18.175: Lecture 29

Still more martingales

Scott Sheffield

MIT

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- **Martingale convergence:** A non-negative martingale almost surely has a limit. Under some conditions (what conditions?) the expectation of the limit is the initial value of the martingale.

Problems

- **Classic brainteaser:** 52 cards (half red, half black) shuffled and face down. I turn them over one at a time. At some point (before last card is turned over) you say "stop". If subsequent card is red, you get one dollar. You do you time your stop to maximize your probability of winning?

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- **Classic question:** Is this also true of the stock market?

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- What's the probability that Cassandra will win? (Give the full range of possibilities.)

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- **Cond. variance theorem:** If X_n is martingale, $EX_n^2 < \infty$ for all n, then $E\big((X_n - X_m)^2|\mathcal{F}_m\big) = E(X_n^2|\mathcal{F}_m) - X_m^2$.

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- **"Accumulated variance" theorems:** Consider martingale X_n with $EX_n^2 < \infty$ for all n. By Doob, can write $X_n^2 = M_n + A_n$ where M_n is a martingale, and

$$A_n = \sum_{m=1}^{n} E(X_m^2|\mathcal{F}_{m-1}) - X_{m-1}^2 = \sum_{m=1}^{n} E\big((X_m - X_{m-1})^2|\mathcal{F}_{m-1}\big).$$

  Then $E(\sup_m |X_m|^2) \le 4EA_\infty$. And $\lim_{n\to\infty} X_n$ exists and is finite a.s. on $\{A_\infty < \infty\}$.

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- **Proof idea:** They all amount to controlling "contribution to expectation from values near infinity".

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- **Proof idea:** First implies second: uniform integrability implies $\sup E|X_n| < \infty$, martingale convergence then implies $X_n \to X$ a.s., and previous result implies $X_n \to X$ in probability. Easier to see second implies third, third implies first.

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  - There is an integrable random variable X so that $X_n = E(X|\mathcal{F}_n)$.
  - In other words, every uniformly integrable martingale can be interpreted as a "revised expectation given latest information" sequence.

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- **Theorem:** $X_{-\infty} = \lim_{n\to-\infty} X_n$ exists a.s. and in L^1.
- **Proof idea:** Use upcrosing inequality to show expected number of upcrossings of any interval is finite. Since $X_n = E(X_0|\mathcal{F}_n)$ the X_n are uniformly integrable, and we can deduce convergence in L^1.

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- **Theorem:** For any stopping time $N \leq \infty$, we have $EX_0 \leq EX_N \leq EX_\infty$ where $X_\infty = \lim X_n$.
- **Fairly general form of optional stopping theorem:** If $L \leq M$ are stopping times and $Y_{M\wedge n}$ is a uniformly integrable submartingale, then $EY_L \leq EY_M$ and $Y_L \leq E(Y_M|\mathcal{F}_L)$.