18.175: Lecture 34

Ergodic theory

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Outline

Recall setup

Birkhoff's ergodic theorem

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Motivating problem

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- We don't have independence. We have translation invariance instead. Is that good enough?
- More general: C_x distributed in *some* translation invariant way, $EC_0 < \infty$. Is mean of C_x (on large box) nearly constant?

Rephrasing problem

- Let θ_x be the translation of the $\mathbb{Z}^2$ that moves 0 to x. Each θ_x induces a measure-preserving translation of Ω. Then $C_x(\omega) = C_0(\theta_{-x}(\omega))$. So summing up the C_x values is the same as summing up the $C_0(\theta_x(\omega))$ value over a range of x.

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- Let's simplify matters still further and consider the one-dimensional problem. In this case, we have a random variable X and we study empirical averages of the form

$$N^{-1} \sum_{n=1}^{N} X(\phi^n \omega).$$

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- If $X_0, X_1, \ldots$ is stationary and $g : \mathbb{R}^{\{0,1,\ldots\}} \to \mathbb{R}$ is measurable, then $Y_k = g(X_k, X_{k+1}, \ldots)$ is stationary.

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- What if X_i are i.i.d. tosses of a p-coin, where p is itself random?

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- **Example:** If $\Omega = \mathbb{R}^{\{0,1,\dots\}}$ and A is invariant, then A is necessarily in tail σ-field $\mathcal{T}$, hence has probability zero or one by Kolmogorov's $0-1$ law. So sequence is ergodic (the shift on sequence space $\mathbb{R}^{\{0,1,2,\dots\}}$ is ergodic.

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- **Other examples:** What about fair coin toss ($\Omega = \{H, T\}$) with $\phi(H) = T$ and $\phi(T) = H$? What about stationary Markov chain sequences?

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- There's this lemma: let A_k be the event the maximum M_k of X_0 and $X_0 + X_1$ up to $X_1 + \ldots + X_{k-1}$ is non-negative. Then $EX_0 1_{A_k} \geq 0$ is non-negative.

Benford's law

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- Does ergodic theorem kind of give a mathematical framework for this law?