# 18.175: Lecture 6

# Laws of large numbers and independence

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# Outline

Definitions

Background results

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# Recall expectation definition

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- Expectation is always defined if $X \geq 0$ a.s., or if integrals of $\max\{X, 0\}$ and $\min\{X, 0\}$ are separately finite.

# Strong law of large numbers

- **Theorem (strong law):** If $X_1, X_2, \ldots$ are i.i.d. real-valued random variables with expectation $m$ and $A_n := n^{-1} \sum_{i=1}^n X_i$ are the *empirical means* then $\lim_{n\to\infty} A_n = m$ almost surely.

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- What does i.i.d. mean?
- Answer: independent and identically distributed.
- Okay, but what does independent mean in this context? And how do you even define an infinite sequence of independent random variables? Is that even possible? It's kind of an empty theorem if it turns out that the hypotheses are never satisfied. And by the way, what measure space and $\sigma$-algebra are we using? And is the event that the limit exists even measurable in this $\sigma$-algebra? Because if it's not, what does it mean to say it has probability one? Also, why do they call it the strong law? Is there also a weak law?

# Independence of two events/random variables/$\sigma$-algebras

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- Random variables $X$ and $Y$ are independent if for all $C, D \in \mathcal{R}$, we have $P(X \in C, Y \in D) = P(X \in C)P(Y \in D)$, i.e., the events $\{X \in C\}$ and $\{Y \in D\}$ are independent.

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- Two $\sigma$-fields $\mathcal{F}$ and $\mathcal{G}$ are independent if $A$ and $B$ are independent whenever $A \in \mathcal{F}$ and $B \in \mathcal{G}$. (This definition also makes sense if $\mathcal{F}$ and $\mathcal{G}$ are arbitrary algebras, semi-algebras, or other collections of measurable sets.)

# Independence of multiple events/random variables/$\sigma$-algebras

- Say events $A_1, A_2, \ldots, A_n$ are independent if for each $I \subset \{1, 2, \ldots, n\}$ we have $P(\cap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$.

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- Say random variables $X_1, X_2, \ldots, X_n$ are independent if for any measurable sets $B_1, B_2, \ldots, B_n$, the events that $X_i \in B_i$ are independent.
- Say $\sigma$-algebras $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ if any collection of events (one from each $\sigma$-algebra) are independent. (This definition also makes sense if the $\mathcal{F}_i$ are algebras, semi-algebras, or other collections of measurable sets.)

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# Extending to $\sigma$-algebras

- **Theorem:** If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are independent, and each $\mathcal{A}_i$ is a $\pi$-system, then $\sigma(\mathcal{A}_1), \ldots, \sigma(\mathcal{A}_n)$ are independent.

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- **Main idea of proof:** Apply the $\pi$-$\lambda$ theorem.

# Kolmogorov's Extension Theorem

- **Task: make sense of this statement.** Let $\Omega$ be the set of all countable sequences $\omega = (\omega_1, \omega_2, \omega_3 \ldots)$ of real numbers. Let $\mathcal{F}$ be the smallest $\sigma$-algebra that makes the maps $\omega \to \omega_i$ measurable. Let $P$ be the probability measure that makes the $\omega_i$ independent identically distributed normals with mean zero, variance one.

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- The $\mathcal{F}$ described above is the natural product $\sigma$-algebra: smallest $\sigma$-algebra generated by the "finite dimensional rectangles" of form $\{\omega : \omega_i \in (a_i, b_i], 1 \le i \le n\}$.

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- We could also ask about i.i.d. sequences of coin tosses or i.i.d. samples from some other space.
- The $\mathcal{F}$ described above is the natural product $\sigma$-algebra: smallest $\sigma$-algebra generated by the "finite dimensional rectangles" of form $\{\omega : \omega_i \in (a_i, b_i], 1 \leq i \leq n\}$.
- Question: what things are in this $\sigma$-algebra? How about the event that the $\omega_i$ converge to a limit?

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- Proved using semi-algebra variant of Carathéeodory's extension theorem.