18.175: Lecture 15

Characteristic functions and central limit theorem

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Outline

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- Characteristic functions are well defined at all t for all random variables X.

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- $Ee^{it(aX+b)} = e^{itb}\phi(at)$

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- **Bilateral exponential:** if $f_X(t) = e^{-|x|}/2$ on $\mathbb{R}$ then $\phi_X(t) = 1/(1 + t^2)$. Use linearity of $f_X \to \phi_X$.

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- **Convolution theorem:** If

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- **Possible application?**

$$\int 1_{[a,b]}(x)f(x)dx = (\widehat{1_{[a,b]}f})(0) = (\hat{f} * \widehat{1_{[a,b]}})(0) = \int \hat{f}(t)\widehat{1_{[a,b]}}(-t)dx.$$

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- Observe that $\frac{e^{-ita}-e^{-itb}}{it} = \int_a^b e^{-ity}dy$ has modulus bounded by $b-a$.
- That means we can use Fubini to compute I_T.

Bochner's theorem

- Given any function ϕ and any points $x_1, \ldots, x_n$, we can consider the matrix with i, j entry given by $\phi(x_i - x_j)$. Call ϕ **positive definite** if this matrix is always positive semidefinite Hermitian.

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- Positive definiteness kind of comes from fact that variances of random variables are non-negative.
- The set of all possible characteristic functions is a pretty nice set.

Continuity theorems

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- **Proof ideas:** First statement easy (since $X_n \implies X$ implies $Eg(X_n) \to Eg(X)$ for any bounded continuous g). To get second statement, first play around with Fubini and establish tightness of the μ_n. Then note that any subsequential limit of the μ_n must be equal to μ. Use this to argue that $\int f d\mu_n$ converges to $\int f d\mu$ for every bounded continuous f.

Moments, derivatives, CLT

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- This and the continuity theorem together imply the central limit theorem.
- **Theorem:** Let $X_1, X_2, \ldots$ by i.i.d. with $EX_i = \mu$, $\mathrm{Var}(X_i) = \sigma^2 \in (0, \infty)$. If $S_n = X_1 + \ldots + X_n$ then $(S_n - n\mu)/(\sigma n^{1/2})$ converges in law to a standard normal.