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Spectral bisection of graphs and connectedness

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ABSTRACT

We present a refinement of the work of Miroslav Fiedler regarding bisections of irreducible matrices. We consider graph bisections as defined by the cut set consisting of characteristic edges of the eigenvector corresponding to the smallest non-zero eigenvalue of the graph Laplacian (the so-called Fiedler vector). We provide a simple and transparent analysis, including the cases when there exist components with value zero. Namely, we extend the class of graphs for which the Fiedler vector is guaranteed to produce connected subgraphs in the bisection. Furthermore, we show that for all connected graphs there exist Fiedler vectors such that connectedness is preserved by the bisection, and estimate the measure of the set of connectedness preserving Fiedler vectors with respect to the set of all Fiedler vectors.

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1. Introduction

The focus of our considerations is the graph bisection problem. In general, a two-way partition (or bisection) of a graph refers to cutting the graph into two parts, where the order (number of vertices) of each subgraph is similar in size, while minimizing the number of edges that connect the two subgraphs. Formally, the goal is to minimize some objective function, most commonly chosen to be either the edge cut, combined with

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strict equality constraints on the order of the two subgraphs, or the ratio or normalized cut, which allows the subgraphs to differ somewhat in order [10]. Finding the optimal bisection is an \mathcal{NP} -complete problem. However, approximations to optimal partitions which can be computed in reasonable time are well known and can be obtained in several different ways. One approach, with very good performance in practice, is the multilevel graph partitioning algorithms introduced in [8,9]. Another approach, which also provides approximation to other important graph quantities is the so-called *spectral bisection* algorithms.

Roughly speaking, the spectral bisection algorithms use the eigenvector of the graph Laplacian matrix corresponding to the algebraic connectivity of the graph. The approximation to the optimal ratio cut is comprised of all the edges where this eigenvector changes sign. Theoretical background for such approximations is based on the works of Fiedler [4,5]. For additional algorithmic and theoretical results, especially for results related to approximating minimal cuts with spectral bisection, we refer to Pothén, Simon and Liou [16], and also to Chan, Ciarlet Jr. and Szeto [1]. For general spectral results with respect to the Laplacian, we refer to Merris [11–13]. In addition to the Laplacian spectral bisection, partitionings based on spectral decomposition of the adjacency matrix have also been investigated by Powers [17].

The works of Fiedler discuss the connectedness of the subgraphs obtained via spectral bisection in the case where the aforementioned eigenvector does not contain any zero components (characteristic vertices). Our results are a refinement of the results by Fiedler and characterize the connectedness of the subgraphs in all cases. In particular, we extend the results of Fiedler for when the eigenvector has zero components. Namely, we extend the class of graphs for which spectral bisection via the cut set of characteristic edges always produces connected subgraphs. In addition, we show that for all graphs there exists an eigenvector corresponding to the algebraic connectivity such that the corresponding spectral bisection preserves connectivity in the subgraphs.

2. Preliminaries

We introduce the necessary background theory with which we will work. Consider the set of connected, undirected graphs with no self-loops, which we will denote by \mathcal{G} . We will represent the set of graphs in \mathcal{G} with $|V| = n$ by \mathcal{G}_n . In the following, the standard (Euclidean) ℓ^2 -inner product is denoted by (\cdot, \cdot) and the corresponding norm by $\|\cdot\|$. The superscript T denotes a transposition (taking an adjoint with respect to the ℓ^2 -inner product). Further, $\mathbf{0}_m$ is the zero element of \mathbb{R}^m , and $\mathbf{1}_m$ is the vector of all ones in \mathbb{R}^m , namely, $\mathbf{1}_m = \underbrace{(1, \dots, 1)}_m^T$. Often, when no confusion is possible, we shall omit the subscripts on $\mathbf{1}_m$ and $\mathbf{0}_m$.

For a given matrix $A \in \mathbb{R}^{n \times n}$ and set $J \subset \{1, 2, \dots, n\}$, we define $A(J)$ to be the matrix generated by A on the indices J , that is, the submatrix of A obtained by deleting all rows and columns of A whose indices are not in J . We define the signature of a ma-

trix A , denoted $s(A) = (p, q)$, to be the row vector containing the number of positive eigenvalues, p , and the number of negative eigenvalues, q , of A .

In addition, let us introduce notation which is helpful when dealing with signs of components of vectors. We introduce, for a given vector $x \in \mathbb{R}^n$, the following subsets of $\{1, \dots, n\}$.

$$\begin{aligned} i_0(x) &= \{j \mid 1 \leq j \leq n, x_j = 0\}, \\ i_-(x) &= \{j \mid 1 \leq j \leq n, x_j < 0\}, \\ i_+(x) &= \{j \mid 1 \leq j \leq n, x_j > 0\}. \end{aligned}$$

For any $x \in \mathbb{R}^n$, we have $i_0(x) \cup i_+(x) \cup i_-(x) = \{1, \dots, n\}$.

With this notation in hand, we recall a few useful results regarding symmetric matrices. We have the following regarding the signature of symmetric matrices from [3], namely, [3, Results (1,10)–(1,12)].

Lemma 2.1. *If a matrix A is symmetric block-diagonal,*

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_k \end{pmatrix},$$

then $s(A) = \sum_{i=1}^k s(A_i)$.

Lemma 2.2. *Let A be symmetric and let A_1 be a principal submatrix of A . If $s(A) = (p, q)$ and $s(A_1) = (p_1, q_1)$, then $p_1 \leq p$ and $q_1 \leq q$.*

Lemma 2.3. *Let*

$$A = \begin{pmatrix} B & c \\ c^T & d \end{pmatrix}$$

be an $n \times n$ matrix, B an $(n-1) \times (n-1)$ matrix. If for some vector x , $Bx = 0$, $c^T x \neq 0$, then $s(A) = s(B) + (1, 1)$.

From Lemmas 2.2 and 2.3, the following two results follow naturally.

Lemma 2.4. *Let A be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Then no principal submatrix of $A - \lambda_s I$ has more than $s - 1$ negative eigenvalues.*

Proof. The matrix $A - \lambda_s I$ has eigenvalues $\lambda_1 - \lambda_s \leq \dots \leq \lambda_{s-1} - \lambda_s \leq 0 \leq \lambda_{s+1} - \lambda_s \leq \dots \leq \lambda_n - \lambda_s$. Therefore, $A - \lambda_s I$ has $s - 1$ negative eigenvalues. Applying Lemma 2.2, we obtain the desired result. \square

Lemma 2.5. *Let*

$$A = \begin{pmatrix} B & C \\ C^T & D \end{pmatrix}$$

be an $n \times n$ matrix, B an $(n-j) \times (n-j)$ matrix. If for some vector x , $Bx = 0$, $C^T x \neq 0$, then $s(A) \geq s(B) + (1, 1)$.

Proof. If $C^T x \neq 0$, then there must be some index i such that the i -th column of C , denoted C_i , satisfies $C_i^T x \neq 0$. Consider the principal submatrix generated by the indices $S = \{1, 2, \dots, n-j, n-j+i\}$,

$$A[S] = \begin{pmatrix} B & C_i \\ C_i^T & D_{ii} \end{pmatrix}.$$

By [Lemma 2.2](#), we have $s(A) \geq s(A[S])$. Applying [Lemma 2.3](#) to $A[S]$, the result follows. \square

In addition, we recall the eigenvalue interlacing theorem, from [\[7, Theorem 2.1\(i\)\]](#).

Theorem 2.6 (*Interlacing theorem*). *Let S be a real $n \times m$ matrix ($n > m$) such that $S^T S = I$ and let A be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Define $B = S^T A S$ and let B have eigenvalues $\mu_1 \leq \mu_2 \leq \dots \leq \mu_m$. Then the eigenvalues of B interlace those of A , namely*

$$\lambda_{n-m+i} \geq \mu_i \geq \lambda_i.$$

We now introduce the graph Laplacian. Let $G = (V, E) \in \mathcal{G}_n$ be a graph. The (graph) Laplacian matrix of G is given by $L(G) = D(G) - A(G)$, where $D(G)$ and $A(G)$ are the diagonal degree and adjacency matrices of G , respectively. We note that $L(G)\mathbf{1}_n = \mathbf{0}_n$. In addition, L is positive semi-definite. It is also well known that for the null-space of L , $\text{Ker}(L(G))$, we have that $\dim(\text{Ker}(L(G)))$ equals the number of connected components in G . Because we only consider connected graphs, the kernel of $L(G)$ is one dimensional and $\text{Ker}(L(G)) = \text{span}\{\mathbf{1}_n\}$. Thus, $\lambda = 0$ is an eigenvalue of L , with corresponding eigenvector proportional to $\mathbf{1}_n$. Let us order the eigenvalues of $L(G)$ as follows: $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n$, and denote by $\varphi_1 = \frac{1}{\sqrt{n}}\mathbf{1}_n, \varphi_2, \dots, \varphi_n$ the corresponding eigenvectors.

The second eigenvalue λ_2 and the corresponding eigenvector φ_2 have special significance and, for this reason, are given special names. The eigenvalue λ_2 is called the algebraic connectivity of the graph and is denoted by $a(G)$. Any eigenvector corresponding to the eigenvalue $a(G)$ is called a *characteristic valuation*, or *Fiedler vector*, of G . For a given characteristic valuation, a vertex which has a zero valuation is called a *characteristic vertex*, and an edge for which the valuation changes sign (taking zero-valuated vertices as positive) is called a *characteristic edge*.

The Fiedler vector proves to be a useful tool for bisecting a graph (partitioning a graph into two parts). This can be seen by noting the connection between the Rayleigh quotient of L and an edge cut, and recalling also that the eigenvector corresponding to λ_2 minimizes the Rayleigh quotient in the subspace $\{x \mid (x, \mathbf{1}) = 0\}$.

We now recall several results from linear algebra related to reducibility. The reducibility of a matrix is an important notion, and, following Varga [18], we have the following definitions.

Definition 2.7. A matrix $A \in \mathbb{R}^{n \times n}$ is reducible if there exists a permutation matrix $\pi \in \mathbb{R}^{n \times n}$ such that

$$\pi A \pi^T = \begin{pmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{pmatrix}.$$

Definition 2.8. A matrix $A \in \mathbb{R}^{n \times n}$ has degree of reducibility r if there exists a permutation matrix $\pi \in \mathbb{R}^{n \times n}$ such that

$$\pi A \pi^T = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,r+1} \\ 0 & A_{2,2} & \cdots & A_{2,r+1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{r+1,r+1} \end{pmatrix}$$

with $A_{i,i}$ irreducible, $1 \leq i \leq r + 1$.

It is well known that a graph G is connected if and only if its corresponding Laplacian matrix $L(G)$ is irreducible. In addition, reducibility is also closely related to articulation points. We have the following definition.

Definition 2.9. Let $G = (V, E) \in \mathcal{G}$ be a graph. If upon the removal of vertex $i \in V$ the resulting graph is disconnected, we say vertex i is an articulation point (or cut vertex) of G . In addition, if the removal of a subset $V_0 \subset V$ results in a disconnected graph, we say that the set of vertices V_0 is an articulation set of G .

The following results on reducibility can be found in Fiedler’s paper [4, Theorem (2,1), Results (1,5), (1,7)].

Theorem 2.10. *Let A be an $n \times n$ non-negative, irreducible, symmetric matrix with eigenvalues $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$. Let v be a vector such that, for fixed $s \geq 2$, $Av \geq \eta_s v$. Then $M = i_+(v) \cup i_0(v)$ is non-void, and the degree of reducibility of $A(M)$ is less than or equal to $s - 2$.*

Lemma 2.11. *If a matrix A with all off-diagonal entries non-positive and all principal minors non-negative is irreducible and singular, then zero is a simple eigenvalue, there*

exists a unique vector $x \neq 0$ such that $Ax = 0$, and this vector is either positive or negative.

Lemma 2.12. *If a symmetric irreducible matrix has all off-diagonal entries non-positive and $Ax = 0$ for a vector $x \neq 0$ that has both positive and negative components, then A is not positive semi-definite.*

We now state a connectivity proposition combining two results from [4], namely, [4, Theorem (3,3)] and [4, Corollary (3,6)].

Theorem 2.13. *Let $G = (V, E) \in \mathcal{G}_n$ and let y be an eigenvector of the Laplacian $L(G)$ corresponding to $a(G)$. Let $V_1 = i_+(y) \cup i_0(y)$. Then*

- i. *The subgraph G_1 of G induced by the subset V_1 of V is connected.*
- ii. *If $i_0(y) = \emptyset$, then the set of all alternating edges, i.e. edges (i, k) for which $y_i y_k < 0$, forms a cut of G such that both banks of G are connected.*

While the above theorem shows connectedness for a spectral bisection with $i_0(y) = \emptyset$, it says nothing about the case when $i_0(y)$ is non-empty. In the following sections we treat the latter case and, for any graph $G \in \mathcal{G}_n$, show the existence of a characteristic valuation such that the spectral bisection preserves connectedness.

3. Properties of characteristic vertices of Fiedler vectors

Consider a graph $G \in \mathcal{G}_n$, and the set of its characteristic valuations $U \subset \mathbb{R}^n$. We are interested in results on connectedness of the subgraphs when the assumption in the second item in Theorem 2.13 does not hold, namely when the characteristic valuations have at least one zero component. To begin, let us define the following equivalence relation:

$$u \sim v \quad \text{if and only if} \quad i_0(u) = i_0(v). \tag{1}$$

This relation splits U into non-overlapping equivalence classes. We allow the following abuse in notation, and denote by $[J]$ the equivalence class $\{u \mid i_0(u) = J\}$. We have either

$$\bigcap_{u \in U} i_0(u) = \emptyset \quad \text{or} \quad \bigcap_{u \in U} i_0(u) \neq \emptyset.$$

In the first case we have the following result, showing that if the intersection of characteristic vertices is empty there exists a characteristic valuation without zero-components.

Lemma 3.1. *Let $G = (V, E) \in \mathcal{G}_n$ be a graph and U be the eigenspace of characteristic valuations of G . If $\bigcap_{u \in U} i_0(u) = \emptyset$, then there exists a vector $w \in U$ such that $i_0(w) = \emptyset$.*

Proof. We will explicitly construct a vector w with non-zero components. Let $m = \dim U$, and $\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ be a basis for U . Clearly, we have $\bigcap_{j=1}^m i_0(\varphi_j) = \emptyset$. Indeed, if this was not true, then there is a k such that $(\varphi_j)_k = 0$ for all $j = 1, \dots, m$. Holding this k fixed, for any $u \in U$ with $u = \sum_{j=1}^m \alpha_j \varphi_j$ we have that the k -th component of u vanishes. This contradicts the condition in the lemma, namely, that $\bigcap_{u \in U} i_0(u) = \emptyset$. We thus conclude that $\bigcap_{j=1}^m i_0(\varphi_j) = \emptyset$. We now consider

$$w = \sum_{j=1}^m \alpha_j \varphi_j, \quad \text{where } \alpha_1 = 1, \quad \text{and}$$

$$\alpha_j = 1 + 2 \max_{k \notin i_0(\varphi_j)} \frac{|\sum_{l=1}^{j-1} \alpha_l (\varphi_l)_k|}{|(\varphi_j)_k|}, \quad j = 2, \dots, m.$$

We aim to show that $|w_\ell| > 0$, for all $\ell = 1, \dots, n$. To do this, we introduce

$$\delta_j = \min_{k \notin i_0(\varphi_j)} |(\varphi_j)_k|.$$

Note that directly from the definition, we have that $\alpha_j \geq 1$ for all $j = 1, \dots, m$. Note also that for $k \notin i_0(\varphi_j)$, we have $|(\varphi_j)_k| \geq \delta_j > 0$ for all $j = 1, \dots, m$. Consider now the set

$$k_\ell = \{j \mid (\varphi_j)_\ell \neq 0\}.$$

It is immediate to see that $k_\ell \neq \emptyset$. Indeed, $k_\ell = \emptyset$ implies that $(\varphi_j)_\ell = 0$ for all $j = 1, \dots, m$ and this in turn is equivalent to saying that $\ell \in \bigcap_{j=1}^m i_0(\varphi_j)$. However, by the considerations given at the beginning of the proof, this is not possible and hence k_ℓ is not empty.

We now denote $k^* = \max k_\ell$ and observe that $(\varphi_j)_\ell = 0$ for all j such that $k^* < j \leq m$. We then have

$$\begin{aligned} |w_\ell| &= \left| \sum_{j=1}^m \alpha_j (\varphi_j)_\ell \right| = \left| \sum_{j=1}^{k^*} \alpha_j (\varphi_j)_\ell \right| \geq \alpha_{k^*} |(\varphi_{k^*})_\ell| - \left| \sum_{j=1}^{k^*-1} \alpha_j (\varphi_j)_\ell \right| \\ &= |(\varphi_{k^*})_\ell| \left(\alpha_{k^*} - \frac{|\sum_{j=1}^{k^*-1} \alpha_j (\varphi_j)_\ell|}{|(\varphi_{k^*})_\ell|} \right) \geq |(\varphi_{k^*})_\ell| \left(\alpha_{k^*} - \max_{k \notin i_0(\varphi_{k^*})} \frac{|\sum_{j=1}^{k^*-1} \alpha_j (\varphi_j)_k|}{|(\varphi_{k^*})_k|} \right) \\ &> |(\varphi_{k^*})_\ell| \left(\alpha_{k^*} - \frac{1}{2} \alpha_{k^*} \right) \geq \frac{1}{2} \delta_{k^*} \alpha_{k^*} > 0. \end{aligned}$$

Since ℓ was arbitrary, this completes the proof. \square

As a consequence of [Lemma 3.1](#), we have a vector $w \in U$ such that $i_0(w) = \emptyset$, and can apply [Theorem 2.13](#). There are other, more straightforward ways to prove such a result, however, we use a constructive proof to illustrate a procedure guaranteed to

find a valuation with no zero components. We can prove an even stronger result. We have the following.

Theorem 3.2. *Let $G = (V, E) \in \mathcal{G}_n$ be a graph, and $U \cong \mathbb{R}^m$ the set of characteristic valuations of G . Suppose $\bigcap_{u \in U} i_0(u) = \emptyset$. Let $W \subset U$ be the set of characteristic valuations such that $i_0(u)$ is non-empty. Then W has zero m -dimensional volume.*

Proof. Let us denote the set of all characteristic valuations u such that $j \in i_0(u)$ by U_j . We have that $W = \bigcup_{j=1}^m U_j$. By Lemma 3.1, there exists some characteristic valuation v such that $v \notin W$. Therefore W is a proper subset of U . Let us consider the set U_j . One can verify that U_j is a subspace of U . Then we may conclude that $U_j \cong \mathbb{R}^{m_j}$, for some $m_j < m$. We may thus conclude that each U_j has zero m -dimensional volume. Therefore, W has zero m -dimensional volume. \square

The second case, when the entire subspace of characteristic valuations has a common zero component, is more intricate. Let us denote the intersection of the zero components of all the characteristic valuations by $i_0(U) := \bigcap_{u \in U} i_0(u)$. With this notation in hand, we now show the existence of a characteristic valuations whose zero components are exactly the common zero components.

Lemma 3.3. *Let $i_0(U) = \bigcap_{u \in U} i_0(u)$. Then there exists a characteristic valuation u such that $i_0(u) = i_0(U)$.*

Proof. Assume that there is no vector for which $i_0(u) = i_0(U)$. This implies that there is no characteristic valuation which is zero-valuated for only the components $i_0(U)$. Therefore, upon removing the components in $i_0(U)$ from every characteristic valuation, we see that the remaining subspace has a basis with an empty intersection of zero-valuated vertices. But, by Lemma 3.1, we can construct a vector w in this subspace which has no zero-valuated vertices. Extending this vector as zero on the vertices from $i_0(U)$ gives us a characteristic valuation such that its zero components are precisely $i_0(U)$. \square

In addition, we aim to show that almost every characteristic valuation is in the equivalence class $[i_0(U)]$. We have the following result.

Theorem 3.4. *Let $G = (V, E) \in \mathcal{G}_n$ be a graph, and $U \cong \mathbb{R}^m$ the set of characteristic valuations of G . Let $i_0(U) = \bigcap_{u \in U} i_0(u)$, and $W \subset U$ be the set of Fiedler vectors u such that $i_0(u)$ is a proper subset of $i_0(U)$. Then the set W has zero m -dimensional volume.*

Proof. Upon the removal of the components $i_0(U)$, $|i_0(U)| = j$, from every characteristic valuation, the remaining subspace has an empty intersection of zero-valuated vertices. Thus, by Theorem 3.2, the set of elements of this subspace which has zero components has zero $(m - j)$ -dimensional volume. Extending the vectors back to U , we can conclude

that the set of elements for which $i_0(U)$ is a proper subset of its zero valuated vertices has zero m -dimensional volume. \square

Based on the results of [Theorem 3.4](#), we can restrict ourselves to the equivalence class corresponding to $i_0(U)$. Without loss of generality we may renumber the vertices of the graph so that $i_0(U) = \{1, \dots, j\}$. We then have the following 2×2 block form of the graph Laplacian

$$L = \begin{pmatrix} L_0 & -X^T \\ -X & \tilde{L} \end{pmatrix} \in \mathbb{R}^{n \times n}, \quad L_0 \in \mathbb{R}^{j \times j}, \quad X \in \mathbb{R}^{(n-j) \times j}, \quad \tilde{L} \in \mathbb{R}^{(n-j) \times (n-j)}.$$

Clearly, for $y \in [i_0(U)]$ we have $y = \begin{pmatrix} \mathbf{0}_j \\ \tilde{y} \end{pmatrix}$, where $\tilde{y} \in \mathbb{R}^{n-j}$, and, moreover, $i_0(\tilde{y}) = \emptyset$. Direct computation shows that $\tilde{L}\tilde{y} = a(G)\tilde{y}$ and $X^T\tilde{y} = 0$.

We consider the reducibility properties of \tilde{L} . We have two cases: (1) the set of zero-valuated vertices is not an articulation set, in which case \tilde{L} is irreducible; (2) the set of zero-valuated vertices is an articulation set, in which case \tilde{L} has degree of reducibility r for some $r \geq 1$.

For the moment we concern ourselves with the latter case. If \tilde{L} has degree of reducibility r , then we have

$$\tilde{L} = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L_{r+1} \end{pmatrix}, \tag{2}$$

with $L_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \dots, r + 1$. Further, the characteristic valuations and block matrix X can be put in a form which is in accordance with the reduced form above, namely,

$$y = \begin{pmatrix} \mathbf{0}_j \\ \tilde{y} \end{pmatrix}, \quad \tilde{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_{r+1} \end{pmatrix}, \quad X = \begin{pmatrix} X_1 \\ \vdots \\ X_{r+1} \end{pmatrix}. \tag{3}$$

We characterize the form that these eigenvectors take explicitly.

Theorem 3.5. *Let $G = (V, E) \in \mathcal{G}_n$ be a graph and U the set of characteristic valuations of G . If $i_0(U)$ is an articulation set with degree of reducibility r , namely, $L(G)$ takes the block form (2), then y , in corresponding block notation (3), is a characteristic valuation of G if and only if $L_i y_i = a(G) y_i$ for all $i = 1, \dots, r + 1$ and $\sum_{i=1}^{r+1} \mathbf{1}^T y_i = 0$. Moreover, $a(G)$ is the minimal eigenvalue for each L_i .*

Proof. We begin by restricting our considerations to the equivalence class $[i_0(U)]$ and take a characteristic valuation $y \in [i_0(U)]$. We recall that $a(G)$ is an eigenvalue of \tilde{L} ,

with corresponding eigenvector \tilde{y} . This implies that $(L_i - a(G)I)$ is singular and $(L_i - a(G)I)y_i = 0$. We first aim to show that none of the vectors y_i has both positive and negative components.

Suppose, to the contrary, that one of the vectors, say y_1 , has both positive and negative components. By Lemma 2.12, this implies that $L_1 - a(G)I$ has at least one negative eigenvalue. However, $\tilde{L} - a(G)I$ is a principal submatrix of $L - a(G)I$. Using Lemma 2.4 with $s = 2$, we see that $\tilde{L} - a(G)I$ has at most one negative eigenvalue, and, therefore, $L_i - a(G)I$ are positive semi-definite for all $i > 1$. We fix one such i , and note that L_i is irreducible. Therefore, by Lemma 2.11, the components of the vector y_i are either all positive or all negative. In addition, L is also irreducible, and, therefore, we have that $X_i \neq 0$. Hence, $X_i^T y_i$ is nonzero. By Lemmas 2.1 and 2.5, we have $s(L - a(G)I) \geq \sum_{i=1}^{r+1} s(L_i - a(G)I) + (1, 1)$. This implies that $L - a(G)I$ has at least two negative eigenvalues. We have reached a contradiction. Therefore, $y_2 = y_3 = \dots = y_r = 0$. However, because we assumed that \tilde{y} has no zero-valuated vertices, we can conclude that our initial assumption that y_1 has both positive and negative components was false. This implies that $a(G)$ is the minimal eigenvalue for each L_i , and furthermore, by the Perron–Frobenius theorem, the corresponding eigenvector y_i is simple, and either positive or negative.

The considerations above readily extend from characteristic valuations from the equivalence class $i_0(U)$ to the entire eigenspace U of characteristic valuations. For any such y there will be a zero block in the decomposition in irreducible components. This zero block corresponds to indices from $i_0(y) \setminus i_0(U)$.

What remains is to show that $y \in U$ if and only if $\mathbf{1}_n^T y = 0$ and $L_i y_i = a(G)y_i$ for all $i = 1, \dots, r + 1$.

First, suppose $y \in U$. This clearly implies $\mathbf{1}_n^T y = 0$, because y is in the eigenspace corresponding to $a(G)$. Writing out $Ly = a(G)y$ block-wise, we easily verify that $L_i y_i = a(G)y_i$ for all $i = 1, \dots, r + 1$.

To show the other direction, suppose $\mathbf{1}_n^T y = 0$ and $L_i y_i = a(G)y_i$ for all $i = 1, \dots, r + 1$. This implies that $\tilde{L}\tilde{y} = a(G)\tilde{y}$ and we have

$$\begin{aligned} y^T Ly &= (\mathbf{0}_j^T, \tilde{y}^T) \begin{pmatrix} L_0 & -X^T \\ -X & \tilde{L} \end{pmatrix} \begin{pmatrix} \mathbf{0}_j \\ \tilde{y} \end{pmatrix} \\ &= (\mathbf{0}_j^T, \tilde{y}^T) \begin{pmatrix} -X^T \tilde{y} \\ \tilde{L}\tilde{y} \end{pmatrix} = \tilde{y}^T \tilde{L}\tilde{y} = a(G)\tilde{y}^T \tilde{y} = a(G)y^T y. \end{aligned}$$

Such an identity can only hold if $y = z + z_0$, where z is an eigenvector of L corresponding to eigenvalue $a(G)$ and z_0 is an element from the null space of L , namely, $z_0 \in \text{span}\{\mathbf{1}_n\}$. Since $\mathbf{1}_n^T y = 0$ we conclude that y must be an eigenvector corresponding to the eigenvalue $a(G)$. The proof is complete. \square

From the above result we have the following corollary regarding the multiplicity of $a(G)$.

Corollary 3.6. *Let $G = (V, E) \in \mathcal{G}_n$ be a graph, and U the set of characteristic valuations of G . If $i_0(U)$ is an articulation set with degree of reducibility r , then the eigenvalue $a(G)$ of L has multiplicity r .*

Proof. Let us fix some $y \in [i_0(U)]$, and construct a basis for U . Consider the vectors $\varphi_j^T = (\mathbf{0}_j^T, \mathbf{0}_{n_1}^T, \dots, -\frac{1^T y_{r+1}}{\mathbf{1}^T y_j} y_j^T, \dots, \mathbf{0}_{n_r}^T, y_{r+1}^T)$, $j = 1, \dots, r$. Each vector is an eigenvector corresponding to $a(G)$. In addition, one can verify by inspection that the set $\{\varphi_j\}_{j=1}^r$ is linearly independent, and spans U . Therefore, $a(G)$ has multiplicity r . \square

4. Bisection and connectedness of the subgraphs

We now investigate the connectivity properties of characteristic valuations under spectral bisections defined by the cut set of characteristic edges. Namely, we aim to extend Fiedler’s [Theorem 2.13](#) to cases where there exist zero components. It suffices to consider graphs for which $i_0(U)$ is non-empty. For graphs in which $i_0(U) = \emptyset$ we have shown, in [Lemma 3.2](#), that almost every characteristic valuation has no zero components. We begin with the case in which $i_0(U)$ is not an articulation set and show that, for this case, $[i_0(U)]$ is connectedness-preserving.

Theorem 4.1. *Let $G = (V, E) \in \mathcal{G}_n$ be a graph, and y a characteristic valuation of G . If G is still connected upon the removal of the vertices $i_0(y) \subset V$, then the graphs generated by $i_+(y) \cup i_0(y)$ and $i_-(y)$ are non-empty and connected.*

Proof. Recall that if $y \in U$ is a characteristic valuation of L then we may reorder the vertices such that $y^T = (\mathbf{0}_j^T, \tilde{y}^T)$, where $i_0(\tilde{y}) = \emptyset$, and $\tilde{L}\tilde{y} = a(G)\tilde{y}$.

Let $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n$ and $\mu_1 \leq \mu_2 \leq \dots \leq \mu_{n-j}$ be the eigenvalues of L and \tilde{L} , respectively. We see $\lambda_1 = 0$ and $\lambda_1 < \lambda_2$ from the properties of the Laplacian of a connected graph. We do not have the same properties for \tilde{L} , because \tilde{L} does not satisfy $\tilde{L}\mathbf{1}_{n-j} = \mathbf{0}_{n-j}$. However, it is immediate to see that \tilde{L} is an M-matrix. From the interlacing [Theorem 2.6](#) we have that

$$0 = \lambda_1 \leq \mu_1 \quad \text{and} \quad a(G) = \lambda_2 \leq \mu_2.$$

From $\tilde{L}\tilde{y} = a(G)\tilde{y}$, we have that $a(G)$ is an eigenvalue and \tilde{y} an eigenvector of \tilde{L} . Therefore, $\mu_1 = a(G)$ or $\mu_2 = a(G)$, or both. By the Perron–Frobenius theorem applied to \tilde{L}^{-1} , (see Perron [[14](#), p. 47], [[15](#), p. 261] and Frobenius [[6](#)]) the eigenvector of μ_1 is either non-negative, or non-positive. Because \tilde{y} contains both positive and negative values (it is, after all, orthogonal to $\mathbf{1}_{n-j}$), \tilde{y} must be an eigenvector of $\mu_2 = a(G)$.

By [Theorem 2.10](#) applied to $gI - \tilde{L}$, where g is the Gershgorin bound $g = 2 \max_{1 \leq i \leq n-j} |\tilde{L}_{ii}|$, we have that the matrix generated by $i_+(\tilde{y}) \cup i_0(\tilde{y})$ is non-void and irreducible. However, we have that $i_0(\tilde{y}) = \emptyset$ and hence $i_+(y) = i_+(\tilde{y})$ also is irreducible. Applying the same considerations to $(-\tilde{y})$, we conclude that both $i_+(\tilde{y})$ and

$i_-(\tilde{y})$ are non-void and generate irreducible matrices. This implies that the subgraphs generated by $i_+(y)$ and $i_-(y)$ are non-empty and connected. Therefore, on the larger graph, represented by L , we have that the graphs generated by $i_+(y) \cup i_0(y)$ and $i_-(y)$ are non-empty and connected. \square

This extends the bisection results of Fiedler to include characteristic valuations for which the zero valuated set is not an articulation set. We can go a step further, and show the existence of a connectedness-preserving characteristic valuation for any graph G .

Theorem 4.2 (*Generalized bisection theorem*). *For any graph $G \in \mathcal{G}_n$ there exists a characteristic valuation u such that the subgraphs generated by $i_+(u) \cup i_0(u)$ and $i_-(u)$ are connected.*

Proof. We proceed with this proof through cases. For a graph G with $i_0(U) = \emptyset$ we have, by Lemma 3.1, the existence of a valuation u such that $i_0(u) = \emptyset$, in which we can apply Theorem 2.13. For the case where $i_0(U) \neq \emptyset$, we first suppose $i_0(U)$ to not be an articulation set. This is covered by Theorem 4.1. Therefore, it suffices to consider the case in which $i_0(U)$ is an articulation set.

Suppose $i_0(U)$ is an articulation set with degree of reducibility $r \geq 1$. We consider the valuation $u^T = (\mathbf{0}_j^T, y_1^T, \dots, y_r^T, \gamma y_{r+1}^T)$, where each y_j is a positive eigenvector of L_j corresponding to $a(G)$ and $\gamma = -\frac{\sum_{j=1}^r \mathbf{1}^T y_j}{\mathbf{1}^T y_{r+1}}$. One can verify that u is a characteristic valuation. The set $i_-(u)$ consists of the components of L_{r+1} . However, L_{r+1} is an irreducible matrix and, therefore, the subgraph generated by $i_-(u)$ is connected. This completes the proof. \square

We now proceed to consider the volume of the connectedness-preserving characteristic valuations of graphs. From Theorem 4.1, we see that for the case when $i_0(U)$ is not an articulation set, we have that the entire set $[i_0(U)]$ is connectedness-preserving.

To consider the case when $i_0(U)$ is an articulation set, we need the following isomorphism. Let y_1, \dots, y_{r+1} be given vectors such that each y_j is an eigenvector of L_j corresponding to $a(G)$, $y_k \in \mathbb{R}^{n_k}$, $\mathbf{1}_{n_k}^T y_k = 1$, $k = 1, \dots, r+1$. We define C to be the natural isomorphism between the Cartesian product of $\text{span}\{y_k\}$ and \mathbb{R}^{r+1} , namely

$$\begin{aligned} C : \text{span}\{y_1\} \times \cdots \times \text{span}\{y_{r+1}\} &\mapsto \mathbb{R}^{r+1}, \\ C(\alpha_1 y_1, \dots, \alpha_{r+1} y_{r+1}) &= (\alpha_1, \dots, \alpha_{r+1})^T. \end{aligned} \quad (4)$$

With this isomorphism, we also associate a set of normalized vectors. For example, C_p will denote the isomorphism given above followed by a normalization in $\ell^p(\mathbb{R}^{r+1})$, namely,

$$C_p(\alpha_1 y_1, \dots, \alpha_{r+1} y_{r+1}) := \frac{(\alpha_1, \dots, \alpha_{r+1})^T}{\|\boldsymbol{\alpha}\|_{\ell^p}}. \quad (5)$$

We now show that the set of connectedness-preserving characteristic valuations has positive volume.

Theorem 4.3. *Let $G = (V, E) \in \mathcal{G}_n$ be a graph, $U \cong \mathbb{R}^m$ the eigenspace of $a(G)$, and $W \subset U$ the set of valuations $u \in U$ such that the subgraphs generated by $i_+(u) \cup i_0(u)$ and $i_-(u)$ are connected. Then W has positive m -dimensional volume.*

Proof. The case when $i_0(U)$ is not an articulation set is covered by [Theorem 4.1](#) and, therefore, it suffices to consider the case when $i_0(U)$ is an articulation set. For $i_0(U)$ an articulation set, we have $m = r$. The proof is completed by considering the mapping $C : U \mapsto \mathbb{R}^{r+1}$ as defined in Eq. (4). The set $C(U)$ is an r -dimensional manifold, which can be defined as $C(U) = \{\alpha \mid \alpha_{r+1} = -\sum_{i=1}^r \alpha_i\}$. Let us consider the set $S = \{\alpha \in C(U) \mid \alpha_j > 0, j = 1, \dots, r\}$. One can verify this to be a subset of $C(W)$. We see immediately that S has positive r -dimensional volume. The proof is complete. \square

In the following lemma we estimate the ratio between the volumes of the ℓ^1 -normalized image of the connectedness-preserving characteristic valuations $C_1(W) \subset \mathbb{R}^r$ and the ℓ^1 -normalized image of the subspace of all characteristic valuations $C_1(U) \subset \mathbb{R}^r$, to give an estimate of what ratio of characteristic valuations preserves connectedness when $i_0(U)$ is an articulation set.

Lemma 4.4. *Let $G \in \mathcal{G}_n$ be a graph, U the set of characteristic valuations of G , and $W \subset U$ the set of elements $u \in U$ such that the subgraphs generated by $i_+(u) \cup i_0(u)$ and $i_-(u)$ are connected. Suppose that $i_0(U)$ is an articulation set with degree of reducibility $r \geq 1$. Then*

$$\rho = \frac{\mu(C_1(W))}{\mu(C_1(U))} = (r + 1) \left[\sum_{i=1}^r \binom{r+1}{i} \binom{r-1}{i-1} \sqrt{\frac{i(r-i+1)}{r}} \right]^{-1}$$

where $\mu(\cdot)$ denotes the $(r - 1)$ -dimensional volume (measure).

Proof. In the following, we denote by χ_S the characteristic function of a set S and we also set $\int := \int_{\mathbb{R}^{r-1}}$ where the integration is with respect to the Lebesgue measure in \mathbb{R}^{r-1} . We introduce the following sets

$$T_i = \{\alpha \in C_1(U) \mid \alpha_1 > 0, \dots, \alpha_i > 0 \text{ and } \alpha_{i+1} < 0, \dots, \alpha_{r+1} < 0\}.$$

We next compute ρ for $r > 1$ (for $r = 1$, one can verify $\rho = 1$). The set $C_1(W)$ consists of all subsets of $C_1(U)$ in which precisely one component is negative. Therefore, ρ takes the form

$$\rho = \frac{\int \chi_{C_1(W)}}{\int \chi_{C_1(U)}} = \frac{(r + 1) \int \chi_{T_r}}{\sum_{i=1}^r \binom{r+1}{i} \int \chi_{T_i}}.$$

Given the signs of the individual components α_i , we can decompose the conditions $\|\alpha\|_{\ell^1} = 1$ and $(\alpha, \mathbf{1}) = 0$ in terms of the sets of positive and negative components, T_+ and T_- (T_+ and T_- partition $\{1, \dots, r + 1\}$), respectively:

$$1 = \|\alpha\|_{\ell^1} = \sum_{i=1}^{r+1} |\alpha_i| = \sum_{i \in T_+} |\alpha_i| + \sum_{j \in T_-} |\alpha_j| = \sum_{i \in T_+} \alpha_i - \sum_{j \in T_-} \alpha_j,$$

$$0 = (\alpha, \mathbf{1}) = \sum_{i=1}^{r+1} \alpha_i = \sum_{i \in T_+} \alpha_i + \sum_{j \in T_-} \alpha_j.$$

We see that $\sum_{i \in T_+} \alpha_i = \frac{1}{2}$ and $\sum_{j \in T_-} \alpha_j = -\frac{1}{2}$. We can now write the expression for ρ in a more explicit manner. Introducing the simplices

$$S_i = \left\{ \alpha \in \mathbb{R}^i \mid \sum_{j=1}^i \alpha_j = \frac{1}{2}, \alpha_1 > 0, \dots, \alpha_i > 0 \right\},$$

we obtain

$$\rho = \frac{(r + 1) \int \chi_{T_r}}{\sum_{i=1}^r \binom{r+1}{i} \int \chi_{T_i}} = \frac{(r + 1) \int \chi_{S_r}}{\sum_{i=1}^r \binom{r+1}{i} (\int_{\mathbb{R}^i} \chi_{S_i}) (\int_{\mathbb{R}^{r+1-i}} \chi_{S_{r+1-i}})}.$$

We recall that the volume of a k -dimensional simplex Δ_k in \mathbb{R}^{k+1}

$$\Delta_k = \left\{ \beta \in \mathbb{R}^{k+1} \mid \sum_{i=1}^{k+1} \beta_i = c, \beta_1 > 0, \dots, \beta_{k+1} > 0 \right\},$$

is given by (see, for example, [2])

$$|\Delta_k| = \int_{\mathbb{R}^k} \chi_{\Delta_k} = \frac{c^k \sqrt{k + 1}}{k!}.$$

This allows us to rewrite ρ in terms of the volumes of simplices.

$$\begin{aligned} \rho &= \frac{(r + 1) |\Delta_{r-1}|}{\sum_{i=1}^r \binom{r+1}{i} |\Delta_{i-1}| |\Delta_{r-i}|} = \frac{\frac{(r+1)\sqrt{r}}{2^{r-1}(r-1)!}}{\sum_{i=1}^r \binom{r+1}{i} \frac{\sqrt{i}}{2^{i-1}(i-1)!} \frac{\sqrt{r-i+1}}{2^{r-i}(r-i)!}} \\ &= (r + 1) \left[\sum_{i=1}^r \binom{r + 1}{i} \binom{r - 1}{i - 1} \sqrt{\frac{i(r - i + 1)}{r}} \right]^{-1}. \end{aligned}$$

This completes the proof. \square

As a corollary from Lemma 4.4, we can see that as the degree of reducibility r increases, the ratio between $C_1(W)$ and $C_1(U)$ decreases exponentially. To show this, we note that

$\frac{i(r-i+1)}{\sum_{i=1}^r \binom{n}{i}} \geq 1$ and $\binom{r-1}{i-1} \geq 1$ for $1 \leq i \leq r$. Using these inequalities and the identity $\sum_{i=1}^n \binom{n}{i} = 2^n$ gives

$$\begin{aligned} \rho &= (r+1) \left[\sum_{i=1}^r \binom{r+1}{i} \binom{r-1}{i-1} \sqrt{\frac{i(r-i+1)}{r}} \right]^{-1} < (r+1) \left(\sum_{i=1}^r \binom{r+1}{i} \right)^{-1} \\ &= \frac{r+1}{2^{r+1} - (r+1)}. \end{aligned}$$

In addition, all ℓ_p -norms in finite-dimensional space are equivalent with constants of equivalence at worst linear in dimension, i.e., for $p > q > 1$

$$\|\beta\|_{\ell_p} \leq \|\beta\|_{\ell_q} \leq n^{(1/q-1/p)} \|\beta\|_{\ell_p}, \quad \text{for all } \beta \in \mathbb{R}^n.$$

Since for $p = 1$ the decay is exponential, this equivalence implies that the ratio of connectedness-preserving characteristic valuations decreases exponentially with degree of reducibility r , irrespective of ℓ^p normalization choice.

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