18.175: Lecture 24

Brownian motion

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Outline

Brownian motion properties and construction

Markov property, Blumenthal's 0-1 law

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- Hmm... does this mean we need to use a σ-algebra in which the event "B_t is continuous" is a measurable?
- Suppose Ω is set of all functions of t, and we use smallest σ-field that makes each B_t a measurable random variable... does that fail?

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- Brownian scaling: fix c, then B_{ct} agrees in law with $c^{1/2}B_t$.
- Another characterization: B is jointly Gaussian, $EB_s = 0$, $EB_sB_t = s \wedge t$, and $t \to B_t$ a.s. continuous.

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- Can prove Hölder continuity using similar estimates (see problem set).
- Can extend to higher dimensions: make each coordinate independent Brownian motion.

Continuity theorem

- **Kolmogorov continuity theorem:** Suppose $E|X_s - X_t|^\beta \leq K|t-s|^{1+\alpha}$ where $\alpha, \beta > 0$. If $\gamma < \alpha/\beta$ then with probability one there is a constant $C(\omega)$ so that $|X(q) - X(r)| \leq C|q-r|^\gamma$ for all $q, r \in \mathbb{Q}_2 \cap [0,1]$.

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- **Proof idea:** First look at values at all multiples of 2^{-0}, then at all multiples of 2^{-1}, then multiples of 2^{-2}, etc.
- At each stage we can draw a nice piecewise linear approximation of the process. How much does the approximation change in supremum norm (or some other Hölder norm) on the ith step? Can we say it probably doesn't change very much? Can we say the sequence of approximations is a.s. Cauchy in the appropriate normed spaced?

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- **Argument from Durrett (Pemantle):** Write

$$G_n = \{|X(i/2^n) - X((i-1)/2^n)|\} \le C|q-r|^\lambda \text{ for } 0 < i \le 2^n\}.$$

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- Chebyshev implies $P(|Y| > a) \le a^{-\beta} E|Y|^\beta$, so if $\lambda = \alpha - \beta\gamma > 0$ then

$$P(G_n^c) \le 2^n \cdot 2^{n\beta\gamma} \cdot E|X(j2^{-n})|^\beta = K2^{-n\lambda}.$$

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- Brownian motion is almost surely not differentiable.
- Brownian motion is almost surely not Lipschitz.
- Kolmogorov-Centsov theorem applies to higher dimensions (with adjusted exponents). One can construct a.s. continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$.

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- Write $\mathcal{F}_s^o = \sigma(B_r : r \leq s)$.
- Write $\mathcal{F}_s^+ = \cap_{t>s} \mathcal{F}_t^o$
- Note right continuity: $\cap_{t>s} \mathcal{F}_t^+ = \mathcal{F}_s^+$.
- $\mathcal{F}_s^+$ allows an "infinitesimal peek at future"

Markov property

- If $s \geq 0$ and Y is bounded and $\mathcal{C}$-measurable, then for all $x \in \mathbb{R}^d$, we have

$$E_x(Y \circ \theta_s | \mathcal{F}_s^+) = E_{B_s} Y,$$

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- **Proof idea:** First establish this for some simple functions Y (depending on finitely many time values) and then use measure theory (monotone class theorem) to extend to general case.

Looking ahead

- **Expectation equivalence theorem** If Z is bounded and measurable then for all $s \geq 0$ and $x \in \mathbb{R}^d$ have

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- **Proof idea:** Consider case that $Z = \sum_{i=1}^m f_m(B(t_m))$ and the f_m are bounded and measurable. Kind of obvious in this case. Then use same measure theory as in Markov property proof to extend general Z.

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- **Observe:** If $Z \in \mathcal{F}_s^+$ then $Z = E_x(Z|\mathcal{F}_s^o)$. Conclude that $\mathcal{F}_s^+$ and $\mathcal{F}_s^o$ agree up to null sets.

Blumenthal's 0-1 law

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- There's nothing you can learn from infinitesimal neighborhood of future.
- **Proof:** If we have $A \in \mathcal{F}_0^+$, then previous theorem implies

$$1_A = E_x(1_A | \mathcal{F}_0^+) = E_x(1_A | \mathcal{F}_0^o) = P_x(A) \quad P_x \mathrm{a.s.}$$

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- If $T_0 = \inf\{t > 0 : B_t = 0\}$ then $P_0(T_0 = 0) = 1$.
- If B_t is Brownian motion started at 0, then so is process defined by $X_0 = 0$ and $X_t = tB(1/t)$. (Proved by checking $E(X_s X_t) = stE(B(1/s)B(1/t)) = s$ when $s < t$. Then check continuity at zero.)

Continuous martingales

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- Do they all kind of look like Brownian motion?