Critical points of complex polynomials from a symplectic viewpoint

PAUL SEIDEL . MIT

Builds on classical work from the 1960s and 1970s (Thom, Milnor, Arnol'd, Brieskorn, A'Campo, Lê, Gabrielov, ...)
We try to integrate insights from modern symplectic topology (Arnol'd, Gromov, Eliashberg, Floer, Hofer, McDuff ...), concentrating on open problems and directions

The situation:

$$p : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$$
$$p \in \mathbb{C}[x_1, \dots, x_{n+1}]$$

or more generally

$$p : X \longrightarrow \mathbb{C}$$
$$p \in \mathbb{C}[X]$$

$X^{n+1} \subset \mathbb{C}^N$ a smooth affine algebraic variety.

most of this talk

We assume that the critical points

$$\{ x \in X : Dp_x = 0 \}$$

are isolated ($\Leftrightarrow$ there are finitely many)

Local questions: How does a critical point x_0, $p(x_0) = z_0$, affect the structure of the fibres $X_z = p^{-1}(z)$, $z \approx z_0$?

Global questions: What is the interplay between the global structure of X and the type of critical points?

Milnor fibre: take $p \in \mathbb{C}[x_1, \dots, x_{n+1}]$ with an isolated critical point at the origin, $p(0, \dots, 0) = 0$.
The Milnor fibre is the complex manifold with boundary

$$\boxed{F = \{ p(x_1, \dots, x_{n+1}) = \delta \quad |x_1|^2 + \dots + |x_{n+1}|^2 \le \varepsilon \}}$$

exact symplectic

$0 < |\delta| \ll \varepsilon \ll 1$. Equip F with the real symplectic form

$$\omega_F = \sum_k d\,re(x_k) \wedge d\,im(x_k) \in \Omega^2(F)$$

$$\omega_F = d\theta_F$$

Monodromy: the space of choices of $\delta \in \mathbb{C}$ has nontrivial topology ($\sim S^1$). From this we obtain the monodromy map

$$\boxed{\varphi : F \xrightarrow[\text{diffeomorphism}]{} F}$$

$\varphi = id$ close to ∂F

$$\begin{cases} \varphi^* \omega_F = \omega_F \\ \varphi^* \theta_F = \theta_F + d(\text{function}) \end{cases}$$

Well-defined isotopy class

$$[\varphi] \in \pi_0(Symp^{ex}(F, \partial F))$$

Dehn twist: For the simplest nontrivial singularity

$$(*) \quad p = x_1^2 + \cdots + x_{n+1}^2$$

we have $F = D^* S^n$ (cotangent vectors of length ≤ 1), and $\varphi = \tau_L$ is the Dehn twist or Picard-Lefschetz transformation associated to the n-dimensional sphere $L = S^n \subset F$.

non-degenerate

Any critical point with $\det(D^2 p \neq 0)$ is locally equivalent to $(*)$

Morsification: Under perturbation of p, any isolated critical point explodes into a finite number of nearby nondegenerate ones $(*)$, so

$$\boxed{\varphi = \tau_{L_1} \cdots \tau_{L_\mu}}$$

Milnor number

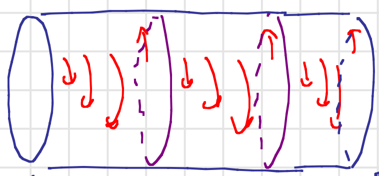
for (non-unique) $L_1, \ldots, L_\mu \subset F$.

Following the low-dimensional model and ideas of Lalonde–McDuff–Polterovich, we have:

<u>Thm</u> (S) $[\varphi] \in \pi_0(\mathrm{Symp}^{ex}(F, \partial F))$ always has infinite order.

The importance of fixed points:

Take $p(x) = x_1^2 + x_2^2$,



$F \cong S^1 \times [0,1]$

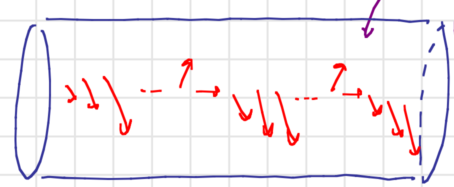
$\varphi^k = \mathrm{id}$ on ∂F

$\varphi^k = \mathrm{id}$ on ∂F

$k-1$ fixed circles in $F \setminus \partial F$

Situation similar to Poincaré's last geometric theorem.

These circles disappear under a small differentiable isotopy



push to the right (decrease to 0 at ∂F)

but under isotopies in $\mathrm{Symp}^{ex}(F, \partial F)$ they at most disintegrate into pairs of isolated fixed points:

Floer cohomology: To any* $[f] \in \pi_0(\mathrm{Symp}^{ex}(F, \partial F))$ we can associate the fixed point Floer cohomology $HF^*(f)$, a finitely generated graded abelian group. Its Euler characteristic is the Lefschetz fixed point number

$$\boxed{\chi(HF^*(f)) = \Lambda(f)}$$

* for $n = 1$, small exceptions apply.

Example $HF^*(\mathrm{id}_F) \cong H^*(F)$.

The underlying chain complex is generated by fixed points of f (after a small perturbation that gets rid of the boundary fixed points), but $HF^*(f)$ is isotopy invariant in $\mathrm{Symp}^{ex}(F, \partial F)$.

Example φ = monodromy of $x_1^2 + x_2^2$

$$HF^*(\varphi^k) = \underbrace{H^*(S^1) \oplus \cdots \oplus H^*(S^1)}_{k-1 \text{ summands}}$$

$(k > 0)$

Multiplicity: $p(x_1,\dots,x_{n+1})$ with an isolated critical point at the origin. Slice through $X = \mathbb{C}^{n+1}$ with a generic line to get

$$p(ta_1,\dots,ta_{n+1}) = \\ = ct^m + \text{higher order}$$

$c \neq 0$. The integer $m \geq 2$ is the multiplicity of $\{p=0\}$ at the origin.

An old theorem of Deligne-Lê-A'Campo says that the Lefschetz number of the monodromy satisfies

$$\Lambda(\varphi^k) = 0 \quad \text{for} \quad 1 \leq k < m$$

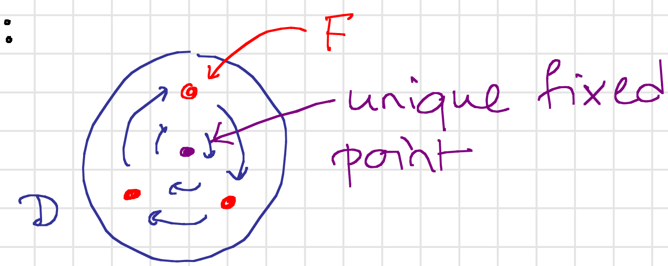
Example $p = x_1^2 + x_2^2$, $\Lambda(\varphi^k) = 0$ $\forall k$ but note that φ^2 has nontrivial Floer cohomology.

$$\boxed{\text{Conjecture } HF^*(\varphi^k) = 0 \text{ for } 1 \leq k < m, \text{ but } HF^*(\varphi^m) \neq 0.}$$

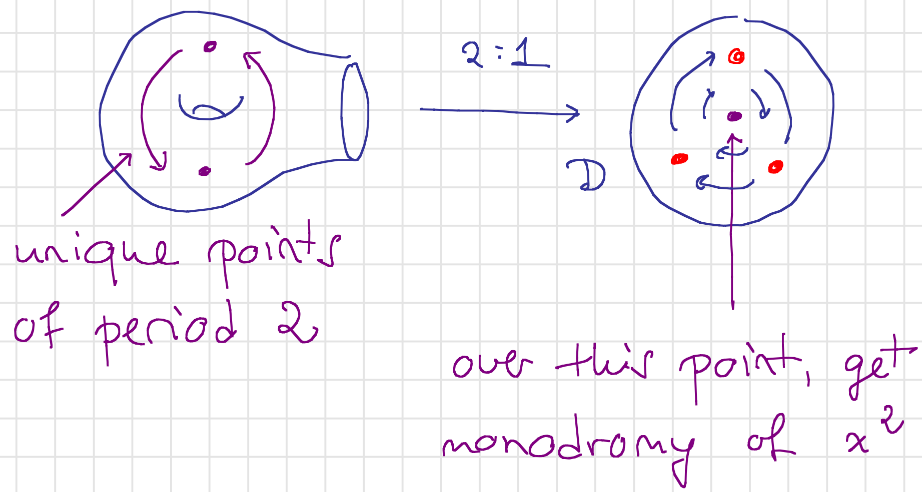
This part is more speculative

Algebraic braids: we explain one approach to the conjecture by induction on n (Lê).

Example $p(x) = x^k$ has $m=k$, $F = \{k \text{ points}\}$ and φ = cyclic permutation $\Rightarrow$ conjecture obvious. φ can naturally be embedded into a diffeomorphism of a disc D:



Example $p = x_1^2 + x_2^k$ has $m=2$. F is a double cover of D branched along k points, and φ is a lift of the previously considered diffeomorphism



Dynamics: the monodromy theorem says that

> the eigenvalues of $\varphi_* : H_*(F) \longrightarrow H_*(F)$ are roots of unity.

This (and a refinement) implies

> the entries of φ_*^k grow at most polynomially (degree $\leq n$) as $k \longrightarrow \infty$

> **Conjecture** The total rank of $HF^*(\varphi^k)$ increases at most polynomially (of degree $\leq n$) as $k \to \infty$

Example For $n=1$, φ is fully reducible in the Thurston classification, and the conjecture follows from that.

Example $p = x_1^2 + \cdots + x_{n+1}^2$, then $\mathrm{rank}(HF^*(\varphi^k))$ grows linearly. Exactly?

Example p (weighted) homogeneous $\Rightarrow$ at most linear growth.

Resolution of singularities This is an approach to the previous conjecture (following classical ideas of Clemens, A'Campo, Fried, and recent work of McLean)

Take an embedded resolution

$$\begin{tikzcd}
\tilde{X}_0 \arrow[r] \arrow[d, phantom, "\cap"] & X_0 \arrow[d, phantom, "\cap"] \\
\tilde{X} \arrow[r, "\rho"] \arrow[dr, "\tilde{p}"'] & X \arrow[d, "p"] \\
& \mathbb{C}
\end{tikzcd}$$

but it won't be!

$\tilde{X}$ is smooth, and $\tilde{X}_0 \subset \tilde{X}$ a divisor with normal crossings. The picture in local complex coordinates is

$$\tilde{p}(\tilde{x}_1, \dots, \tilde{x}_{n+1}) = \tilde{x}_1^{a_1} \cdots \tilde{x}_{n+1}^{a_{n+1}}$$

If the symplectic form was the standard (constant) one, the monodromy would rotate each of the tori

$$|\tilde{x}_1| = r_1, \dots, |\tilde{x}_{n+1}| = r_1$$
$$\tilde{x}_1^{a_1} \cdots \tilde{x}_{n+1}^{a_{n+1}} = \varepsilon$$

by some amount.

__Discrepancy and degrees__ As before, take a resolution $\rho : \tilde{X} \to X$, and write $\rho^{-1}(0) = \bigcup_i D_i$ (irreducible components). Take the meromorphic complex volume form

$$\eta = \frac{dx_1 \wedge \dots \wedge dx_{n+1}}{p}$$

log-canonical singularity

which has a pole along X_0; write

$$d_i = \mathrm{ord}_{D_i}(\rho^* \eta) \in \mathbb{Z}.$$

($d_i > 0$ means a zero)

__Example__ p homogeneous of degree k; $\tilde{X} \to X$ is a single blowup, $D = \mathbb{CP}^n$ and $d = n-k$.

The log minimal discrepancy (an invariant of the critical point) is defined as

$$MD = \begin{cases} -\infty & \text{some } d_i < -1, \\ \min\{d_i\} & \text{otherwise} \end{cases}$$

$$\boxed{\text{\underline{Conjecture} If } MD \geq -1, \text{ all groups } HF^*(\varphi^k),\ k \geq 0, \text{ vanish in degrees } * > n.}$$

<u>Example</u> $p = x_1^2 + \dots + x_{n+1}^2$, $F \cong D^* S^n$.

$$HF^*(\varphi^{2k}) = H^{*+2k(1-n)}(F) \underbrace{\oplus H^{*+(2k-2)(1-n)}(\partial F) \oplus H^{*+(2k-4)(1-n)}(\partial F) \oplus \dots \oplus H^{*+2(1-n)}(\partial F)}_{(k-1) \text{ copies of } H^*(\partial F)}$$

If $n=1$ (the borderline case $MD = -1$), these groups are concentrated in degrees $\{0,1\}$. If $n>1$ ($MD > -1$), the number of generators in each degree is bounded independently of k.

There is a direct geometric approach to the conjecture, by looking at how monodromy acts on the holomorphic volume form on F

Compare McLean's recent work on log Kodaira dimension

But there is also a formalism for computing $HF^*(\varphi^k)$, which may yield another approach.

Fukaya categories: recall that

$$\varphi = \tau_{L_1} \cdots \tau_{L_\mu}$$

for some spheres $L_1, \dots, L_\mu \subset F$. These are Lagrangian submanifolds and have their own version of Floer cohomology $HF^*(L_i, L_j)$ which is based on their (essential) intersections.

- $\chi(HF^*(L_i, L_j)) = \pm L_i \cdot L_j$
  algebraic intersection number
- For $n=1$ and $i \neq j$,
  rank $HF^*(L_i, L_j)$ = geometric intersection number

One defines a category $\mathcal{A}$ with

- Objects $L_1, \dots, L_\mu$
- Nontrivial morphisms $L_i \to L_j$ $(i<j)$ given by elements of $HF^*(L_i, L_j)$ (or rather, the underlying chain complex)

Hochschild cohomology: One can associate to $\mathcal{A}$ two finite-dimensional graded vector spaces

$HH_*(\mathcal{A},\mathcal{A})$ Hochschild homology

$HH^*(\mathcal{A},\mathcal{A})$ Hochschild cohomology

$HH_*(\mathcal{A},\mathcal{A})$ is always of rank μ, but:

Conjecture There is a long exact sequence

$$\to HH^{*-1}(\mathcal{A},\mathcal{A}) \to HF^*(\varphi) \longrightarrow H^*(B^{2n+2}) \to \cdots$$

The conjecture is based on an insight of Donaldson, that nontrivial fixed points of

$$\varphi = \tau_{L_1} \cdots \tau_{L_m}$$

correspond to chains of intersections

$$(L_{i_1} \cap L_{i_2}) \times \cdots \times (L_{i_r} \cap L_{i_1})$$
$$1 \le i_1 < \cdots < i_r \le \mu$$

rather trivial in our case $(HF^*(\varphi) \underset{?}{\cong} 0)$

The conjecture (in more general form) can be applied to yield the Floer cohomology of

$$\varphi^k = \tau_{L_1} \cdots \tau_{L_\mu} \tau_{L_1} \cdots \tau_{L_\mu} \cdots$$

for any k.

Corollary (of the conjecture) The sequence of graded vector spaces $HF^*(\varphi^k)$ is computable.

The computability result is quite general, but for monodromy maps, it would be better to have a purely algebro-geometric formula, maybe in terms of rational curves in $\tilde{X}_0 \longrightarrow X_0 = p^{-1}(0)$ (compare: Diogo's work on symplectic cohomology; and Denef-Loeser-Nicaise-Sebag).

Two proofs of a closely related result announced (Perutz, Ma'u)

A sample "global" question:

Recall from Morse theory–

Lemma Let M^{n+1} be a manifold. There is a C^∞ function $M \longrightarrow \mathbb{R}$ with nondegenerate critical points and $\leq n+1$ critical values.

This is unlike the number of critical points, which depends on $H_*(M)$ etc.

Question Given $p: X \longrightarrow \mathbb{C}$ and its deformations $(p_t)_{t \in T}$, what is the minimal number of critical values for p_t?

Take $\mathcal{A}$ as before; $HH^*(\mathcal{A},\mathcal{A})$ is a graded commutative ring.

Thm (S) Suppose that there are nilpotent elements $h_1, \dots, h_r \in HH^*(\mathcal{A},\mathcal{A})$, whose product is nonzero. Then as long as the critical points remain nondegenerate,

$$\#\text{critical values} \geq r+1.$$

Conjecture The nondegeneracy condition in the theorem can be weakened to having isolated critical points.

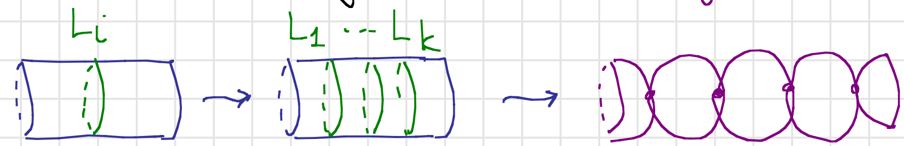
This would follow from two of the earlier conjectures.

Example Take $q \in \mathbb{C}[x_3]$ degree k polynomial; consider

$$X = \{ x_1 x_2 + q(x_3) = 0 \} \xrightarrow{p = x_3} \mathbb{C}$$

Then $p^{-1}(z) \cong S^1 \times \mathbb{R}$, and $L_1 = \cdots = L_k = S^1 \times \{0\}$

Purely topologically, one could deform this to having just one singular fibre:



However, in symplectic topology this is impossible because of areas. Indeed, there is a nilpotent $h \in HH^0(A, A)$ with $h^{k-1} \neq 0$.