HAMILTON POWERS OF EULERIAN DIGRAPHS

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Abstract. In this note, we prove that the $\lceil \frac{1}{2}\sqrt{n}\log_2^2 n\rceil^{th}$ power of a connected n-vertex Eulerian digraph is Hamiltonian, and provide an infinite family of digraphs for which the $\lfloor \sqrt{n}/2 \rfloor^{th}$ power is not.

1. Preliminaries

The k^{th} power of a (directed or undirected) graph G, denoted G^k, is the graph on the vertices of G in which there is an edge from a vertex u to a vertex v if there exists a uv-path in G of length at most k. It is well-known that the cube of any connected undirected graph is Hamiltonian (see [6, 11], also [3, Ex 10-14]). In 1974, Fleischner proved that the square of any two-connected undirected graph is Hamiltonian, solving the Plummer-Nash-Williams conjecture [4] (see [5] for a much simpler proof). Unfortunately, strongly-connected directed graphs (digraphs) may require the $\lceil n/2 \rceil^{th}$ power to be Hamiltonian; even k-strong connectedness is only sufficient for guaranteeing that the $\lceil n/(2k) \rceil^{th}$ power is Hamiltonian [10]. For a general survey on Hamilton cycles in digraphs, we refer the reader to [7]. Interestingly, results for Eulerian digraphs are not nearly so bleak[1]. Through the study of minimally Eulerian digraphs (connected Eulerian digraphs with no proper connected Eulerian subgraph), we prove that

Theorem 1.1. *The $\lceil \frac{1}{2}\sqrt{n}\log_2^2 n\rceil^{th}$ power of any n-vertex connected Eulerian digraph is Hamiltonian.*

In fact, we prove an even stronger result (in Theorem 2.1) that, given a minimally Eulerian digraph $G = (V, A)$, specifies an ordering $v_1, ..., v_n$ of V and an edge-disjoint directed path (dipath) decomposition $P_1, ..., P_n$ of G, such that each P_i is a $v_i v_{i+1}$-dipath ($v_{n+1} := v_1$) of length at most $\lceil \frac{1}{2}\sqrt{n}\log_2^2 n\rceil$. In addition, we provide an infinite family of minimally Eulerian digraphs for which the $\lfloor \sqrt{n}/2 \rfloor^{th}$ power is not Hamiltonian (Example 2.2). For details regarding the importance of minimally Eulerian digraphs and their connection to the traveling salesman problem, we refer the reader to [2, 8].

1.1. **Notation, Definitions, and Basic Results.** Let $G = (V, A)$ be a simple digraph. If G contains a spanning directed cycle (dicycle), then G is Hamiltonian. If G contains an Euler circuit (a circuit containing every edge), then G is Eulerian. If G is connected, this is equivalent to the condition that, for every vertex $v \in V$, the indegree $d^-(v)$ equals the outdegree $d^+(v)$. If G is a connected Eulerian digraph and contains no proper connected Eulerian subgraph on the vertices of G, then G is minimally Eulerian; equivalently, a connected Eulerian digraph G is minimally Eulerian if, for any dicycle C of G, the graph $G - C := (V, A - A(C))$ is disconnected. If G contains no dicycle,

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[1]The first notable example of a class of digraphs requiring a "non-trivial" (say, $o(n)$) Hamiltonicity exponent are cacti, see [9] for details.

then G is acyclic. For more details regarding graph theoretic definitions and notation, we refer the reader to [1]. Let

$$p_{\#}(G) := \frac{1}{2} \sum_{u \in V} |d^+(u) - d^-(u)|,$$

a measure of how "close" to Eulerian a digraph is, and a key ingredient in our proof. The quantity $p_{\#}(G)$ is exactly the minimal number of dipaths required in an edge-disjoint decomposition of G into dipaths and dicycles. That $p_{\#}(G)$ dipaths are required follows immediately from the definition of $p_{\#}(G)$ above. That $p_{\#}(G)$ dipaths are sufficient follows from a simple greedy algorithm (iteratively perform walks from vertices u with $d^+(u) > d^-(u)$, removing dicycles when they are formed, and only removing the dipath when a vertex v with $d^+(v) = 0$ is reached). The size of an acyclic digraph G is immediately bounded above by $p_{\#}(G)\,(|V| - 1)$, and an even tighter estimate can be obtained relatively quickly:

Proposition 1.2. *Let $G = (V, A)$ be an acyclic digraph. Then $|A| \leq \sqrt{2p_{\#}(G)}\,|V|$.*

Proof. If $p_{\#}(G) = 0, 1, 2$, the result follows immediately, as $|A| \leq p_{\#}(G)(|V| - 1)$. Now, let $p_{\#}(G) > 2$, $V = \{v_1, ..., v_n\}$ be a topological sorting of G (i.e., $v_i v_j \in A$ implies that $i < j$), $k \in \mathbb{N}$ be the smallest number such that $p_{\#}(G) \leq \binom{k}{2}$, $\ell = \lceil n/k \rceil$, and $V_i = \{v_{(i-1)k+1}, ..., v_{ik}\}$, $i = 1, ..., \ell - 1$, $V_\ell = \{v_{(\ell-1)k+1}, ..., v_n\}$. There are at most $\binom{k}{2}$ edges within each of the subsets V_i, $i = 1, ..., \ell - 1$, and at most $\binom{n-k(\ell-1)}{2}$ within the subset V_ℓ. Our digraph G can be decomposed into $p_{\#}(G)$ edge-disjoint dipaths, and, by the topological sorting of V, each of the aforementioned $p_{\#}(G)$ dipaths has at most $\ell - 1$ edges between the subsets $V_1, ..., V_\ell$. Therefore, there are at most $(\ell - 1)p_{\#}(G)$ total edges between the subsets $V_1, ..., V_\ell$. Combining these estimates gives

$$|A| \leq \big(\ell - 1\big)\Big[\tbinom{k}{2} + p_{\#}(G)\Big] + \tbinom{n-k(\ell-1)}{2}.$$

Dividing by $\sqrt{p_{\#}(G)}\,n$, we have

$$\frac{|A|}{\sqrt{p_{\#}(G)}\,n} \leq \frac{\ell - 1}{n}\left(\frac{\binom{k}{2}}{\sqrt{p_{\#}(G)}} + \sqrt{p_{\#}(G)}\right) + \frac{\binom{n-k(\ell-1)}{2}}{\sqrt{p_{\#}(G)}n}.$$

The right hand side is convex w.r.t. $p_{\#}(G)$ and maximized when $p_{\#}(G)$ is as small as possible. We note that, by the definition of k, $p_{\#}(G) > \binom{k-1}{2}$. So the right hand side can be bounded above by replacing $p_{\#}(G)$ by $\binom{k-1}{2}$, giving

$$\frac{|A|}{\sqrt{p_{\#}(G)}\,n} < \frac{\ell - 1}{n}\frac{(k-1)^2}{\binom{k-1}{2}^{1/2}} + \frac{\big(n - k(\ell - 1)\big)\big(n - k(\ell - 1) - 1\big)}{2\binom{k-1}{2}^{1/2} n}.$$

The right hand side is a convex quadratic function in the term ℓ (treating ℓ as a variable independent of n and k), and therefore achieves its maximum at one of the endpoints of the interval $[n/k, n/k + 1]$. Setting $\ell = n/k$ gives

$$\frac{\ell - 1}{n}\frac{(k-1)^2}{\binom{k-1}{2}^{1/2}} + \frac{\big(n - k(\ell - 1)\big)\big(n - k(\ell - 1) - 1\big)}{2\binom{k-1}{2}^{1/2} n} = \frac{(k-1)^2}{k\binom{k-1}{2}^{1/2}} - \frac{k^2 - 3k + 2}{2n\binom{k-1}{2}^{1/2}},$$

and setting $\ell = n/k + 1$ gives

$$\frac{\ell-1}{n}\,\frac{(k-1)^2}{\binom{k-1}{2}^{1/2}} + \frac{\big(n-k(\ell-1)\big)\big(n-k(\ell-1)-1\big)}{2\binom{k-1}{2}^{1/2} n} = \frac{(k-1)^2}{k\binom{k-1}{2}^{1/2}}.$$

Noting that $k^2 - 3k + 2 \geq 0$ for all $k \in \mathbb{N}$, we conclude that the maximum over the interval $[n/k, n/k+1]$ is obtained at $\ell = n/k+1$. Replacing ℓ by $n/k+1$, we have

$$|A| < \frac{(k-1)^2}{k\binom{k-1}{2}^{1/2}}\sqrt{p_\#(G)}\,n = \frac{(k-1)^{3/2}}{k(k-2)^{1/2}}\sqrt{2p_\#(G)}\,n \leq \sqrt{2p_\#(G)}\,n,$$

for $k \geq 3$ (recall, $p_\#(G) > 2$). □

From Proposition 1.2 we immediately obtain a bound (tight up to a multiplicative constant; see Example 2.2) on the maximum size of a minimally Eulerian digraph:

Proposition 1.3. *Let $G = (V, A)$ be a minimally Eulerian digraph. Then $|A| \leq \sqrt{2(|V|-1)}\,|V| + |V| - 1$.*

Proof. G is a connected Eulerian digraph, so it admits a rooted, directed subgraph T of G in which there is a unique path (in T) from the root to any other vertex of G. Every dicycle of G must intersect an edge of T, as the removal of any dicycle from a minimally Eulerian graph disconnects it. Therefore, $G - T$ is acyclic, and by Proposition 1.2, $|A| \leq |A(G-T)| + |A(T)| \leq \sqrt{2(|V|-1)}\,|V| + |V| - 1$. □

2. A Proof of Theorem 1.1 and a Lower Bound

To prove Theorem 1.1, we show an even stronger statement regarding minimally Eulerian digraphs.

Theorem 2.1. *Let $G = (V, A)$, $|V| = n > 1$, be a minimally Eulerian graph. Then there exists an ordering $v_1, ..., v_n$ of V and an n-dipath edge-disjoint decomposition $P_1, ..., P_n$ of G such that each P_i is a v_iv_{i+1}-dipath ($v_{n+1} := v_1$) of length at most $\lceil f(n)\sqrt{n}\,\rceil$, where*

$$f(n) = (\log_2 n)^{\log_{3/2} 2 + o(1)} \leq \frac{1}{2}\log_2^2 n.$$

Proof. We first show that there exists an ordering $v_1, ..., v_n$ of $V(G)$ such that there is an n-dipath edge-disjoint decomposition $P_1, ..., P_n$ of G such that each P_i is a v_iv_{i+1}-dipath. This ordering and decomposition can be constructed by picking a base vertex $v_1 \in V(G)$ and considering an Eulerian circuit W of G starting at v_1, ordering the remaining vertices based on the order of first appearance in this circuit, and taking each dipath P_i to be the walk in W between the first appearance of v_i and the first appearance of v_{i+1}. As G is minimally Eulerian, each such walk is a dipath. It suffices to consider $n \geq 6388$, as the length of a dipath is at most $n-1$ and $\lceil \frac{1}{2}\sqrt{n}\log_2^2 n\,\rceil \geq n-1$ for $n = 1, ..., 6387$.

Let $v_1, ..., v_n$ be an ordering of $V(G)$ and $P_1, ..., P_n$ a decomposition of G into edge-disjoint v_iv_{i+1}-dipaths P_i. We choose this ordering and decomposition so that the elements of the set $\{|A(P_1)|, ..., |A(P_n)|\}$ are lexicographically minimized (i.e., minimizes the length of the longest dipath, minimizes the length of the 2^{nd} longest dipath conditional on the minimality of the longest dipath, etc). Let $\widehat{P}$ be the longest dipath in the set $\{P_1, ..., P_n\}$, with length $|A(\widehat{P})| = \alpha\sqrt{n}$ for some $\alpha \geq \frac{1}{2}\big[\log_2 n\big]^{\log_{3/2} 2}$. We aim to build a sequence of subgraphs $H_0(:= \widehat{P}) \subset H_1 \subset H_2 \subset ...$, bound the order of each subgraph from below using the lexicographic minimality of path lengths, and conclude that if α is too large then some H_i contains too many vertices, thus producing an upper bound on α.

Let $H_0 = \widehat{P}$. Let H_ℓ, $\ell > 0$, be the union of all P_i satisfying both $|A(P_i)| \geq \alpha\sqrt{n}/2^\ell$ and $\{v_i, v_{i+1}\} \cap V(H_{\ell-1}) \neq \varnothing$. Let n_ℓ, m_ℓ, and k_ℓ be the number of vertices, edges, and dipaths P_i in H_ℓ. We have $n_0 = \alpha\sqrt{n} + 1$, $m_0 = \alpha\sqrt{n}$, $k_0 = 1$ and, by construction, $m_\ell \geq k_\ell m_0/2^\ell$ for all $\ell \geq 0$.

We may produce a lower bound for the size of each H_ℓ by our lexicographic minimality condition. We claim that every vertex of H_ℓ is either the start- or end-vertex of a dipath P_i of length at least $m_0/2^{\ell+1}$. Suppose, to the contrary, that some $v_i \in V(H_\ell)$ satisfies $|A(P_{i-1})|, |A(P_i)| < m_0/2^{\ell+1}$. Let P_j be a dipath in H_ℓ containing v_i, and let us denote the $v_j v_i$ (resp. $v_i v_{j+1}$) portion of this path by P_j^1 (resp. P_j^2). By removing P_i, P_{i+1}, and P_j from our set $\{P_1, ..., P_n\}$ and replacing them with P_j^1, P_j^2, and $P_i \cup P_{i+1}$, we have replaced a path of length $|A(P_j)|$ ($|A(P_j)| \geq m_0/2^\ell$) with paths all of length strictly less than $|A(P_j)|$, a contradiction. Therefore, $k_{\ell+1} \geq n_\ell/2$ for all $\ell \geq 0$, as every vertex in $V(H_\ell)$ is the start- or end-vertex of a dipath P_i in $H_{\ell+1}$, and each dipath P_i has only one start- and one end-vertex.

The graph H_ℓ can be decomposed into the edge-disjoint union of two graphs $H_{\ell,a}$ and $H_{\ell,e}$, where $H_{\ell,a}$ is acyclic with $p_\#(H_{\ell,a}) \leq k_\ell$ (as H_ℓ is the edge-disjoint union of k_ℓ paths) and $H_{\ell,e}$ is the vertex-disjoint union of minimally Eulerian graphs $H_{\ell,e}^{(1)}, ..., H_{\ell,e}^{(p_\ell)}$ for some p_ℓ (if the Eulerian graph $H_{\ell,e}^{(j)}$ is not minimal, neither is G). By Proposition 1.2, $H_{\ell,a}$ has at most $\sqrt{2k_\ell}\, n_\ell$ edges. By Proposition 1.3, $H_{\ell,e}$ has at most

$$\sum_{j=1}^{p_\ell} \left(\sqrt{2(n_\ell^{(j)} - 1)}\, n_\ell^{(j)} + n_\ell^{(j)} - 1 \right) \leq \sqrt{2(n_\ell - 1)}\, n_\ell + n_\ell - 1$$

edges, where $n_\ell^{(j)} := |V(H_{\ell,e}^{(j)})|$, $j = 1, ..., p_\ell$. Therefore,

$$m_\ell \leq \sqrt{2k_\ell}\, n_\ell + \sqrt{2(n_\ell - 1)}\, n_\ell + n_\ell - 1.$$

Combining this inequality with the bound $m_\ell \geq k_\ell m_0/2^\ell$, we have

$$k_\ell m_0/2^\ell \leq \sqrt{2k_\ell}\, n_\ell + \sqrt{2(n_\ell - 1)}\, n_\ell + n_\ell - 1. \tag{1}$$

Using Inequality (1), we produce a recursive lower bound on n_ℓ that gives an upper bound on α. In particular, we aim to show that

$$n_\ell \geq \left(\frac{n_{\ell-1} m_0}{5 \times 2^\ell} \right)^{2/3} \quad \text{for all } \ell \leq \log_2(5^2\alpha). \tag{2}$$

If $n_\ell \geq \sqrt{2k_\ell}\, m_0/2^\ell$, then Inequality (2) immediately holds, as

$$n_\ell \geq \frac{\sqrt{2k_\ell}\, m_0}{2^\ell} = \left[\left(\frac{n_{\ell-1} m_0}{5 \times 2^\ell} \right)^2 \left(\frac{(2k_\ell)^{3/2} n^{1/2}}{n_{\ell-1}^2} \right) \left(\frac{5^2\, \alpha}{2^\ell} \right) \right]^{1/3} \geq \left(\frac{n_{\ell-1} m_0}{5 \times 2^\ell} \right)^{2/3}$$

for $\alpha \geq 2^\ell/5^2$. Now, suppose that $n_\ell < \sqrt{2k_\ell}\, m_0/2^\ell$. Then $k_\ell m_0/2^\ell - \sqrt{2k_\ell} n_\ell$ is monotonically increasing with respect to k_ℓ. Combining this fact with the bound $k_\ell \geq n_{\ell-1}/2$ and Inequality (1), we obtain

$$n_{\ell-1} m_0/2^{\ell+1} - \sqrt{n_{\ell-1}}\, n_\ell \leq k_\ell m_0/2^\ell - \sqrt{2k_\ell}\, n_\ell \leq \sqrt{2(n_\ell - 1)}\, n_\ell + n_\ell - 1.$$

This implies that

$$n_{\ell-1} m_0 / 2^{\ell+1} \leq \sqrt{2(n_\ell - 1)}\, n_\ell + \sqrt{n_{\ell-1}}\, n_\ell + n_\ell - 1 < \frac{5}{2} n_\ell^{3/2},$$

for $n \geq 6388$, as $n_\ell \geq n_0 = \alpha\sqrt{n} + 1$, and so the claim holds in this case as well.

Using the initial bound $n_0 > m_0$ and Inequality (2), we obtain

$$\begin{aligned} n \geq n_\ell &\geq n_0^{(2/3)^\ell} \prod_{i=1}^{\ell} \left(\frac{m_0}{5 \times 2^{\ell+1-i}} \right)^{(2/3)^i} \\ &= \frac{n_0^{(2/3)^\ell}}{2^{2\ell}} \left(\frac{16 m_0^2}{25} \right)^{1-(2/3)^\ell} \\ &> \frac{16\, m_0^{2-(2/3)^\ell}}{25 \times 2^{2\ell}} \\ &= \frac{16\, \alpha^{2-(2/3)^\ell} n^{1-\frac{1}{2}(2/3)^\ell}}{25 \times 2^{2\ell}} \end{aligned}$$

for $\ell \leq \log_2(5^2\alpha)$. Taking the logarithm of both sides, we obtain the inequality

$$\log_2 \alpha < \frac{1}{2-(2/3)^\ell} \big(\log_2(25/16) + 2\ell + \tfrac{1}{2}(2/3)^\ell \log_2 n \big). \tag{3}$$

Setting $\ell = \lceil \log_{3/2} \big(\frac{3}{11} \log_2 n \big) \rceil$, we have $\ell < \log_2(5^2\alpha)$, as

$$\begin{aligned} \log_{3/2} \big(\tfrac{3}{11} \log_2 n \big) + 1 &= (\log_{3/2} 2) \log_2(\log_2 n) + \log_{3/2}(3/11) + 1 \\ &< (\log_{3/2} 2) \log_2(\log_2 n) + 2 \log_2(5) - 1 \\ &= \log_2 \big(\tfrac{5^2}{2} \log_2^{\log_{3/2} 2} n \big). \end{aligned}$$

For $\ell = \lceil \log_{3/2} \big(\frac{3}{11} \log_2 n \big) \rceil$, Inequality 3 implies that

$$\begin{aligned} \log_2 \alpha &< \frac{\log_2(25/16) + 2\big[\log_{3/2} \big(\frac{3}{11} \log_2 n \big) + 1 \big] + \frac{1}{2}(2/3)^{\log_{3/2} \left(\frac{3}{11} \log_2 n \right)} \log_2 n}{2 - (2/3)^{\log_{3/2} \left(\frac{3}{11} \log_2 n \right)}} \\ &= \frac{1}{1 - \frac{11}{6 \log_2 n}} \big[\log_2(5/2) + \log_{3/2} \big(\tfrac{3}{11} \log_2 n \big) + \tfrac{11}{12} \big]. \end{aligned}$$

Taking the (base two) exponential of both sides, we obtain

$$\alpha < 2^{\frac{\log_2(5/2) - \log_{3/2}(11/3) + 11/12}{1 - 11/(6 \log_2 6388)}} \big[\log_2 n \big]^{\frac{\log_{3/2} 2}{1 - 11/6 \log_2 6388}} \leq .46 \big[\log_2 n \big]^{1.9995}.$$

This completes the proof. $\square$

Finally, we give the following infinite class of digraphs to illustrate that Theorem 1.1 is tight up to a logarithmic factor.

Example 2.2. *Let* $G_k = (V_k, A_k)$, $k \in \mathbb{N}$, $k \geq 4$, *where* $V_k = \{u_1, ..., u_{\ell-1}, v_1, ..., v_\ell\}$, $\ell := k(k+1)/2$, *and* $u_i u_j \in A_k$ *for* $0 < j - i \leq k$, *and* $u_{\ell-\phi(i)} v_i, v_i u_{\phi(i)} \in A_k$ *for all* $i = 1, ..., \ell$, *where* $\phi(i)$ *is the smallest number* $p \in \mathbb{N}$ *such that* $\sum_{j=1}^{p} (k+1-j) \geq i$. *This digraph is minimally Eulerian, as every dicycle contains some vertex* v_i *and* $d^+(v_i) = d^-(v_i) = 1$ *for all* i. *There are* $n = k^2 + k - 1$ *vertices and* $k(k^2 + 2k - 1)/2$ *edges (i.e., about* $n^{3/2}/2$*). The distance between any pair* v_i, v_j *in the graph is at least*

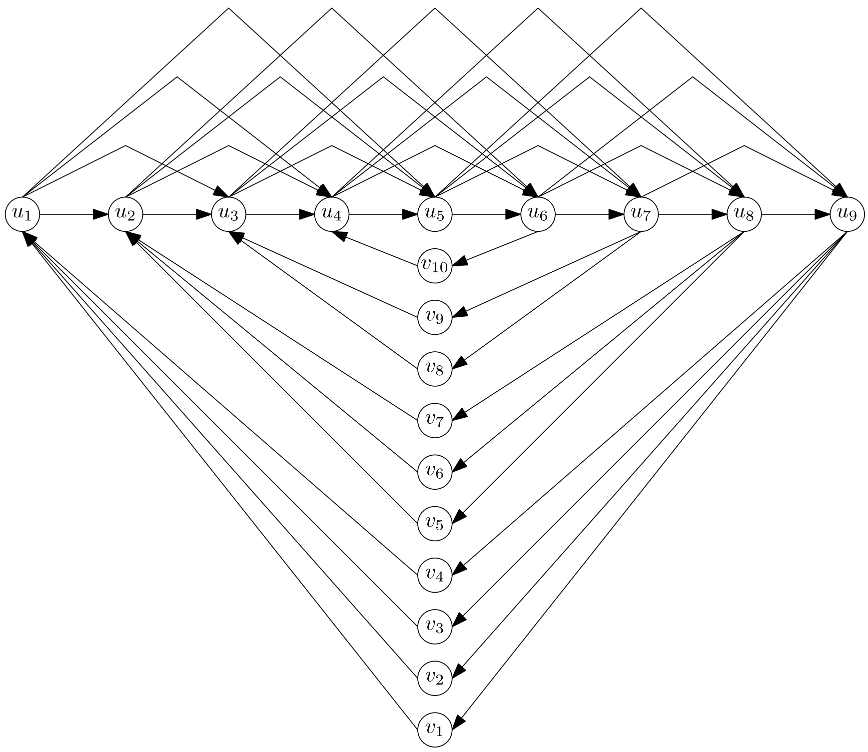

FIGURE 1. The minimally Eulerian graph G_k from Example 2.2 for $k = 4$.

$\lceil(\ell + 1)/k\rceil = \lceil k/2\rceil + 1 \geq \lfloor\sqrt{n}/2\rfloor + 1$. *In any Hamiltonian dicycle of a power of* G_k*, some pair* v_i, v_j *must be adjacent, and so at least the* $[\lfloor\sqrt{n}/2\rfloor + 1]^{th}$ *power is required. See Figure 1 for a visual example for* $k = 4$*.*

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