18.175: Lecture 13

Infinite divisibility and Lévy processes

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Outline

Poisson random variable convergence

Extend CLT idea to stable random variables

Infinite divisibility

Higher dimensional CFs and CLTs

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- **Key idea for all these examples:** Divide time into large number of small increments. Assume that during each increment, there is some small probability of thing happening (independently of other increments).

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- Use Taylor expansion $e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$.

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- Then $\mathrm{Var}[X] = E[X^2] - E[X]^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$.

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- **Theorem:** Let $X_{n,m}$ be independent $\{0,1\}$-valued random variables with $P(X_{n,m}=1)=p_{n,m}$. Suppose $\sum_{m=1}^n p_{n,m} \to \lambda$ and $\max_{1 \le m \le n} p_{n,m} \to 0$. Then $S_n = X_{n,1} + \ldots + X_{n,n} \implies Z$ were Z is Poisson(λ).

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- **Proof idea:** Just write down the log characteristic functions for Bernoulli and Poisson random variables. Check the conditions of the continuity theorem.

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Recall continuity theorem

- **Strong continuity theorem:** If $\mu_n \implies \mu_\infty$ then $\phi_n(t) \to \phi_\infty(t)$ for all t. Conversely, if $\phi_n(t)$ converges to a limit that is continuous at 0, then the associated sequence of distributions μ_n is tight and converges weakly to a measure μ with characteristic function ϕ.

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- Let's look up stable distributions. Up to affine transformations, this is just a two-parameter family with characteristic functions $\exp[-|t|^\alpha(1 - i\beta\mathrm{sgn}(t)\Phi)]$ where $\Phi = \tan(\pi\alpha/2)$ where $\beta \in [-1, 1]$ and $\alpha \in (0, 2]$.

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- More generally $\{m \leq n : X_m/n^{1/\alpha} \in (a,b)\}$ converges in law to Poisson with mean $\int_A \frac{\alpha}{2|x|^{\alpha+1}} dx < \infty$.

Domain of attraction to stable random variable

- More generality: suppose that $\lim_{x\to\infty} P(X_1 > x)/P(|X_1| > x) = \theta \in [0,1]$ and $P(|X_1| > x) = x^{-\alpha} L(x)$ where L is *slowly varying* (which means $\lim_{x\to\infty} L(tx)/L(x) = 1$ for all $t > 0$).

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- **Theorem:** Then $(S_n - b_n)/a_n$ converges in law to limiting random variable, for appropriate a_n and b_n values.

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- More general constructions are possible via Lévy Khintchine representation.

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- The inversion theorems and continuity theorems that apply here are essentially the same as in the one-dimensional case.