18.175: Lecture 25

Reflections and martingales

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Outline

Conditional expectation

Martingales

Arcsin law, other SRW stories

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Conditional expectation

- Say we're given a probability space $(\Omega, \mathcal{F}_0, P)$ and a σ-field $\mathcal{F} \subset \mathcal{F}_0$ and a random variable X measurable w.r.t. $\mathcal{F}_0$, with $E|X| < \infty$. The **conditional expectation of X given $\mathcal{F}$** is a new random variable, which we can denote by $Y = E(X|\mathcal{F})$.

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- Any Y satisfying these properties is called a **version** of $E(X|\mathcal{F})$.
- Is it possible that there exists more than one version of $E(X|\mathcal{F})$ (which would mean that in some sense the conditional expectation is not canonically defined)?
- Is there some sense in which $E(X|\mathcal{F})$ always exists and is always uniquely defined (maybe up to set of measure zero)?

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- **Proof:** let $A = \{Y > 0\} \in \mathcal{F}$ and observe: $\int_A YdP \int_A XdP \leq \int_A |X|dP$. By similarly argument, $\int_{A^c} -YdP \leq \int_{A^c} |X|dP$.

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- **Uniqueness of** Y: Suppose Y' is $\mathcal{F}$-measurable and satisfies $\int_A Y'dP = \int_A XdP = \int_A YdP$ for all $A \in \mathcal{F}$. Then consider the set $Y - Y' \geq \epsilon\}$. Integrating over that gives zero. Must hold for any ϵ. Conclude that $Y = Y'$ almost everywhere.

Radon-Nikodym theorem

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- Recall **Radon-Nikodym theorem:** If μ and ν are σ-finite measures on $(\Omega, \mathcal{F})$ and ν is absolutely continuous w.r.t. μ, then there exists a measurable $f : \Omega \to [0, \infty)$ such that $\nu(A) = \int_A f d\mu$.

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- Observe: this theorem implies existence of conditional expectation.

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Two big results

- **Optional stopping theorem:** Can't make money in expectation by timing sale of asset whose price is non-negative martingale.

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- **Martingale convergence:** A non-negative martingale almost surely has a limit.

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- **Wald's second equation:** Let X_i be i.i.d. with $E|X_i| = 0$ and $EX_i^2 = \sigma^2 < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = \sigma^2 EN$.

Wald applications to SRW

- $S_0 = a \in \mathbb{Z}$ and at each time step S_j independently changes by ± 1 according to a fair coin toss. Fix $A \in \mathbb{Z}$ and let $N = \inf\{k : S_k \in \{0, A\}$. What is $\mathbb{E}S_N$?

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Reflection principle

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- How many walks from $(0, x)$ to (n, y) that don't cross the horizontal axis?
- Try counting walks that *do* cross by giving bijection to walks from $(0, -x)$ to (n, y).

Ballot Theorem

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- Answer: $(\alpha - \beta)/(\alpha + \beta)$. Can be proved using reflection principle.

Arcsin theorem

- Theorem for last hitting time.

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- Theorem for amount of positive positive time.