18.175: Lecture 10

Zero-one laws and maximal inequalities

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Outline

Recollections

Kolmogorov zero-one law and three-series theorem

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Kolmogorov zero-one law and three-series theorem

Recall Borel-Cantelli lemmas

- **First Borel-Cantelli lemma:** If $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P(A_n \text{ i.o.}) = 0$.

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- **First Borel-Cantelli lemma:** If $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P(A_n \text{ i.o.}) = 0$.
- **Second Borel-Cantelli lemma:** If A_n are independent, then $\sum_{n=1}^{\infty} P(A_n) = \infty$ implies $P(A_n \text{ i.o.}) = 1$.

Recall strong law of large numbers

- **Theorem (strong law):** If $X_1, X_2, \dots$ are i.i.d. real-valued random variables with expectation m and $A_n := n^{-1}\sum_{i=1}^n X_i$ are the *empirical means* then $\lim_{n\to\infty} A_n = m$ almost surely.

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Kolmogorov zero-one law

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- Recall theorem that if $\mathcal{A}_i$ are independent π-systems, then σA_i are independent.
- Deduce that $\sigma(X_1, X_2, \dots, X_n)$ and $\sigma(X_{n+1}, X_{n+1}, \dots)$ are independent. Then deduce that $\sigma(X_1, X_2, \dots)$ and $\mathcal{T}$ are independent, using fact that $\cup_k \sigma(X_1, \dots, X_k)$ and $\mathcal{T}$ are π-systems.

Kolmogorov maximal inequality

- **Thoerem:** Suppose X_i are independent with mean zero and finite variances, and $S_n = \sum_{i=1}^n X_n$. Then

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- **Main idea of proof:** Consider first time maximum is exceeded. Bound below the expected square sum on that event.