18.175: Lecture 13

More large deviations

Scott Sheffield

MIT

Outline

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- What's the higher dimensional analog of rolling the tangent line?

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- If X takes both positive and negative values with positive probability then $M(t)$ grows at least exponentially fast in $|t|$ as $|t| \to \infty$.

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- Answer: M_X^n.

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- Kind of a quantitative form of the weak law of large numbers. The empirical average A_n is *very* unlikely to ϵ away from its expected value (where "very" means with probability less than some exponentially decaying function of n).

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- Replace $I(x)$ by $I(x) - (\lambda, x)$? What is $\inf_x I(x) - (\lambda, x)$?

Cramer's theorem

- Let μ_n be law of empirical mean $A_n = \frac{1}{n}\sum_{j=1}^n X_j$ for i.i.d. vectors $X_1, X_2, \ldots, X_n$ in $\mathbb{R}^d$ with same law as X.

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- General Γ: cut into finitely many pieces, bound each piece?

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- But by how much did we have to modify the measure to make this typical? Not more than by factor $e^{-n \inf_{x \in \Gamma^0} I(x)}$.