18.175: Lecture 35

Ergodic theory

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Outline

Recall setup

Birkhoff's ergodic theorem

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- **Example:** If $\Omega = \mathbb{R}^{\{0,1,\ldots\}}$ and A is invariant, then A is necessarily in tail σ-field $\mathcal{T}$, hence has probability zero or one by Kolmogorov's $0 - 1$ law. So sequence is ergodic (the shift on sequence space $\mathbb{R}^{\{0,1,2,\ldots\}}$ is ergodic.

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- **Other examples:** What about fair coin toss ($\Omega = \{H, T\}$) with $\phi(H) = T$ and $\phi(T) = H$? What about stationary Markov chain sequences?

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- There's this lemma: let A_k be the event the maximum M_k of X_0 and $X_0 + X_1$ up to $X_1 + \ldots + X_{k-1}$ is non-negative. Then $EX_0 1_{A_k} \geq 0$ is non-negative.

Benford's law

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- Does ergodic theorem kind of give a mathematical framework for this law?