18.175: Lecture 18

Poisson random variables

Scott Sheffield

MIT

Outline

Extend CLT idea to stable random variables

Outline

Extend CLT idea to stable random variables

Recall continuity theorem

- **Strong continuity theorem:** If $\mu_n \implies \mu_\infty$ then $\phi_n(t) \to \phi_\infty(t)$ for all t. Conversely, if $\phi_n(t)$ converges to a limit that is continuous at 0, then the associated sequence of distributions μ_n is tight and converges weakly to a measure μ with characteristic function ϕ.

Recall CLT idea

- Let X be a random variable.

Recall CLT idea

- Let X be a random variable.
- The **characteristic function** of X is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$.

Recall CLT idea

- Let X be a random variable.
- The **characteristic function** of X is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$.
- And if X has an mth moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.

Recall CLT idea

- Let X be a random variable.
- The **characteristic function** of X is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$.
- And if X has an mth moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
- In particular, if $E[X] = 0$ and $E[X^2] = 1$ then $\phi_X(0) = 1$ and $\phi_X'(0) = 0$ and $\phi_X''(0) = -1$.

Recall CLT idea

- Let X be a random variable.
- The **characteristic function** of X is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$.
- And if X has an mth moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
- In particular, if $E[X] = 0$ and $E[X^2] = 1$ then $\phi_X(0) = 1$ and $\phi_X'(0) = 0$ and $\phi_X''(0) = -1$.
- Write $L_X := -\log \phi_X$. Then $L_X(0) = 0$ and $L_X'(0) = -\phi_X'(0)/\phi_X(0) = 0$ and $L_X'' = -(\phi_X''(0)\phi_X(0) - \phi_X'(0)^2)/\ \phi_X(0)^2 = 1$.

Recall CLT idea

- Let X be a random variable.
- The **characteristic function** of X is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$.
- And if X has an mth moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
- In particular, if $E[X] = 0$ and $E[X^2] = 1$ then $\phi_X(0) = 1$ and $\phi_X'(0) = 0$ and $\phi_X''(0) = -1$.
- Write $L_X := -\log \phi_X$. Then $L_X(0) = 0$ and $L_X'(0) = -\phi_X'(0)/\phi_X(0) = 0$ and $L_X'' = -(\phi_X''(0)\phi_X(0) - \phi_X'(0)^2)/\ \phi_X(0)^2 = 1$.
- If $V_n = n^{-1/2}\sum_{i=1}^n X_i$ where X_i are i.i.d. with law of X, then $L_{V_n}(t) = nL_X(n^{-1/2}t)$.

Recall CLT idea

- Let X be a random variable.
- The **characteristic function** of X is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$.
- And if X has an mth moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
- In particular, if $E[X] = 0$ and $E[X^2] = 1$ then $\phi_X(0) = 1$ and $\phi_X'(0) = 0$ and $\phi_X''(0) = -1$.
- Write $L_X := -\log \phi_X$. Then $L_X(0) = 0$ and $L_X'(0) = -\phi_X'(0)/\phi_X(0) = 0$ and $L_X'' = -(\phi_X''(0)\phi_X(0) - \phi_X'(0)^2)/\ \phi_X(0)^2 = 1$.
- If $V_n = n^{-1/2} \sum_{i=1}^n X_i$ where X_i are i.i.d. with law of X, then $L_{V_n}(t) = nL_X(n^{-1/2}t)$.
- When we zoom in on a twice differentiable function near zero (scaling vertically by n and horizontally by $\sqrt{n}$) the picture looks increasingly like a parabola.

Stable laws

- Question? Is it possible for something like a CLT to hold if X has infinite variance? Say we write $V_n = n^{-a} \sum_{i=1}^n X_i$ for some a. Could the law of these guys converge to something non-Gaussian?

Stable laws

- Question? Is it possible for something like a CLT to hold if X has infinite variance? Say we write $V_n = n^{-a} \sum_{i=1}^{n} X_i$ for some a. Could the law of these guys converge to something non-Gaussian?
- What if the L_{V_n} converge to something else as we increase n, maybe to some other power of $|t|$ instead of $|t|^2$?

Stable laws

- Question? Is it possible for something like a CLT to hold if X has infinite variance? Say we write $V_n = n^{-a}\sum_{i=1}^n X_i$ for some a. Could the law of these guys converge to something non-Gaussian?
- What if the L_{V_n} converge to something else as we increase n, maybe to some other power of $|t|$ instead of $|t|^2$?
- The the appropriately normalized sum should be converge in law to something with characteristic function $e^{-|t|^\alpha}$ instead of $e^{-|t|^2}$.

Stable laws

- Question? Is it possible for something like a CLT to hold if X has infinite variance? Say we write $V_n = n^{-a}\sum_{i=1}^n X_i$ for some a. Could the law of these guys converge to something non-Gaussian?
- What if the L_{V_n} converge to something else as we increase n, maybe to some other power of $|t|$ instead of $|t|^2$?
- The the appropriately normalized sum should be converge in law to something with characteristic function $e^{-|t|^\alpha}$ instead of $e^{-|t|^2}$.
- We already saw that this should work for Cauchy random variables.

Stable laws

- Example: Suppose that $P(X_1 > x) = P(X_1 < -x) = x^{-\alpha}/2$ for $0 < \alpha < 2$. This is a random variable with a "power law tail".

Stable laws

- Example: Suppose that $P(X_1 > x) = P(X_1 < -x) = x^{-\alpha}/2$ for $0 < \alpha < 2$. This is a random variable with a "power law tail".
- Compute $1 - \phi(t) \approx C|t|^\alpha$ when $|t|$ is large.

Stable laws

- Example: Suppose that $P(X_1 > x) = P(X_1 < -x) = x^{-\alpha}/2$ for $0 < \alpha < 2$. This is a random variable with a "power law tail".
- Compute $1 - \phi(t) \approx C|t|^\alpha$ when $|t|$ is large.
- If $X_1, X_2, \ldots$ have same law as X_1 then we have $E\exp(itS_n/n^{1/\alpha}) = \phi(t/n^\alpha)^n = \big(1 - (1 - \phi(t/n^{1/\alpha}))\big)$. As $n \to \infty$, this converges pointwise to $\exp(-C|t|^\alpha)$.

Stable laws

- Example: Suppose that $P(X_1 > x) = P(X_1 < -x) = x^{-\alpha}/2$ for $0 < \alpha < 2$. This is a random variable with a "power law tail".
- Compute $1 - \phi(t) \approx C|t|^\alpha$ when $|t|$ is large.
- If $X_1, X_2, \dots$ have same law as X_1 then we have $E\exp(itS_n/n^{1/\alpha}) = \phi(t/n^\alpha)^n = \big(1 - (1 - \phi(t/n^{1/\alpha}))\big)$. As $n \to \infty$, this converges pointwise to $\exp(-C|t|^\alpha)$.
- Conclude by continuity theorems that $X_n/n^{1/\alpha} \implies Y$ where Y is a random variable with $\phi_Y(t) = \exp(-C|t|^\alpha)$

Stable laws

- Example: Suppose that $P(X_1 > x) = P(X_1 < -x) = x^{-\alpha}/2$ for $0 < \alpha < 2$. This is a random variable with a "power law tail".
- Compute $1 - \phi(t) \approx C|t|^\alpha$ when $|t|$ is large.
- If $X_1, X_2, \ldots$ have same law as X_1 then we have $E\exp(itS_n/n^{1/\alpha}) = \phi(t/n^\alpha)^n = \big(1 - (1 - \phi(t/n^{1/\alpha}))\big)$. As $n \to \infty$, this converges pointwise to $\exp(-C|t|^\alpha)$.
- Conclude by continuity theorems that $X_n/n^{1/\alpha} \implies Y$ where Y is a random variable with $\phi_Y(t) = \exp(-C|t|^\alpha)$
- Let's look up stable distributions. Up to affine transformations, this is just a two-parameter family with characteristic functions $\exp[-|t|^\alpha(1 - i\beta\mathrm{sgn}(t)\Phi)]$ where $\Phi = \tan(\pi\alpha/2)$ where $\beta \in [-1, 1]$ and $\alpha \in (0, 2]$.

Stable-Poisson connection

- Let's think some more about this example, where $P(X_1 > x) = P(X_1 < -x) = x^{-\alpha}/2$ for $0 < \alpha < 2$ and $X_1, X_2, \ldots$ are i.i.d.

Stable-Poisson connection

- Let's think some more about this example, where $P(X_1 > x) = P(X_1 < -x) = x^{-\alpha}/2$ for $0 < \alpha < 2$ and $X_1, X_2, \ldots$ are i.i.d.
- Now $P(an^{1/\alpha} < X_1 < bn^{1\alpha} = \frac{1}{2}(a^{-\alpha} - b^{-\alpha})n^{-1}$.

Stable-Poisson connection

- Let's think some more about this example, where $P(X_1 > x) = P(X_1 < -x) = x^{-\alpha}/2$ for $0 < \alpha < 2$ and $X_1, X_2, \ldots$ are i.i.d.
- Now $P(an^{1/\alpha} < X_1 < bn^{1\alpha} = \frac{1}{2}(a^{-\alpha} - b^{-\alpha})n^{-1}$.
- So $\{m \le n : X_m/n^{1/\alpha} \in (a, b)\}$ converges to a Poisson distribution with mean $(a^{-\alpha} - b^{-\alpha})/2$.

Stable-Poisson connection

- Let's think some more about this example, where $P(X_1 > x) = P(X_1 < -x) = x^{-\alpha}/2$ for $0 < \alpha < 2$ and $X_1, X_2, \ldots$ are i.i.d.
- Now $P(an^{1/\alpha} < X_1 < bn^{1\alpha} = \frac{1}{2}(a^{-\alpha} - b^{-\alpha})n^{-1}$.
- So $\{m \leq n : X_m/n^{1/\alpha} \in (a,b)\}$ converges to a Poisson distribution with mean $(a^{-\alpha} - b^{-\alpha})/2$.
- More generally $\{m \leq n : X_m/n^{1/\alpha} \in (a,b)\}$ converges in law to Poisson with mean $\int_A \frac{\alpha}{2|x|^{\alpha+1}} dx < \infty$.

Domain of attraction to stable random variable

- More generality: suppose that $\lim_{x\to\infty} P(X_1 > x)/P(|X_1| > x) = \theta \in [0,1]$ and $P(|X_1| > x) = x^{-\alpha}L(x)$ where L is *slowly varying* (which means $\lim_{x\to\infty} L(tx)/L(x) = 1$ for all $t > 0$).

Domain of attraction to stable random variable

- More generality: suppose that $\lim_{x\to\infty} P(X_1 > x)/P(|X_1| > x) = \theta \in [0,1]$ and $P(|X_1| > x) = x^{-\alpha}L(x)$ where L is *slowly varying* (which means $\lim_{x\to\infty} L(tx)/L(x) = 1$ for all $t > 0$).
- **Theorem:** Then $(S_n - b_n)/a_n$ converges in law to limiting random variable, for appropriate a_n and b_n values.

Infinitely divisible laws

- Say a random variable X is **infinitely divisible**, for each n, there is a random variable Y such that X has the same law as the sum of n i.i.d. copies of Y.

Infinitely divisible laws

- Say a random variable X is **infinitely divisible**, for each n, there is a random variable Y such that X has the same law as the sum of n i.i.d. copies of Y.
- What random variables are infinitely divisible?

Infinitely divisible laws

- Say a random variable X is **infinitely divisible**, for each n, there is a random variable Y such that X has the same law as the sum of n i.i.d. copies of Y.
- What random variables are infinitely divisible?
- Poisson, Cauchy, normal, stable, etc.

Infinitely divisible laws

- Say a random variable X is **infinitely divisible**, for each n, there is a random variable Y such that X has the same law as the sum of n i.i.d. copies of Y.
- What random variables are infinitely divisible?
- Poisson, Cauchy, normal, stable, etc.
- Let's look at the characteristic functions of these objects. What about compound Poisson random variables (linear combinations of Poisson random variables)? What are their characteristic functions like?

Infinitely divisible laws

- Say a random variable X is **infinitely divisible**, for each n, there is a random variable Y such that X has the same law as the sum of n i.i.d. copies of Y.
- What random variables are infinitely divisible?
- Poisson, Cauchy, normal, stable, etc.
- Let's look at the characteristic functions of these objects. What about compound Poisson random variables (linear combinations of Poisson random variables)? What are their characteristic functions like?
- More general constructions are possible via Lévy Khintchine representation.

Higher dimensional limit theorems

- Much of the CLT story generalizes to higher dimensional random variables.

Higher dimensional limit theorems

- Much of the CLT story generalizes to higher dimensional random variables.
- For example, given a random vector (X, Y, Z), we can define $\phi(a, b, c) = Ee^{i(aX+bY+cZ)}$.

Higher dimensional limit theorems

- Much of the CLT story generalizes to higher dimensional random variables.
- For example, given a random vector (X, Y, Z), we can define $\phi(a, b, c) = Ee^{i(aX+bY+cZ)}$.
- This is just a higher dimensional Fourier transform of the density function.

Higher dimensional limit theorems

- Much of the CLT story generalizes to higher dimensional random variables.
- For example, given a random vector (X, Y, Z), we can define $\phi(a, b, c) = Ee^{i(aX+bY+cZ)}$.
- This is just a higher dimensional Fourier transform of the density function.
- The inversion theorems and continuity theorems that apply here are essentially the same as in the one-dimensional case.