathsf{G}_D^m)}.$$

The result follows immediately from applying Theorem 1 on each pair of graphs G_D^i and G_D^{i+1}. $\qquad\square$

REMARK 1 *A direct consequence of Theorem 5 is $a(\mathsf{G}_D) \leq a(\mathsf{G})$.*

Similarly, we also have the following result concerning the minimal eigenvalue of the random walk Laplacian after disaggregating several vertices. Denote $\mathsf{G}_D^0 = \mathsf{G}$, and denote the graph after disaggregating vertex i by G_D^i. Note that $\mathsf{G}_D^m = \mathsf{G}_D$. The local subgraph corresponding to disaggregating vertex i is denoted by G_a^i.

THEOREM 6 *Let ν_2 be the second smallest eigenvalue of the random walk Laplacian associated with a graph $\mathsf{G} \in \mathcal{G}$ and ν_2^D be the second smallest eigenvalue of the random walk Laplacian corresponding to the disaggregated graph $\mathsf{G}_D \in \mathcal{G}$. Suppose we disaggregated m vertices and each of them are disaggregated into $d_i > 1$ vertices, $i = 1, 2, \ldots, m$. We have*

$$\frac{\nu_2}{\nu_2^D} \geq \prod_{i=1}^{m} \left(1 + \frac{2\omega_{total}(\mathsf{G}_D^{i-1})\,\omega_{total}(\mathsf{G}_a^i)}{\omega_{total}(\mathsf{G}_D^i)} \phi_i^2 \right).$$

Consequently, we have $\nu_2^D = \alpha\nu_2$, where

$$\alpha := \left[\prod_{i=1}^{m} \left(1 + \frac{2\omega_{total}(\mathsf{G}_D^{i-1})\omega_{total}(\mathsf{G}_a^i)}{\omega_{total}(\mathsf{G}_D^i)} \phi_i^2 \right) \right]^{-1} \leq 1. \qquad (3.5)$$

Proof. The result follows by applying Theorem 2 recursively. $\qquad\square$

Based on the estimates on the eigenvalues of normalized graph Laplacian, we can estimate the Cheeger constants as follows.

THEOREM 7   *For the Cheeger constant of the original graph* G *and the disaggregated and simply connected graph* G_D, *we have*

$$h(\mathsf{G}_D) \leq \sqrt{1 - (1 - 2\alpha h(\mathsf{G}))^2},$$

where α *is defined by* (3.5). *If* $h(\mathsf{G}) \geq \frac{4\alpha}{4\alpha^2+1}$, *then* $h(\mathsf{G}_D) \leq h(\mathsf{G})$.

Proof. The proof is the same as the proof of Theorem 3. □

4.   Preconditioning Using Disaggregated Graph

We first aim to show eigenvalue interlacing between L and a new matrix, which is obtained by scaling L_D appropriately. We can rescale P by introducing $\widetilde{P} = D_s P$, where

$$D_s = \text{diag}(d_1^{-1/2} I_{d_1 \times d_1}, \dots, d_m^{-1/2} I_{d_m \times d_m}, I_{n0 \times n0}) \tag{4.1}$$

is a diagonal scaling matrix, giving us $\widetilde{P}^T \widetilde{P} = I$. Based on the scaled prolongation, we are able to show the eigenvalues of diagonal scaled matrix

$$\widetilde{L}_D := D_s^{-1} L_D D_s^{-1} \tag{4.2}$$

interlaces with L. First, let us recall the interlacing theorem.

THEOREM 8 (Interlacing Theorem [5], Vol. 1, Chap. I)   *Let* $S \in \mathbb{R}^{n \times m}$ *be such that* $S^T S = I_{m \times m}$, $m < n$ *and let* $B \in \mathbb{R}^{n \times n}$ *be symmetric, with eigenvalues* $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. *Define* $A = S^T B S$ *and let* A *have eigenvalues* $\mu_1 \leq \mu_2 \leq \dots \leq \mu_m$. *Then* $\lambda_i \leq \mu_i \leq \lambda_{n-m+i}$.

From here, we have the following.

THEOREM 9   *Let* L *have eigenvalues* $\lambda_1(L) \leq \lambda_2(L) \leq \dots \leq \lambda_n(L)$ *and* $\widetilde{L}_D = D_s^{-1} L_D D_s^{-1}$ *have eigenvalues* $\lambda_1(\widetilde{L}_D) \leq \lambda_2(\widetilde{L}_D) \leq \dots \leq \lambda_N(\widetilde{L}_D)$. *Then*

$$\lambda_i(\widetilde{L}_D) \leq \lambda_i(L) \leq \lambda_{N-n+i}(\widetilde{L}_D).$$

Proof. From the above Lemma, we have

$$L = P^T L_D P = P^T D_s D_s^{-1} L_D D_s^{-1} D_s P = \widetilde{P}^T D_s^{-1} L_D D_s^{-1} \widetilde{P} = \widetilde{P}^T \widetilde{L}_D \widetilde{P}.$$

As $\widetilde{P}^T \widetilde{P} = I_{n \times n}$, by the Interlacing Theorem 8, the eigenvalues of L and $\widetilde{L}_D$ interlace. □

We now discuss how to use the disaggregated graph G_D to solve the graph Laplacian on the original graph G. Here, we will use $\widetilde{L}_D$ as the auxiliary problem and design a preconditioner based on the Fictitious Space Lemma [14] and auxiliary space framework [20]. Because L and $\widetilde{L}_D$ are both symmetric positive semi-definite, we first state the refined version of the Fictitious Space Lemma proposed in [7].

THEOREM 10 (Theorem 6.3 and 6.4 in [7])   *Let $\widetilde{V}$ and V be two Hilbert spaces and $\Pi : \widetilde{V} \mapsto V$ be a surjective map. Suppose that $\widetilde{\mathcal{A}} : \widetilde{V} \mapsto \widetilde{V}'$ and $\mathcal{A} : V \mapsto V'$ are symmetric semi-definite operators. Moreover, suppose*

$$\Pi(N(\widetilde{\mathcal{A}})) = N(\mathcal{A}), \tag{4.3}$$

$$\|\Pi\, \widetilde{v}\|_A \leq c_1 \|\widetilde{v}\|_{\widetilde{A}}, \quad \forall\, \widetilde{v} \in \widetilde{V}, \tag{4.4}$$

for any $v \in V$ there exists $\widetilde{v} \in \widetilde{V}$ such that $\Pi\widetilde{v} = v$ and $\|\widetilde{v}\|_{\widetilde{A}} \leq c_0 \|v\|_A$,   (4.5)

then for any symmetric positive definite operator $\widetilde{\mathcal{B}} : \widetilde{V}' \mapsto \widetilde{V}$, we have that for $\mathcal{B} = \Pi\,\widetilde{\mathcal{B}}\,\Pi^T$,

$$\kappa(\mathcal{B}\mathcal{A}) \leq \left(\frac{c_1}{c_0}\right)^2 \kappa(\widetilde{\mathcal{B}}\widetilde{\mathcal{A}}).$$

Applying the above theory to our disaggregation framework, we take $\mathcal{A} = L$, $\widetilde{\mathcal{A}} = \widetilde{L}_D$, and $\Pi = \widetilde{P}^T$. Noting that the null space of $\widetilde{L}_D$ is spanned by $D_s\mathbf{1}_N$, we have

$$\widetilde{P}^T D_s \mathbf{1}_N = P^T D_s^2 \mathbf{1}_N = \mathbf{1}_n$$

which verifies (4.3). Naturally, $\widetilde{P}^T$ is surjective. Using a preconditioner $\widetilde{B}_D$ of $\widetilde{L}_D$, we can define a preconditioner

$$B = \widetilde{P}^T \widetilde{B}_D \widetilde{P}$$

for L. We give the following results concerning the quality of the preconditioner B.

COROLLARY 1   *Let L be the graph Laplacian corresponding to the graph $\mathsf{G} \in \mathcal{G}$ and L_D be the graph Laplacian corresponding to the disaggregated graph $\mathsf{G}_D \in \mathcal{G}$. Let D_s be defined by (4.1) and $\widetilde{L}_D$ be defined by (4.2). If*

$$\|\widetilde{P}^T \widetilde{\boldsymbol{v}}\|_L \leq c_1 \|\widetilde{\boldsymbol{v}}\|_{\widetilde{L}_D}, \quad \forall\, \widetilde{\boldsymbol{v}} \in \widetilde{V} \tag{4.6}$$

and for any $\boldsymbol{v} \in V$, there exist a $\widetilde{\boldsymbol{v}} \in \widetilde{V}$ such that $\widetilde{P}^T \widetilde{\boldsymbol{v}} = \boldsymbol{v}$ and

$$\|\widetilde{\boldsymbol{v}}\|_{\widetilde{L}_D} \leq c_0 \|\boldsymbol{v}\|_L. \tag{4.7}$$

Then for the preconditioner $B = \widetilde{P}^T \widetilde{B}_D \widetilde{P}$, we have

$$\kappa(BL) \leq \left(\frac{c_1}{c_0}\right)^2 \kappa(\widetilde{B}_D \widetilde{L}_D).$$

We need to verify that conditions (4.6) and (4.7) hold for $\widetilde{P} = D_s P$. For condition (4.7), we choose $\widetilde{\boldsymbol{v}} = \widetilde{P}^T \boldsymbol{v}$ for any $\boldsymbol{v} \in V$, giving $\widetilde{P}^T \widetilde{\boldsymbol{v}} = \widetilde{P}^T \widetilde{P} \boldsymbol{v} = \boldsymbol{v}$ since $\widetilde{P}^T \widetilde{P} = I$. Note that

$$\|\widetilde{\boldsymbol{v}}\|_{\widetilde{L}_D}^2 = \langle \widetilde{L}_D \widetilde{P} \boldsymbol{v}, \widetilde{P} \boldsymbol{v}\rangle = \|\boldsymbol{v}\|_L^2, \tag{4.8}$$

which implies condition (4.7) holds with $c_0 = 1$.

To show that condition (4.6) holds, we use the following result.

LEMMA 6   *Let $L \in \mathbb{R}^{n \times n}$ be the graph Laplacian corresponding to a graph $\mathsf{G} \in \mathcal{G}$ with n vertices. For all $i \in \{1, \ldots, n\}$ and $\boldsymbol{u} \in \mathbb{R}^n$ we have*

$$\frac{1}{n}(\boldsymbol{u}, \boldsymbol{1}_n) - (\boldsymbol{u}, \boldsymbol{e}_i) = \frac{1}{n}(L_i^{-1}\boldsymbol{1}_n, L\boldsymbol{u}),$$

where $L_i = (L + \boldsymbol{e}_i \boldsymbol{e}_i^T)$.

Proof. First, we note that for all $i \in \{1, \ldots, n\}$ the matrices L_i are invertible, because they all are irreducibly diagonally dominant M-matrices. We refer to Varga [19] for this classical result.

Next, observe that $L_i \boldsymbol{1}_n = \boldsymbol{e}_i$, and hence, $L_i^{-1}\boldsymbol{e}_i = \boldsymbol{1}_n$. Therefore, we have that

$$
\begin{aligned}
(L_i^{-1}\boldsymbol{1}_n, L\boldsymbol{u}) &= (L_i^{-1}\boldsymbol{1}_n, (L_i - \boldsymbol{e}_i\boldsymbol{e}_i^T)\boldsymbol{u}) = (\boldsymbol{1}_n, \boldsymbol{u}) - (\boldsymbol{u}, \boldsymbol{e}_i)(L_i^{-1}\boldsymbol{1}_n, \boldsymbol{e}_i) \\
&= (\boldsymbol{1}_n, \boldsymbol{u}) - (\boldsymbol{u}, \boldsymbol{e}_i)(\boldsymbol{1}_n, L_i^{-1}\boldsymbol{e}_i) = (\boldsymbol{1}_n, \boldsymbol{u}) - (\boldsymbol{u}, \boldsymbol{e}_i)(\boldsymbol{1}_n, \boldsymbol{1}_n).
\end{aligned}
$$

As $(\boldsymbol{1}_n, \boldsymbol{1}_n) = n$, this completes the proof. $\square$

The result shown in Lemma 6 is also found in [3, Lemma 3.2], but is included for completeness.

Let $\mathsf{G}_a^k = (V_a^k, E_a^k, \omega_a^k)$, $k = 1, 2, \cdots, m$, be the m local subgraphs created by disaggregating each of the m vertices. We now apply Lemma 6 to $\mathsf{G}_a^k = (V_a^k, E_a^k, \omega_a^k)$, $k = 1, 2, \cdots, m$, with $\boldsymbol{u} = \widetilde{\boldsymbol{v}}_k$, the restriction of $\widetilde{\boldsymbol{v}}$ on G_a^k. For $j \in V_a^k$, we have,

$$\widetilde{v}_j = \frac{1}{d_k} \sum_{p \in V_a^k} \widetilde{v}_p - \frac{1}{d_k} \langle L_{k,j}^{-1}\boldsymbol{1}_{d_k}, L_k\widetilde{\boldsymbol{v}}_k \rangle, \tag{4.9}$$

where L_k is the unweighted graph Laplacian of the local graph G_a^k and $L_{k,j}$ is defined in accordance with Lemma 6: $L_{k,j} = L_k + \boldsymbol{e}_j^k \left(\boldsymbol{e}_j^k\right)^T$, for $j \in V_a^k$. Setting $W_k^j := \frac{1}{d_k^2}\|L_{k,j}^{-1}\boldsymbol{1}_{d_k}\|_{L_k}^2$, $j \in V_a^k$, and denoting

$$
\begin{aligned}
E_D^0 &:= \{e = (i, j) \in E_D, i, j \in V^0\}, \\
E_D^1 &:= \{e = (i, j) \in E_D, i \in V^0, j \in V_a^k, k = 1, 2, \cdots, m\}, \\
E_D^2 &:= \{e = (i, j) \in E_D, i \in V_a^k, j \in V_a^\ell, k, \ell = 1, 2, \cdots, m, k \neq \ell\},
\end{aligned}
$$

we are ready to present the following lemma related to the condition (4.6).

LEMMA 7   *For each disaggregated local subgraph G_a^k, if, for an edge $e' = (p, q) \in E_a^k$, we assign a weight $\omega_{e'}$ such that*

$$\omega_{e'} \geq W_{e'} := (1+\epsilon^{-1}) \left[\sum_{\substack{e=(i,j) \in E_D^1 \\ i \in V^0, \, j \in V_a^k}} \omega_e W_k^j + 2\sum_{\ell=1}^m \left(\sum_{\substack{e=(i,j) \in E_D^2 \\ i \in V_a^k, \, j \in V_a^\ell}} \omega_e W_k^i + \sum_{\substack{e=(i,j) \in E_D^2 \\ i \in V_a^\ell, \, j \in V_a^k}} \omega_e W_k^j \right) \right], \tag{4.10}$$

then we have

$$\|\widetilde{P}^T\widetilde{\boldsymbol{v}}\|_L^2 \leq (1+\epsilon)\|\widetilde{\boldsymbol{v}}\|_{\widetilde{L}_D}^2, \quad \forall\, \widetilde{v} \in \widetilde{V}, \tag{4.11}$$

where $\epsilon > 0$.

Proof. The proof is quite tedious, and, therefore, is left to Appendix A. □

Lemma 7 shows that the constant c_1 can be made arbitrarily close to 1 if the weights on the internal edges of the disaggregation are chosen to be large enough. As an immediate consequence, we have the following theorem for the preconditioner B.

THEOREM 11 *Under the assumptions of Corollary 1 and Lemma 7, for the preconditioner $B = \widetilde{P}^T\widetilde{B}_D\widetilde{P}$, we have*

$$\kappa(BL) \leq (1+\epsilon)\,\kappa(\widetilde{B}_D\widetilde{L}_D). \tag{4.12}$$

Proof. The relation (4.12) follows from Corollary 1 since $c_0 = 1$ in (4.8) and $c_1 = (1+\epsilon)^{1/2}$ in Lemma 7. □

Finally, since $\widetilde{L}_D := D_s^{-1}L_D D_s^{-1}$, if we have a preconditioner B_D for L_D and define $\widetilde{B}_D = D_s L_D D_s$, then it is easy to verify that $\kappa(\widetilde{B}_D\widetilde{L}_D) = \kappa(B_D L_D)$. We have the following theorem showing that the preconditioned operator BL has a condition number comparable to the condition number of $B_D L_D$.

THEOREM 12 *Under the assumptions of Corollary 1 and Lemma 7 and let $\widetilde{B}_D = D_s B_D D_s$, for the preconditioner $B = \widetilde{P}^T\widetilde{B}_D\widetilde{P}$, we have*

$$\kappa(BL) \leq (1+\epsilon)\,\kappa(B_D L_D). \tag{4.13}$$

Proof. (4.13) follows from Theorem 11 and the fact that $\kappa(\widetilde{B}_D\widetilde{L}_D) = \kappa(B_D L_D)$. □

Clearly, Theorems 4.12 and 4.13 imply that, when the weights on the internal edges of the disaggregation are chosen to be large enough, preconditioners for disaggregated graph provide effective preconditioners for the original graph, which indirectly supports the technique suggested in [11].

Acknowledgements

Appendix A. Proof of Lemma 7

We denote $\widetilde{\boldsymbol{u}} = \widetilde{P}^T \widetilde{P} \widetilde{\boldsymbol{v}}$, and we have,

$$
\begin{aligned}
\|\widetilde{P}^T \widetilde{\boldsymbol{v}}\|_L^2 &= \langle \widetilde{L}_D \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}} \rangle = \sum_{e=(i,j)\in E_D} \omega_e (\widetilde{u}_i - \widetilde{u}_j)^2 \\
&= \sum_{e=(i,j)\in E_D^0} \omega_e (\widetilde{u}_i - \widetilde{u}_j)^2 + \sum_{k=1}^m \sum_{e=(i,j)\in E_a^k} \omega_e (\widetilde{u}_i - \widetilde{u}_j)^2 \\
&\quad + \sum_{e=(i,j)\in E_D^1} \omega_e (\widetilde{u}_i - \widetilde{u}_j)^2 + \sum_{e=(i,j)\in E_D^2} \omega_e (\widetilde{u}_i - \widetilde{u}_j)^2 \\
&=: \ I_0 + I_1 + I_2.
\end{aligned}
$$

Here, we have set $I_0 = \sum_{e=(i,j)\in E_D^0} \omega_e (\widetilde{v}_i - \widetilde{v}_j)^2$,

$$
I_1 = \sum_{k=1}^m \sum_{\substack{e=(i,j)\in E_D^1 \\ i\in V^0,\ j\in V_a^k}} \omega_e \left(\widetilde{v}_i - \frac{1}{d_k} \sum_{p\in V_a^k} \widetilde{v}_p \right)^2,
$$

and

$$
I_2 = \sum_{k=1}^m \sum_{\ell=1}^m \sum_{\substack{e=(i,j)\in E_D^2 \\ i\in V_a^k,\ j\in V_a^\ell}} \omega_e \left(\frac{1}{d_k} \sum_{p\in V_a^k} \widetilde{v}_p - \frac{1}{d_\ell} \sum_{q\in V_a^\ell} \widetilde{v}_q \right)^2.
$$

Next, we estimate I_1 and I_2 on the right-hand side. For $e = (i,j) \in E_D^1$, $i \in V^0$ and $j \in V_a^k$, using (4.9), we have

$$
\begin{aligned}
\left(\widetilde{v}_i - \frac{1}{d_k} \sum_{p\in V_a^k} \widetilde{v}_p \right)^2 &= \left(\widetilde{v}_i - \widetilde{v}_j - \frac{1}{d_k} \langle L_{k,j}^{-1} \mathbf{1}_{d_k}, L_k \widetilde{\boldsymbol{v}}_k \rangle \right)^2 \\
&\leq (1+\epsilon)(\widetilde{v}_i - \widetilde{v}_j)^2 + (1+\epsilon^{-1}) \frac{1}{d_k^2} \|L_{k,j}^{-1} \mathbf{1}_{d_k}\|_{L_k}^2 \|\widetilde{\boldsymbol{v}}_k\|_{L_k}^2 \\
&= (1+\epsilon)(\widetilde{v}_i - \widetilde{v}_j)^2 + \sum_{e'=(p,q)\in E_a^k} \left[(1+\epsilon^{-1}) W_k^j \right] (\widetilde{v}_p - \widetilde{v}_q)^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
I_1 \;\leq\; & \sum_{k=1}^{m} \sum_{\substack{e=(i,j)\in E_D^1 \\ i\in V^0,\ j\in V_a^k}} \omega_e \left\{ (1+\epsilon)\,(\widetilde{v}_i-\widetilde{v}_j)^2 + \sum_{e'=(p,q)\in E_a^k} \left[(1+\epsilon^{-1})\,W_k^j\right](\widetilde{v}_p-\widetilde{v}_q)^2 \right\} \\
=\; & (1+\epsilon)\sum_{k=1}^{m}\sum_{\substack{e=(i,j)\in E_D^1 \\ i\in V^0,\ j\in V_a^k}} \omega_e\,(\widetilde{v}_i-\widetilde{v}_j)^2 \\
& + \sum_{k=1}^{m}\sum_{e'=(p,q)\in E_a^k}\left[(1+\epsilon^{-1})\sum_{\substack{e=(i,j)\in E_D^1 \\ i\in V^0,\ j\in V_a^k}} \omega_e W_k^j\right](\widetilde{v}_p-\widetilde{v}_q)^2.
\end{aligned}
$$

Next, using (4.9), for $e=(i,j)\in E_D^2$, $i\in V_a^k$ and $j\in V_a^\ell$ we have

$$
\begin{aligned}
\left(\frac{1}{d_k}\sum_{p\in V_a^k}\widetilde{v}_p - \frac{1}{d_\ell}\sum_{q\in V_a^\ell}\widetilde{v}_q\right)^2
&= \left(\widetilde{v}_i-\widetilde{v}_j + \frac{1}{d_k}\langle L_{k,i}^{-1}\mathbf{1}_{d_k},L_k\widetilde{\boldsymbol{v}}_k\rangle - \frac{1}{d_\ell}\langle L_{\ell,j}^{-1}\mathbf{1}_{d_\ell},L_\ell\widetilde{\boldsymbol{v}}_\ell\rangle\right)^2 \\
&\leq (1+\epsilon)\,(\widetilde{v}_i-\widetilde{v}_j)^2 + 2(1+\epsilon^{-1})\frac{1}{d_k^2}\|L_{k,i}^{-1}\mathbf{1}_{d_k}\|_{L_k}^2\|\widetilde{\boldsymbol{v}}_k\|_{L_k}^2 \\
&\quad + 2(1+\epsilon^{-1})\frac{1}{d_\ell^2}\|L_{\ell,j}^{-1}\mathbf{1}_{d_\ell}\|_{L_\ell}\|\widetilde{\boldsymbol{v}}_\ell\|_{L_\ell}^2 \\
&= (1+\epsilon)\,(\widetilde{v}_i-\widetilde{v}_j)^2 + \sum_{e'=(p,q)\in E_a^k}\left[2\left(1+\epsilon^{-1}\right)W_k^i\right](\widetilde{v}_p-\widetilde{v}_q)^2 \\
&\quad + \sum_{e'=(p,q)\in E_a^\ell}\left[2\left(1+\epsilon^{-1}\right)W_\ell^j\right](\widetilde{v}_p-\widetilde{v}_q)^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
I_2 \;\leq\; & \sum_{k=1}^{m}\sum_{\ell=1}^{m}\sum_{\substack{e=(i,j)\in E_D^2 \\ i\in V_a^k,\ j\in V_a^\ell}} \omega_e \left\{ (1+\epsilon)\,(\widetilde{v}_i-\widetilde{v}_j)^2 + \sum_{e'=(p,q)\in E_a^k}\left[2\left(1+\epsilon^{-1}\right)W_k^i\right](\widetilde{v}_p-\widetilde{v}_q)^2 \right. \\
& \left. + \sum_{e'=(p,q)\in E_a^\ell}\left[2\left(1+\epsilon^{-1}\right)W_\ell^j\right](\widetilde{v}_p-\widetilde{v}_q)^2 \right\}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
I_2 \;\le\;& (1+\epsilon)\sum_{k=1}^{m}\sum_{\ell=1}^{m}\sum_{\substack{e=(i,j)\in E_D^2 \\ i\in V_a^k,\; j\in V_a^\ell}} \omega_e\left(\widetilde{v}_i-\widetilde{v}_j\right)^2 \\
&+\sum_{k=1}^{m}\sum_{e'=(p,q)\in E_a^k}\left[\sum_{l=1}^{m}\sum_{\substack{e=(i,j)\in E_D^2 \\ i\in V_a^k,\; j\in V_a^\ell}} 2(1+\epsilon^{-1})\omega_e W_k^i\right]\left(\widetilde{v}_p-\widetilde{v}_q\right)^2 \\
&+\sum_{\ell=1}^{m}\sum_{e'=(p,q)\in E_a^\ell}\left[\sum_{k=1}^{m}\sum_{\substack{e=(i,j)\in E_D^2 \\ i\in V_a^k,\; j\in V_a^\ell}} 2(1+\epsilon^{-1})\omega_e W_\ell^j\right]\left(\widetilde{v}_p-\widetilde{v}_q\right)^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I_2 \;\le\;& (1+\epsilon)\sum_{k=1}^{m}\sum_{\ell=1}^{m}\sum_{\substack{e=(i,j)\in E_D^2 \\ i\in V_a^k,\; j\in V_a^\ell}} \omega_e\left(\widetilde{v}_i-\widetilde{v}_j\right)^2 \\
&+\sum_{k=1}^{m}\sum_{e'=(p,q)\in E_a^k}\left[2(1+\epsilon^{-1})\left(\sum_{\ell=1}^{m}\sum_{\substack{e=(i,j)\in E_D^2 \\ i\in V_a^k,\; j\in V_a^\ell}} \omega_e W_k^i + \sum_{\ell=1}^{m}\sum_{\substack{e=(i,j)\in E_D^2 \\ i\in V_a^\ell,\; j\in V_a^k}} \omega_e W_k^j\right)\right]\left(\widetilde{v}_p-\widetilde{v}_q\right)^2.
\end{aligned}
$$

Now, we use the definition of $W_{e'}$ (4.10) and the estimates on I_1 and I_2 to obtain that

$$
\begin{aligned}
\|\widetilde{P}^T\widetilde{\boldsymbol{v}}\|_L^2 \;\le\;& \sum_{e\in E_D^0}\omega_e\left(\widetilde{v}_i-\widetilde{v}_j\right)^2 + (1+\epsilon)\sum_{k=1}^{m}\sum_{\substack{e=(i,j)\in E_D^1 \\ i\in V^0,\; j\in V_a^k}} \omega_e\left(\widetilde{v}_i-\widetilde{v}_j\right)^2 \\
&+(1+\epsilon)\sum_{k=1}^{m}\sum_{\ell=1}^{m}\sum_{\substack{e=(i,j)\in E_D^2 \\ i\in V_a^k,\; j\in V_a^\ell}} \omega_e\left(\widetilde{v}_i-\widetilde{v}_j\right)^2 + \sum_{k=1}^{m}\sum_{e'=(p,q)\in E_a^k} W_{e'}\left(\widetilde{v}_p-\widetilde{v}_q\right)^2.
\end{aligned}
$$

Due to (4.10), we have that $\omega_{e'}\ge W_{e'}$ and (4.11) follows. This completes the proof.

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