Publications

Richard P. Stanley

ABSTRACT. A brief discussion of my published papers.

(1) Algorithmic Complexity, NASA Report No. 32-999 (September 1, 1966).
See the discussion under [8].

(2) Zero-square rings, *Pacific J. Math.* **30** (1969), 811–824.
My first published paper, written when I was a junior at Caltech for the E. T. Bell Prize, under the guidance of Richard Dean, with whom I was taking a second year course in algebra. I think that the paper arose from a homework problem, namely, to show that if R is a ring (not necessarily commutative) generated by n elements and $x^2 = 0$ for all $x \in R$, then any product of $n+1$ elements is 0. Dean suggested that I submit the paper to *Pacific J. Math.* Even back then I thought that the paper was rather routine, except for one wrinkle. One main object was to determine the least number of generators of the additive group R_+ of a ring R such that $x^2 = 0$ for all $x \in R$, and such that there exists n elements in R whose product was nonzero. To rule out one case it was necessary to use the fact (Lemma 5.1) that a symmetric matrix of odd order over $\mathbb{F}_2$ with 0's on the main diagonal is singular.

(3) On the number of open sets of finite topologies, *J. Combinatorial Theory* **10** (1971), 74–79.
When I was a graduate student at Harvard I took a course in algebraic topology from Albrecht Dold, who was visiting Harvard at the time. One of the course requirements was a paper. Mine was on the singular homology of finite topological spaces, based on a paper by Michael C. McCord, *Duke Math. J.* **33** (1966), 465–474. (Incidentally, beginning just three pages after the end of McCord's paper were two papers by Frank Ryan, the only NFL player to obtain a Ph.D. in mathematics.) As a result of this course paper, I had a good understanding of the basic combinatorics of finite topological spaces and their relation to posets. See [70], 2nd ed., Exercise 3.3. Thus when I ran across papers by Sharp and by Stephens on finite topologies, I could immediately see that the connection with posets would lead to stronger results.

2000 *Mathematics Subject Classification.* Primary 01A75.
Partially supported by NSF grant DMS-1068625.

(4) The conjugate trace and trace of a plane partition, *J. Combinatorial Theory* **14** (1973), 53–65.

Plane partitions fit nicely into the general theory developed in my thesis though they have special properties that require other techniques, such as the RSK algorithm. MacMahon gave the famous generating function $\prod_{i\geq 1}(1-x^i)^{-i}$ for plane partitions. Someone (I forget who) asked if there was a nice combinatorial interpretation of the coefficients of $\prod_{i\geq 1}(1-qx^i)^{-i}$. I realized that the connection between plane partitions and RSK developed by Edward Bender and Donald Knuth could be used to answer this question.

(5) Structure of incidence algebras and their automorphism groups, *Bull. Amer. Math. Soc.* **76** (1970), 1236–1239.

The incidence algebra $I(P)$ of a locally finite poset P over a field K was a hot topic when I was in graduate school, and someone mentioned to me the question of whether $I(P)$ determined P. Since I liked algebra I took a look at this question and was able to solve it affirmatively. It suggested the new question of whether every K-automorphism of $I(P)$ was induced by an automorphism of P. Somewhat surprisingly, the answer turned out to be negative, though the "bad" automorphisms are quite special.

(6) Modular elements of geometric lattices, *Algebra Universalis* **1** (1971), 214–217.

I discovered the main result of this paper when thinking about extending some results of my thesis from finite distributive lattices (whose maximal chains correspond to linear extensions of a poset) to more general posets. The first draft of the paper had some additional properties of modular elements of geometric lattices, but Garrett Birkhoff told me that the paper should stay focused on a single result.

(7) A chromatic-like polynomial for ordered sets, in *Proc. Second Chapel Hill Conference on Combinatorial Mathematics and Its Applications* (May, 1970), pp. 421–427.

This paper was based on a portion of my Ph.D. thesis. The analogy between order polynomials of posets and chromatic polynomials of graphs is further brought out in [18].

(8) The following papers appeared in the Jet Propulsion Laboratory *Space Programs Summary* or *Deep Space Network* journals:
   - New results on algorithmic complexity, JPL *SPS* 37-34, Vol. IV.
   - Further results on the algorithmic complexity of (p, q) automata, JPL *SPS* 37-35, Vol. IV.
   - The notion of a (p, q, r) automaton, JPL *SPS* 37-35, Vol. IV.
   - Enumeration of a special class of permutations, JPL *SPS* 37-40, Vol. IV (1966), 208–214.
   - Moments of weight distributions, JPL *SPS* 37-40, Vol. IV (1966), 214–216.
   - Some results on the capacity of graphs (with R. J. McEliece and H. Taylor), JPL *SPS* 37-61, Vol. III (1970), 51–54.
   - A study of Varshamov codes for asymmetric channels (with M. F. Yoder), JPL Technical Report 32-1526, *DSM*, Vol. XIV (1973), 117–123.

These papers, together with [1] and [17], were written when I worked in the group at JPL responsible for developing codes for transmitting data to and from unmanned spacecraft. For some further details see [162]. The paper on Varshamov codes used a technique that I found useful later, especially in [108].

(9) Ordered structures and partitions (revision of 1971 Harvard University thesis), *Memoirs of the Amer. Math. Soc.*, no. 119 (1972), iii + 104 pages.

My original thesis was about 250 pages, so this revision was greatly abridged. Gian-Carlo Rota had 200 copies of my thesis xeroxed, considerably overestimating the demand. About 150 copies continued to moulder in my office until 2013, when the M.I.T. Mathematics Department moved to a new temporary location, giving me an excuse to get rid of them.

When I was working on my thesis I had come up with many conjectures about the form of certain generating functions. The big stumbling block was whether for each linear extension $a_1, \ldots, a_p$ of a partial ordering P of $1, 2, \ldots, p$ there was a way of choosing a subset S of $\{1, 2, \ldots, p-1\}$ with certain desirable properties. At first I tried to prove the existence of S by induction. I couldn't get that to work, so I tried a double induction. It looked like infinitely many induction arguments $I_1, I_2, \ldots$ might be needed, with an additional induction argument proving the induction step for each I_i. I could never get this horrific mess to work out. Sitting one day in the Harvard mathematics library (in Two Divinity Avenue, the former location of the mathematics department), it came to me in a flash that if the elements of P respected the usual ordering of integers (i.e., $1, 2, \ldots, p$ is a linear extension of P) then one could simply take $S = \{i \colon a_i > a_{i+1}\}$, the descent set of the permutation $a_1, \ldots, a_p$. It was immediately clear that this would solve all my conjectures. This was the strongest mathematical epiphany of my career.

When I later was explaining some of these ideas to Bob McEliece, he asked what would happen with arbitrary partial orderings of $1, 2, \ldots, p$, not just those that respected the usual ordering $1 < 2 < \cdots < p$. I had another revelation and instantaneously realized that the whole theory would carry over, thus giving rise to the general theory of (P, ω)-partitions.

(10) On the foundations of combinatorial theory (VI): The idea of generating function (with G.-C. Rota and P. Doubilet), in *Sixth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. II: *Probability Theory*, University of California, 1972, pp. 267–318.

My only joint paper with Rota. The paper [162] explains how I got interested in the topic of "explaining" why certain generating functions like $\sum a_n x^n$ and $\sum a_n \frac{x^n}{n!}$ are ubiquitous in combinatorics, while other types like $\sum a_n \frac{x^n}{(n+1)^n}$ never occur. Peter Doubilet contributed to this paper when he was a graduate student of Rota. After receiving his degree he left mathematics and went to medical school.

(11) Supersolvable lattices, *Algebra Universalis* **2** (1972), 197–217.

A sequel to [6]. For the present paper I received the most helpful referee's suggestion of my career. (Most likely the referee was Robert Dilworth.) My original definition of supersolvable was in terms of modular elements, which led to some quite technical proofs. The referee suggested the definition used in the published version, thereby greatly simplifying the proofs and making them more conceptual.

(12) Theory and application of plane partitions, Parts 1 and 2, *Studies in Applied Math.* **50** (1971), 167–188, 259–279.

I was rather proud of this paper, because I wrote it as a graduate student and it got a lot of attention. There aren't so many expository papers written by graduate students. The original title was "Symmetric functions and plane partitions." Gian-Carlo Rota, however, was angling to have me become part of the applied math group within the M.I.T. Math. Dept. (because that is where combinatorics was situated in the department) so he gave it a more "applied" title. It was also published in the journal *Studies in Applied Mathematics*, which was operated by the M.I.T. applied math group. Part 1 is an exposition of Philip Hall's paper "The algebra of partitions" on symmetric functions, while Part 2 gives some combinatorial applications. Hall's paper was published in an obscure location and almost unknown, but Bob McEliece, one of my JPL colleagues, had spent a year visiting Philip Hall so could give me a copy. Hall gives no proofs, so I had to work them all out myself. I remember getting some assistance from Gene Rodemich. Several years later, when Ian Macdonald was getting interested in symmetric functions, he asked me to send him a copy of Hall's paper because it was otherwise inaccessible.

(13) An extremal problem for finite topologies and distributive lattices, *J. Combinatorial Theory* **14** (1973), 209–214.

This paper was inspired by the Fibonacci lattice $\boldsymbol{F_1}$, which I later discussed in [14]. $\boldsymbol{F_1}$ has the property that it is a locally finite distributive lattice with $\hat{0}$, with exactly two join-irreducibles of each positive rank, and moreover it has the maximum number of elements of each rank among all such lattices. Thus it seemed natural to generalize the condition that there are exactly two join-irreducibles of each rank.

(14) The Fibonacci lattice, *Fibonacci Quarterly* **13** (1975), 215–232.

In my Ph.D. thesis a certain infinite graded distributive lattice $\boldsymbol{F_1}$ arose as a nice example. The number of elements of $\boldsymbol{F_1}$ of rank n is the Fibonacci number F_{n+1}, whence the term *Fibonacci lattice*. I thought that it would be a nice topic for a paper in *Fibonacci Quarterly*. The paper begins with a general exposition of the combinatorics of graded distributive lattices with finitely many elements of each rank and then specializes to $\boldsymbol{F_1}$.

(15) A Brylawski decomposition for finite ordered sets, *Discrete Math.* **4** (1973), 77–82.

I knew from my thesis and [7] that the order polynomial of a poset was a kind of analogue of the chromatic polynomial of a graph. Thus it was natural to ask whether there was an "order analogue" of the deletion-contraction recurrence for order polynomials. A general theory of the deletion-contraction recurrence was worked out by Tom Brylawski, a *de facto* student of Rota. This explains the terminology "Brylawski decomposition." Tom Brylawski was very pleased to see his name in the title of a paper.

(16) Review of Claude Berge, *Principles of Combinatorics*, in *Bull. Amer. Math. Soc.* **77** (1971), 685–689.

The book under review has quite an interesting selection of topics, especially for such an early publication date. It is probably the first book that treats Möbius inversion on a locally finite poset and the permutohedron, for instance.

(17) A combinatorial packing problem (with five other authors), *SIAM-AMS Proc.* **4** (1971), 97–108.

From JPL. It is devoted to the Shannon capacity of a graph. The real breakthrough in the area is due to László Lovász in 1978.

(18) Acyclic orientations of graphs, *Discrete Math.* **5** (1973), 171–178.

An interesting example of what I call "wishful thinking as a proof technique." If P is a p-element poset, then for every bijection (or *labeling*) $\omega\colon P \to \{1, \ldots, p\}$ we can define a certain set $\mathcal{A}(P, \omega)$ whose elements are called (P, ω)-*partitions*. It seemed natural to ask when $\mathcal{A}(P, \omega) = \mathcal{A}(P, \omega')$. It wasn't hard to see that for fixed P, the number of *distinct* sets $\mathcal{A}(P, \omega)$ is equal to the number $\mathfrak{o}(H(P))$ of acyclic orientations of the Hasse diagram $H(P)$ of P (considered as a graph). Thus I became curious about what could be said about the number $\mathfrak{o}(G)$ of acyclic orientations of any finite graph G. There didn't seem to be any kind of simple formula. This is where the wishful thinking came in. The best situation would be that $\mathfrak{o}(G)$ satisfied the deletion-contraction recurrence. I didn't have any reason to believe this was true and expected to immediately find a counterexample, but in this endeavor I failed. Since deletion-contraction invariants were completely classified by the work of Tutte and Brylawski, it was then routine to see that $\mathfrak{o}(G) = (-1)^p \chi_G(-1)$, where p is the number of vertices of G and χ_G the chromatic polynomial of G. It is quite surprising that this elegant result had not been previously discovered, especially since there was already a paper in the literature devoted to connections between acyclic orientations and chromatic polynomials!

(19) Linear homogeneous diophantine equations and magic labelings of graphs, *Duke Math. J.* **40** (1973), 607–632.

This is my first paper connected to commutative algebra. David Smith had brought to my attention a tantalizing conjecture by Anand, Dumir, and Gupta. In particular, the number $H_n(r)$ of $n \times n$ matrices of nonnegative integers with every row and column summing to r (called "magic squares," though this definition is much weaker than the classical definition of magic squares) was conjectured to be a polynomial in r for fixed n. I immediately became fascinated by this conjecture, which seemed rather similar to my earlier work on P-partitions. The case $n = 3$ had been solved by MacMahon using his "syzygetic method." (David Smith himself did the case $n = 4$.) After much effort trying to read MacMahon and trying to understand what syzygies were, I finally realized that the conjecture could be essentially solved by applying the Hilbert syzygy theorem to a certain ring associated with $n \times n$ magic squares. In fact, $H_n(r)$ was the Hilbert polynomial of this ring. It also is the Ehrhart polynomial of the convex polytope of $n \times n$ doubly-stochastic matrices. Thus my paper led me into much further research related to Hilbert functions and Ehrhart theory.

For some properties of the polynomial $H_n(r)$ also conjectured by Anand, Dumir, and Gupta, I used an algorithm in MacMahon's masterwork *Combinatory Analysis* later called the *Elliott-MacMahon algorithm.* This algorithm received no further attention until it was brilliantly revived by George Andrews and many subsequent collaborators in a series of papers entitled "MacMahon's partition analysis."

(20) Enumeration of posets generated by disjoint unions and ordinal sums, *Proc. Amer. Math. Soc.* **45** (1974), 295–299.

These posets are now called *series-parallel posets.* They are in some sense the "simplest" posets, and it seemed like a natural problem to enumerate them.

(21) Finite lattices and Jordan-Hölder sets, *Algebra Universalis* **4** (1974), 361–371.

This paper was motivated by the result in my thesis that (in current terminology) the flag h-vector $\beta_L(S)$ of a finite distributive lattice is nonnegative. I believe it was Ken Bogart who originally asked me if this result still held for finite semimodular lattices, which I was able to prove in my paper. The concept of admissible lattice was later superseded by the beautiful theory of lexicographic shellability developed by Anders Björner and later Björner and Michelle Wachs.

(22) Combinatorial reciprocity theorems, *Advances in Math.* **14** (1974), 194–253.

In my thesis I proved a reciprocity theorem for P-partitions, which was generalized in [19] to homogeneous linear diophantine equations, or equivalently, to integer points in rational polyhedral cones. I was interested in how far I could push this reciprocity theory. In particular, what could be said about *inhomogeneous* linear diophantine equations? The paper culminates in a "monster reciprocity theorem," which gives a sufficient condition for reciprocity in the inhomogeneous case. The proof involves a horrendous computation involving multiple contour integration. Later in [51] I found a more definitive theorem which gives the deviation from reciprocity for *any* inhomogeneous system of linear diophantine equations. The two proofs provide an interesting connection between commutative algebra and complex analysis (more specifically, residues of multivariate rational functions).

(23) Combinatorial reciprocity theorems, in *Combinatorics (Part 2)* (M. Hall, Jr., and J. H. Van Lint, eds.), Mathematical Centre Tracts **56**, Mathematisch Centrum, Amsterdam, 1974, pp. 107–118.

An exposition of some of the results of [22], based on a talk given at a conference in Nijenrode Castle, The Netherlands, in 1974. I remember being pleased that my undergraduate adviser Marshall Hall was in the audience and afterwards praised my talk.

(24) Generating functions, in *Studies in Combinatorics* (G.-C. Rota, ed.), Mathematical Association of America, 1978, pp. 100–141.

A survey which helped organize the material that later appeared in my two books [70] and [113].

(25) Cohen-Macaulay rings and constructible polytopes, *Bull. Amer. Math. Soc.* **81** (1975), 133–135.

When I wrote this paper I had worked out the connection between f-vectors of simplicial complexes and Cohen-Macaulay rings and was able to give some applications. After I submitted the paper but before I received the galley proofs Gerald Reisner proved his famous characterization of Cohen-Macaulay face rings (now often called "Stanley-Reisner rings") of simplicial complexes. Thus I could say in a note added in proof that the Upper Bound Conjecture for spheres was a theorem, though my paper does not actually state the conjecture. Further details on the history of the Upper Bound Conjecture appear in [162].

(26) Branchings and partitions (with L. Carlitz), *Proc. Amer. Math. Soc.* **53** (1975), 246–249.

I was the referee of a paper by Leonard Carlitz, in which he obtained a generating function related to "complete branchings" (equivalent to what are now called *Gelfand-Tsetlin patterns*). I pointed out in my report a simple bijection between complete branchings and plane partitions, and that Carlitz's result was equivalent to a well-known plane partition generating function. The editor suggested that I become a coauthor of the paper.

(27) The Upper Bound Conjecture and Cohen-Macaulay rings, *Studies in Applied Math.* **54** (1975), 135–142.

Perhaps my most significant paper, not only because of the results and techniques themselves, but also because it was instrumental in my receiving tenure at M.I.T. For the history of this result see [162].

(28) Binomial posets, Möbius inversion, and permutation enumeration, *J. Combinatorial Theory (A)* **20** (1976), 336–356.

The techniques of [10] allow certain invariants, such as the Möbius function, of a class of posets known as *binomial posets*, to be expressed in terms of generating functions. I realized that in certain cases these invariants also had an interpretation in terms of permutation statistics. Thus I could give some new generating functions for some permutation statistics.

(29) Stirling polynomials (with I. Gessel), *J. Combinatorial Theory (A)* **24** (1978), 24–33.

In general, if $p(n)$ is a polynomial of degree d, then there is a polynomial $F_p(x)$ such that $\sum_{n\geq 0} p(n)x^n = F_p(x)/(1-x)^{d+1}$. Often the coefficients of $F_p(x)$ are of combinatorial interest. The classical case is when $p(n) = n^d$ and $F_p(x)$ is the Eulerian polynomial $A_d(x)$. Thus I decided to look at the polynomial $S(n+d,n)$, where $S(n,k)$ is a Stirling number of the second kind. The theory, developed in collaboration with Ira Gessel, turned out to be quite interesting. This was my only research paper with Ira, who was my first Ph.D. student to obtain a degree.

(30) Hilbert functions of graded algebras, *Advances in Math.* **28** (1978), 57–83.

The work I had done in writing [19] and [27] made me very aware of the connection between Hilbert functions and combinatorics. In particular, there were some fundamental results going back to Macaulay that were not widely known. Thus I decided to write an exposition of this subject, with a few new results and new proofs. According to Google Scholar this is my most cited paper.

(31) Magic labelings of graphs, symmetric magic squares, systems of parameters, and Cohen-Macaulay rings, *Duke Math. J.* **43** (1976), 511–531.

In [19] I proved that if $S_n(r)$ is the number of $n \times n$ symmetric matrices of nonnegative integers with every row and column summing to r, then $S_n(r) = P_n(r) + (-1)^r Q_n(r)$, where $P_n(r)$ is a polynomial of degree $\binom{n}{2}$ and $Q_n(r)$ is a polynomial of degree at most $\binom{n}{2}$. This suggested the very interesting problem of determining $\deg Q_n(r)$, which measures how far $S_n(r)$ is from being a polynomial. Using a lot of machinery from commutative algebra, the present paper shows that $\deg Q_n(r) \leq \binom{n-1}{2} - 1$ if n is odd, and $\deg Q_n(r) \leq \binom{n-2}{2} - 1$ if n is even. Moreover, I conjectured that equality holds. This conjecture was later proved by Rong-Qing Jia using spline theory.

(32) Relative invariants of finite groups generated by pseudo-reflections, *J. Algebra* **49** (1977), 134–148.

The ring R^G of invariants of a finite group (acting linearly on R), or more generally, the R^G-module of invariants relative to a character of degree one, affords a nice example of the interplay between commutative algebra and combinatorics. The "nicest" rings R^G are just polynomial rings, which occur if and only if G is generated by pseudo-reflections (semisimple linear transformations with exactly one eigenvalue not equal to one). Using Hilbert function techniques I proved some results about the relative invariants of such groups.

(33) Some combinatorial aspects of the Schubert calculus, in *Combinatoire et Réprésentation du Groupe Symétrique (Strasbourg, 1976)*, Lecture Notes in Math., No. 579, Springer-Verlag, Berlin, 1977, pp. 217–251.

Philip Hall, in his paper "The algebra of partitions," mentions a connection between symmetric functions and the combinatorics of Schubert varieties (in the Grassmannian). At that time I didn't have the background to understand this comment. Shortly thereafter appeared an expository paper on Schubert calculus by my colleague Steve Kleiman and his student Dan Laksov. This paper was quite difficult for me, but eventually I felt I understood the subject well enough to write a paper on it from a combinatorial perspective. I did make an unfortunate error in thinking that I had proved Theorem 4.5, which gives the number of points of a skew Schubert variety over $\mathbb{F}_q$. The result is true for ordinary Schubert varieties but not for skew ones. In fact, the correct number is essentially given by the R-polynomial of Kazhdan and Lusztig. Thus I missed a chance to investigate this polynomial well before Kazhdan and Lusztig, though it is extremely unlikely that I would have discovered the Kazhdan-Lusztig polynomials themselves, even in type A.

(34) Cohen-Macaulay complexes, in *Higher Combinatorics* (M. Aigner, ed.), Reidel, Dordrecht/Boston, 1977, pp. 51–62.

Development of further aspects of face rings of simplicial complexes. Perhaps the most noteworthy aspect of this paper is the concept of a *level ring* (a property in between Cohen-Macaulay and Gorenstein), which motivated the conjecture that the h-vector of matroid complex is the f-vector of a pure multicomplex (or order ideal of monomials). I have been surprised by how much attention this conjecture has received and how much has been shown, though I am not at all confident that it is actually true.

(34a) Eulerian partitions of a unit hypercube, in *Higher Combinatorics* (M. Aigner, ed.), Reidel, Dordrecht/Boston, 1977, p. 49.

My shortest paper. At a conference in Berlin which I attended, Dominique Foata raised the question of whether there was a certain kind of geometric proof of a result of Laplace on the volume of a hypersimplex (a certain slice of the unit cube). I was able to find an elegant solution, so Foata suggested that I include it as an addendum to his paper. It received a surprising amount of attention for such a brief paper.

(35) Enumeration of power sums modulo a prime (with A. Odlyzko), *J. Number Theory* **10** (1978), 263–272.

My paper [8] (item 7) with Michael Yoder involved such problems as computing the number of subsets of $\mathbb{Z}/n\mathbb{Z}$ whose elements sum to some fixed element of $\mathbb{Z}/n\mathbb{Z}$. During a consulting visit to Bell Labs I discussed with Andrew Odlyzko some possible generalizations, resulting in the present paper. This paper also established my Erdős number as two.

(36) Balanced Cohen-Macaulay complexes, *Trans. Amer. Math. Soc.* **249** (1979), 139–157.

Balanced Cohen-Macaulay complexes Δ have the property that we can grade their face ring A_Δ by $\mathbb{N}^k$ for some $k \geq 1$ such that their exists a system of parameters that is homogeneous with respect to this grading. When $k > 1$ this allows refinements of the usual combinatorial properties of A_Δ.

In Corollary 4.5 I give a necessary condition on the "fine h-vector" (a refinement of the usual h-vector) for balanced Cohen-Macaulay complexes. In the next paragraph I give an example showing that this condition is not sufficient. Unfortunately my "counterexample" had an error, and in fact the necessary condition is also sufficient, and in fact is not difficult to prove! This is pointed out in Section 6.7 of [68].

(37) Exponential structures, *Studies in Applied Math.* **59** (1978), 73–82.

The motivation for this paper is the idea that a "nice" q-analogue of partitions of a set is a decomposition of a vector space into a direct sum of subspaces. How can this idea be generalized, and what kind of combinatorics emerges?

(38) Invariants of finite groups and their applications to combinatorics, *Bull. Amer. Math. Soc.* (new series) **1** (1979), 475–511.

A survey paper well-described by the title. This paper was reprinted in the same journal in 2011.

(39) Combinatorics and invariant theory, in *Relations between Combinatorics and Other Parts of Mathematics* (D. K. Ray-Chaudhuri, ed.), Proc. Symposium in Pure Math. **34** (1979), American Mathematical Society, Providence, Rhode Island, pp. 345–355.

This paper is primarily concerned with extending techniques for finite groups discussed in [38] to compact Lie groups. The situation is more difficult combinatorially because the formula for the Hilbert series of the ring of invariants or module of relative invariants is a finite sum for finite groups (Molien's theorem), but an integral in the continuous case. I prove a criterion for the module of relative invariants of a torus to be Cohen-Macaulay and discuss a conjectured generalization for any compact group motivated by an analysis of the integral form of Molien's theorem. This conjecture was proved by Michel Van den Bergh in 1991.

(40) Decompositions of rational convex polytopes, *Ann. Discrete Math.* **6** (1980), 333–342.

Some theorems on Ehrhart polynomials are proved by using shelling to decompose a rational convex polytope into "nice" pieces. As an aside it is noted that the pulling triangulation of certain polytopes including the Birkhoff polytope of doubly stochastic matrices is unimodular. This result turned out to be of interest in the theory of Gröbner bases.

(41) Unimodal sequences arising from Lie algebras, in *Young Day Proceedings* (T. V. Narayana, R. M. Mathsen, and J. G. Williams, eds.), Dekker, New York/Basel, 1980, pp. 127–136.

I had been aware from conversations with Lie theorists that Dynkin had shown the every irreducible representation of a complex semisimple Lie algebra gives rise to a unimodal sequence of integers. Moreover, J. W. B. Hughes had shown that a special case was the coefficients of the polynomial $(1+q)(1+q^2)\cdots(1+q^n)$. It took a fair amount of work for me to understand this well enough to state a purely combinatorial formulation of Dynkin's result in terms of root systems.

(42) Weyl groups, the hard Lefschetz theorem, and the Sperner property, *SIAM J. Algebraic and Discrete Methods* **1** (1980), 168–184.

I had the idea that if one could show that a certain poset $L(m,n)$ had the Sperner property (or some stronger versions of it), then perhaps one could deduce the hard Lefschetz theorem for the Grassmannian $\mathrm{Gr}(m+n,m)$ of m-dimensional subspaces of $\mathbb{C}^{m+n}$. Then in a conversation with Ken Baclawski about Rota's linear algebraic approach toward matching theory, I realized in an instant that the argument actually goes the other way: the hard Lefschetz theorem for $\mathrm{Gr}(m,m+n)$ implies that $L(m,n)$ is Sperner! This argument can be generalized quite a bit, and Bob Proctor pointed out that one of the generalizations had been shown by Bernd Lindström to imply an old conjecture of Erdős and Moser. Thus as a bonus I obtained a proof of the Erdős-Moser conjecture. Alas, this particular conjecture was not one for which Erdős offered one of his legendary monetary awards.

(43) An application of algebraic geometry to extremal set theory, in *Ring Theory and Algebra III* (B. R. McDonald, ed.), Dekker, New York/Basel, 1980, pp. 415–422.

An exposition of the proof in [42] of the Erdős-Moser conjecture.

(44) The character generator of $SU(n)$, *J. Math. Physics* **21** (1980), 2321–2326.

This paper answers a question asked me by Jiří Patera.

(45) Differentiably finite power series, *European J. Combinatorics* **1** (1980), 175–188.

Differentiably finite (or D-finite, also called holonomic) power series had appeared in many combinatorial and other contexts, but no systematic discussion of their properties had been given. I could not find in the literature even such basic results as the fact that D-finite series form a ring. Thus I was inspired to write the present paper. The subject has blossomed in many directions, and it still can be a fascinating problem to decide whether some power series is D-finite.

(46) The number of faces of a simplicial convex polytope, *Advances in Math.* **35** (1980), 236–238.

A spectacular result, but the proof consists of little more than searching the literature and asking some experts the right questions. A brief history of the proof appears at the end of [54].

(47) Factorization of permutations into n-cycles, *Discrete Math.* **37** (1981), 255–262.

For a finite group G with conjugacy classes $K_1, \dots, K_k$, the *connection coefficient* $c^j_{i_1,\dots,i_m}$ is the number of m-tuples $(w_1, \dots, w_m) \in G^m$ such that $w_r \in K_{i_r}$ and $w_1 \cdots w_m$ is some fixed element of K_j. Burnside gave a formula for $c^j_{i_1,\dots,i_m}$ in terms of the irreducible characters of G. I realized that this formula could be used to give some information about the connection coefficients of the symmetric group $\mathfrak{S}_n$ since a lot was known about its irreducible characters. Subsequently Goulden, Jackson, and others developed this approach much more extensively.

(48) Two combinatorial applications of the Aleksandrov-Fenchel inequalities, *J. Combinatorial Theory (A)* **31** (1981), 56–65.

My work related to the hard Lefschetz theorem got me interested in the subject of unimodality and log-concavity. When I learned of the Aleksandrov-Fenchel inequalities, which state that certain sequences related to the volumes of convex bodies are log-concave, I tried to think of some combinatorial applications. This accounted for Section 2 of the present paper. On a visit to Bell Labs, Ron Graham mentioned to me a log-concavity conjecture of Fan Chung, Peter Fishburn, and himself related to linear extensions of posets. It didn't hurt to see if the Aleksandrov-Fenchel inequalities were relevant, though I had little reason to believe so. Amazingly enough, they were exactly what was needed to prove the conjecture of Chung *et al.*

(49) Review of I. G. Macdonald, *Symmetric functions and Hall polynomials*, in *Bull. Amer. Math. Soc.* (new series) **4** (1981), 254–265.

I have always admired this book (and the second edition of 1995) for the huge amount of interesting information packed between its pages. I was very happy to have a chance to write this review.

(50) Some aspects of groups acting on finite posets, *J. Combinatorial Theory (A)* **32** (1982), 132–161.

The flag h-vector of a finite graded poset P can be interpreted as a (reduced) Euler characteristic of a certain simplicial complex. If G is a group of automorphisms of P, then it follows that the flag h-vector can be refined into a virtual representation of G. If P is Cohen-Macaulay, then this virtual representation is an actual representation. This general theory is rather straightforward, but it remains to work out some interesting examples. The most interesting example in my paper is the lattice of partitions of an n-set. Connections were later found with free Lie algebras and other topics, along with much further work on the action of $\mathfrak{S}_n$ on rank-selected homology.

(51) Linear diophantine equations and local cohomology, *Inventiones Math.* **68** (1982), 175–193.

Once I finally understood a proof of Reisner's theorem using local cohomology due to Hochster, I realized that the same technique could be applied to solutions to linear inhomogeneous equations in nonnegative integers, the main topic of [22]. In particular, I was able to give a definitive reciprocity theorem for any such system of equations. What has received the most attention, however, is a side conjecture about what can take the place of a homogeneous system of parameters when we have an $\mathbb{N}^d$-grading. (When $d > 1$, there need not exist a system of parameters homogeneous with respect to this grading.) This led other researchers to define "Stanley depth" and "Stanley decomposition." I was pleased to see an article "What is Stanley depth?" by Pournaki *et al.* in a 2009 issue of the *Notices of the AMS*. In 2015 Art Duval, Bennet Goeckner, Carly Klivans, and Jeremy Martin found a counterexample to my conjecture.

(52) An introduction to the theory of Cohen-Macaulay partially ordered sets (with A. Björner and A. Garsia), in *Ordered Sets* (I. Rival, ed.), Reidel, Dordrecht/Boston/London, 1982, pp. 583–615.

In 1981 my two coauthors and I were speakers at a Symposium on Partially Ordered Sets, organized by Ivan Rival, in Banff, Canada. We decided to give a three-lecture introduction (one lecture each) on Cohen-Macaulay posets. I made a tee shirt which said "Cohen-Macaulay posets" on the front. Adriano was so impressed that he went into Banff and had two identical tee shirts made for him and Anders!

(53) *Combinatorics and Commutative Algebra*, Progress in Mathematics, vol. 41, Birkhäuser, Boston/Basel/Stuttgart, 1983, viii + 88 pages; second edition, 1996, ix + 164 pages.

The first edition was written by Anders Björner based on eight lectures that I gave at Stockholm University (Stockholms universitet) in 1981. The second edition was written after some prodding by Ann Kostant. The second edition includes a new Chapter 0 devoted to background information and terminology. After the book was published I received a strange letter from someone who was incensed that a book should have a Chapter 0.

(54) The number of faces of simplicial polytopes and spheres, in *Discrete Geometry and Convexity* (J. E. Goodman, et al., eds.), Ann. New York Acad. Sci., vol. 440 (1985), pp. 212–223.

A survey paper including a short history on how the necessity of McMullen's g-conjecture for simplicial polytopes was proved.

(55) An introduction to combinatorial commutative algebra, in *Enumeration and Design* (D. M. Jackson and S. A. Vanstone, eds.), Academic Press, Toronto/Orlando, 1984, pp. 3–18.

An exposition of some basic properties of graded algebra. Homological algebra is avoided by the use of Lemma 12. Kaplansky takes a similar course in his book *Commutative Rings*, where he calls the non-graded analogue of Lemma 12 one of the most useful facts about commutative rings.

(56) On the number of reduced decompositions of elements of Coxeter groups, *European J. Combinatorics* **5** (1984), 359–372.

This paper had its origins in a letter (back in ancient times when people still used postal mail) from Paul Edelman pointing out that the number of reduced decompositions of the longest element w_0 of $\mathfrak{S}_n$ factors into small primes for $n \leq 5$. I found this extremely intriguing and worked intensely for several months. This paper is another example of "proof by wishful thinking." For each $w \in \mathfrak{S}_n$ I defined a certain formal power series F_w (which in later terminology was clearly a quasi-symmetric function) in variables $x_1, x_2, \ldots$ that was homogeneous of degree $\ell(w)$ (the number of inversions of w) and for which the coefficient of $x_1 x_2 \cdots x_n$ was the number $r(w)$ of reduced decompositions of w. I didn't see how anything could be done with F_w unless by some miracle it was actually a symmetric function. In fact, this turned out to be the case! It wasn't at all easy to prove, and I don't know whether anyone has actually checked all the details of my ugly proof. Ezra Miller later gave a more tractable geometric version of the proof based on "pipe dreams." Then an exceptionally elegant proof based on the nilCoxeter algebra of $\mathfrak{S}_n$ was given by Sergey Fomin and appears in [94]. I was disappointed that I was unable to prove that F_w was Schur positive but was very happy to see the beautiful proof later given by Paul Edelman and Curtis Greene using a new version of the RSK algorithm. When I wrote the present paper, I had no inkling that F_w would later turn out to be a stable Schubert polynomial (now also called a "Stanley symmetric function") and would be connected with numerous other topics and generalized in a number of directions.

(57) $GL(n, \mathbf{C})$ for combinatorialists, in *Surveys in Combinatorics* (E. K. Lloyd, ed.), London Math. Soc. Lecture Note Series, vol. 82, Cambridge University Press, Cambridge, 1983, pp. 187–199.

Perhaps the most difficult result for students in my symmetric function courses is that the Schur functions $s_\lambda(x_1, \ldots, x_n)$, where $\ell(\lambda) \leq n$, are the polynomial characters of $\mathrm{GL}(n, \mathbb{C})$. The present paper is an exposition of the significance (but not the proof) of this result, with some examples and applications.

(58) Combinatorial aspects of the hard Lefschetz theorem, in *Proc. International Congress of Mathematicians (Warsaw, 1983)*, North-Holland, Amsterdam (1984), pp. 447–453.

A survey of results from [42] and [46], based on an invited talk in the algebra section of the 1983 Warsaw ICM. After agreeing to give this talk, I was sent a form asking me if I had any requests about scheduling my talk. I stated that I wanted only *not* to be scheduled at the same time as Dominique Foata, who was giving a talk in the combinatorics section. So what do you think happened?

(59) The stable behavior of some characters of $SL(n, \mathbf{C})$, *Linear and Multilinear Algebra* **16** (1984), 29–34.

This paper arose from some discussions with Ranee Gupta (Brylinski). It has been subsequently superseded by numerous developments. The polynomial $P_{\alpha\beta}(q)$ turned out to be equal to $K_{\alpha\beta}(q, q)$, where $K_{\alpha\beta}(q, t)$ is a (q, t)-Kostka polynomial. Thus Conjecture 8.3, that $P_{\alpha\beta}(q)$ has nonnegative coefficients, follows from Haiman's deep result that $K_{\alpha\beta}(q, t)$ has nonnegative coefficients.

(60) Quotients of Peck posets, *Order* **1** (1984), 29–34.

I use some simple linear algebra to prove slight strengthenings of results of Pouzet and Rosenberg and of Harper concerning posets with the Sperner and related properties. A special case (Theorem 2(b)) shows that q-binomial coefficients have unimodal coefficients. I show that the same technique gives an alternative proof of a result of Lovász that the edge-reconstruction conjecture holds for graphs with n vertices and at least $\frac{1}{2}\binom{n}{2}$ edges. My argument was generalized by Krasikov and Roddity to the case of at least $n \log n$ edges. (Previously Müller had generalized Lovász's argument to this case.) Much of this material is included in my undergraduate textbook [159].

(61) The q-Dyson conjecture, generalized exponents, and the internal product of Schur functions, in *Combinatorics and Algebra* (C. Greene, ed.), Contemporary Math., vol. 34, American Math. Society, Providence, RI (1984), 81–94.

A summary of [59] based on a talk at a conference in Boulder, Colorado, in 1983.

(62) Unimodality and Lie superalgebras, *Studies in Applied Math.* **72** (1985), 263–281.

Victor Kac brought to my attention that the unimodality results of [41] have a "superanalogue." The present paper is an exposition for combinatorialists with some further applications and remarks.

(63) Reconstruction from vertex-switching, *J. Combinatorial Theory (B)* **38** (1985), 132–138.

I realized that the technique used in [60] to prove Lovász's edge-reconstruction result could also be applied to a new kind of reconstruction problem which I called *vertex-switching*. Further results were obtained by Krasikov, Roddity, and others.

(64) On dimer coverings of rectangles of fixed width, *Discrete Applied Math.* **12** (1985), 81–87.

David Klarner once brought to my attention that for fixed k, the generating function $F_k(x)$ for the number of dimer coverings of a $k \times n$ rectangle (computed for small k by the transfer-matrix method) appeared to have some unexpected symmetry properties. At that time I was completely mystified, but later it occurred to me that Kasteleyn's famous explicit formula for the number of dimer coverings of a rectangle could be used rather straightforwardly to resolve these symmetry conjectures.

(65) Symmetries of plane partitions, *J. Combinatorial Theory (A)* **43** (1986), 103–113. Erratum, **44** (1987), 310.

Various people beginning with MacMahon considered the enumeration of symmetry classes of plane partitions, but no one had systematically listed and discussed all the classes before. It turned out there were ten of them. I was pleased that I was able to enumerate one of them (self-complementary plane partitions) that had not been considered before. Subsequently all the open problems in my paper were solved by various people.

(66) Two poset polytopes, *Discrete Comput. Geom.* **1** (1986), 9–23.

The order polytope of a poset is implicit in my thesis and was also considered explicitly by Ladnor Geissinger (unpublished). I recall mentioning to Richard Dean that I did not know the number $e(Z_n)$ of linear extensions of the fence (or zigzag poset) Z_n. He expressed surprise that the number of linear extensions wasn't known for such a simple poset. A little later I realized that $e(Z_n)$ is just the number E_n of alternating permutations of $1, 2 \ldots, n$, about which much was known. In particular, the volume of the order polytope $\mathcal{O}(Z_n)$ is $E_n/n!$. In *SIAM Review* **27** (1985), 579–580, I solved a problem proposed by E. E. Doberkat by showing that the volume of some other polytope was also $E_n/n!$. I realized that this other polytope could be generalized to the chain polytope of any finite poset, eventually leading to the present paper.

(67) A baker's dozen of conjectures concerning plane partitions, in *Combinatoire Énumérative* (G. Labelle and P. Leroux, eds.), Lecture Notes in Math., no. 1234, Springer-Verlag, Berlin/Heidelberg/New York, 1986, pp. 285–293.

A summary of some open problems involving the enumeration of plane partitions and alternating sign matrices. The latter are in bijection with monotone triangles, which may be regarded as a variant of plane partitions. I was completely enthralled when I first heard the alternating sign matrix conjecture of Mills, Robbins, and Rumsey. My first thought was that this must be equivalent to a known result, since how could there be anything really new in the much-studied subject of plane partitions? However, I was never able to prove this conjecture. It was first given a famous proof by Doron Zeilberger, who farmed out pieces of the proof to many of his colleagues (including myself). I was assigned a combinatorial identity equivalent to Schur's product formula for the sum of all Schur functions, so I escaped with an easy task. Later a more direct proof was given by Greg Kuperberg. Since I had already thought about symmetries of plane partitions in [65], it was natural to extend this idea to symmetries of alternating sign matrices. Mill, Robbins, and Rumsey looked at the resulting problems and came up with many conjectures included in the present paper. I believe that all the conjectures have now been proved.

(68) The number of faces of balanced Cohen-Macaulay complexes and a generalized Macaulay theorem (with A. Björner and P. Frankl), *Combinatorica* **7** (1987), 23–34.

An outgrowth of [36]. A more definitive result was later obtained by Frankl, Füredi, and Kalai.

(69) Generalized h-vectors, intersection cohomology of toric varieties, and related results, in *Commutative Algebra and Combinatorics* (M. Nagata

and H. Matsumura, eds.), Advanced Studies in Pure Mathematics **11**, Kinokuniya, Tokyo, and North-Holland, Amsterdam/New York, 1987, pp. 187–213.

I learned about the generalized h-vector (now called the toric h-vector) of a (rational) convex polytope from Askold Khovanskii at the International Congress of Mathematicians, Warsaw, 1983. (Incidentally, this meeting was a fantastic opportunity to meet Russian mathematicians who were barred from traveling to the West. In particular, I had the great experience of being invited to the hotel room of Khovanskii and Alexander Varchenko in order to share some food they had brought with them. They did not have the funds to eat at restaurants.) The fundamental duality result $h_i = h_{d-i}$ for toric h-vectors was a combinatorial statement, but for Khovanskii it was a consequence of Poincaré duality for the intersection cohomology of a toric variety. Naturally I wanted to find a combinatorial proof, and in the greatest generality I could think of, which turned out to be Eulerian posets.

There is a strong analogy between the Hilbert polynomial of the face ring of a simplicial complex and the Ehrhart polynomial of an integral convex polytope. In the present paper I develop a theory of "acceptable functions on a lower Eulerian poset" in order to give an Ehrhart analogue of the toric h-vector.

(70) *Enumerative Combinatorics*, vol. 1, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986, xi + 306 pages; second printing, Cambridge University Press, New York/Cambridge, 1996. Second edition, Cambridge University Press, 2012.

This book had its genesis on the beaches of San Diego during the academic year 1978–79, when Pierre Leroux and I were both visiting UCSD. Little did I realize what a monstrous project was being unleashed! The original plan was that I would write the text and Pierre the exercises. I soon realized, however, that I wanted to do the whole thing myself. I had been collecting mathematical facts that I found interesting on 3×5 index cards, which became the foundation for the many exercises in EC1. Teaching courses on enumerative combinatorics at MIT was also essential for gathering and organizing the material. A very nice editor named John Kimmel convinced me to publish the book in a new Wadsworth & Brooks Cole Mathematics Series. When this series was discontinued the copyright was transferred to Cambridge University Press.

(71) A bound on the spectral radius of graphs with e edges, *Linear Alg. Appl.* **87** (1987), 267–269.

The idea for this paper came to me from listening to a lecture, probably by Richard Brualdi.

(72) Unimodal and log-concave sequences in algebra, combinatorics, and geometry, in *Graph Theory and Its Applications: East and West*, Ann. New York Acad. Sci., vol. 576, 1989, pp. 500–535.

A survey paper discussing the surprising diversity of methods for showing that sequences are unimodal or log-concave. I first started gathering this material systematically for an invited hour address I gave at the AMS annual meeting in Cincinnati, 1982.

(73) Some combinatorial properties of Jack symmetric functions, *Advances in Math.* **77** (1989), 76–115.

I learned of Jack symmetric functions, along with some intriguing open problems, from Ian Macdonald. During the 1986 winter and spring quarters at Caltech I was a Sherman Fairchild Distinguished Scholar, which meant I had a lot of time on my hands for research. I decided that I would take a look at Jack symmetric functions, using Phil Hanlon as a sounding board. Slowly I developed the necessary machinery until finally the last cog fell into place.

(74) On the number of faces of centrally-symmetric simplicial polytopes, *Graphs and Combinatorics* **3** (1987), 55–66.

This my only paper for which all the research was done on a plane. Anders Björner had made some intriguing conjectures about h-vectors (or f-vectors) of centrally symmetric simplicial polytopes. I realized that for such polytopes the group $\mathbb{Z}/2\mathbb{Z}$ acts on the face ring. Thus the face ring becomes a vector space direct sum of invariants and anti-invariants for this group action. Could this simple decomposition be of any use? This is another example of wishful thinking. I didn't think that such a simplistic approach would be at all fruitful, but I went ahead and did the computation. It turned out to give exactly the conjectured results!

(75) Plane partitions: past, present, and future, in *Combinatorial Mathematics: Proceedings of the Third International Conference*, Ann. New York Acad. Sci., vol. 555, pp. 397–401.

Essentially an extended abstract of a talk at the Third International Conference on Combinatorial Mathematics.

(76) Robinson-Schensted algorithms for skew tableaux (with B. Sagan), *J. Combinatorial Theory (A)* **55** (1990), 161–193.

When applying the theory of differential posets [77] to Young's lattice, one obtains some Cauchy-like identities involving skew Young tableaux. Naturally a combinatorial proof involving some analogue of the RSK algorithm is desired, since the RSK algorithm itself is used to give a bijective proof of the classical Cauchy identity for Schur functions. Bruce Sagan was a natural collaborator for this project since his thesis and some later work involved analogues of the RSK algorithm.

(77) Differential posets, *J. Amer. Math. Soc.* **1** (1988), 919–961.

At some point I realized that a number of identities involving Young tableaux and some of their variants were formal consequences of the trivial operator identity $\frac{\partial}{\partial p_1}p_1 - p_1\frac{\partial}{\partial p_1} = I$, the identity operator. Here $p_1 = x_1 + x_2 + \cdots$, the first power sum symmetric function. The theory of differential posets arose from a desire to generalize this observation as much as possible. One can think of this theory as the combinatorial theory of the Weyl relation $UD - DU = I$. Around the same time Sergey Fomin was developing a more general theory of "dual graded graphs."

(78) Variations on differential posets, in *Invariant Theory and Tableaux* (D. Stanton, ed.), The IMA Volumes in Mathematics and Its Applications, vol. 19, Springer-Verlag, New York, 1990, pp. 145–165.

A sequel to [77].

(79) Algebraic enumeration (with I. Gessel), in *Handbook of Combinatorics*, vol. 2 (R. L. Graham, M. Grötschel, and L. Lovász, eds.), Elsevier, Amsterdam, and MIT Press, Cambridge, 1995, pp. 1021–1061.

After agreeing to write with Ira Gessel this survey of algebraic enumeration, I became temporarily nonfunctional due to some back problems. Ira ended up writing the entire paper and generously insisted that I remain a coauthor.

(80) Further combinatorial properties of two Fibonacci lattices, *European J. Combinatorics* **11** (1990), 181–188.

In this paper I consider two lattices $\mathrm{Fib}(r)$ and $Z(r)$, $r \geq 1$. They are both (infinite) graded posets with a Fibonacci number F_{n-1} elements of rank n. $\mathrm{Fib}(1)$ had earlier been considered in [14] (using the notation $\boldsymbol{F_1}$) and $Z(r)$ in [77]. The posets $\mathrm{Fib}(r)$ and $Z(r)$ have a remarkable number of combinatorial properties in common, some which were not so easy to prove. It would be nice to find a more conceptual explanation of this phenomenon and to find some generalizations.

(81) A recurrence for linear extensions (with P. Edelman and T. Hibi), *Order* **6** (1989), 15–18.

A cute application of promotion of linear extensions. I later gave a survey of this theory in [146]. My student Benjamin Iriarte gave a completely different proof of the main result in his 2015 Ph.D. thesis.

(82) f-vectors and h-vectors of simplicial posets, *J. Pure Appl. Alg.* **71** (1991), 319–331.

Simplicial posets may be regarded as a generalization of simplicial complexes. In a simplicial poset, two faces can intersect in any subcomplex, not just a single face. A theory of h-vectors based on face rings can be developed analogously to simplicial complexes, but the theory is less rich and the proofs are easier, essentially because it is much easier to construct simplicial posets with desired properties. I wasn't quite able to complete the characterization of h-vectors of Gorenstein simplicial posets; this was done by Mikiya Masuda in 2003.

(83) A zonotope associated with graphical degree sequences, in *Applied Geometry and Discrete Combinatorics*, DIMACS Series in Discrete Mathematics, vol. 4, 1991, pp. 555–570.

I was aware since [40, Example 3.1] of a formula for the Ehrhart polynomial of an integral zonotope. At some point I realized that the polytope of degree sequences, first considered by Michael Koren after a suggestion from Micha Perles, is a zonotope $\mathcal{Z}_n$ and that the Ehrhart polynomial formula can be used to count the number $f(n)$ of ordered degree sequences of simple graphs on the vertex set $1, 2, \ldots, n$. Moreover, the f-vector of $\mathcal{Z}_n$ can be computed using results of Zaslavsky on signed graphs. The generating function $F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}$ is one of my favorite enumerative generating functions because the coefficient $f(n)$ has a simple combinatorial description, the formula for $F(x)$ looks very mysterious, the only known proof involves zonotope theory and linear algebra, and once the theory is understood each piece of the generating function has a nice conceptual explanation.

(84) Some applications of algebra to combinatorics, *Discrete Applied Math.* **34** (1991), 241–277.

A survey of some topics in algebraic combinatorics discussed in a series of lectures at a conference at George Washington University in 1989. The paper was actually written by Rodica Simion based on her notes from my lectures.

(85) On the Hilbert function of a graded Cohen-Macaulay domain, *J. Pure Appl. Algebra* **73** (1991), 307–314.

A complete characterization of the Hilbert function of a graded Cohen-Macaulay integral domain, generated in degree one, seems hopeless. In this paper I give a necessary condition based on the fact that such a ring R contains its canonical module as a graded ideal. The proof works if we assume only that R has a homogeneous system of parameters of degree one.

(86) Derangements on the n-cube (with W. Y. C. Chen), *Discrete Math.* **115** (1993), 64–75.

This paper was written with my academic brother William Yongchuan Chen when he was a graduate student at M.I.T. I was pleased to see him create a successful Center for Combinatorics at Nankai University that I have enjoyed visiting many times.

(87) On immanants of Jacobi-Trudi matrices and permutations with restricted position (with J. Stembridge), *J. Combinatorial Theory (A)* **62** (1993), 261–279.

Immanants are a generalization of determinants due to D. E. Littlewood. Ian Goulden and David Jackson in 1992 were the first to look at immanants of Jacobi-Trudi matrices. Further work of Curtis Greene and Mark Haiman led to some intriguing conjectures by John Stembridge. Here Stembridge and I make some progress on one of these conjectures. For further information, see the discussion below at [100].

(88) A monotonicity property of h-vectors and h^*-vectors, *European J. Combinatorics* **14** (1993), 251–258.

Gil Kalai had proved using his theory of algebraic shifting that if Δ is a d-dimensional Cohen-Macaulay simplicial complex and Δ' is a d-dimensional Cohen-Macaulay subcomplex of Δ, then $h_i(\Delta') \leq h_i(\Delta)$ for all i. Here I use commutative algebra techniques to give a simpler proof of a more general result (where we can take $\dim \Delta' < \dim \Delta$ under a suitable hypothesis) and extend the result to the "arithmetic" setting, that is, to h^* (or δ) vectors of Ehrhart polynomials. The proofs involve the interesting technique of using "special" systems of parameters with certain properties; it does not suffice to take a generic system of parameters.

(89) Subdivisions and local h-vectors, *J. Amer. Math. Soc.* **5** (1992), 805–851.

Gil Kalai and I had discussed the effect of subdivision on f-vectors and h-vectors. For instance, does a subdivision of a Cohen-Macaulay complex increase the h-vector? The present paper is devoted to this theory. The first part deals with simplicial complexes, and the second part to a very general setting that includes Ehrhart polynomials and Kazhdan-Lusztig polynomials. For simplicial complexes it turns out that there are several notions of subdivisions with quite different behaviors with respect to f-vectors. In fact I was originally unaware of this phenomenon until Clara Chan found a counterexample to one of the main results in the first draft. In regard to the proofs in the first part, it is necessary as in [88] to work with a special system of parameters. A key definition is that of the *local h-polynomial* $\ell_V(\Gamma, x)$ of a subdivision Γ of a simplex on the vertex set V. I find it interesting that characterizing when $\ell_V(\Gamma, x) = 0$ seems to be a very difficult problem. The second part includes a conjecture (Conjecture 7.14) on an Ehrhart generalization of the nonnegativity of local h-vectors. This conjecture was proved by Kalle Karu in 2006.

(90) A combinatorial decomposition of acyclic simplicial complexes, *Discrete Math.* **120** (1993), 175–182.

Gil Kalai used algebraic shifting to characterize f-vectors of acyclic (i.e., vanishing reduced homology) simplicial complexes. Kalai and Björner went on to characterize f-vectors of simplicial complexes (and more general types of cell complexes) with prescribed Betti numbers. I use some simple exterior algebra to obtain a combinatorial decomposition of acyclic complexes that implies Kalai's original result. I conjecture a generalization for any simplicial complex that would imply the Kalai-Björner result. I could only prove a weak form of my conjecture, but Art Duval managed to prove the entire conjecture one year later. A further generalization of the acyclic decomposition theorem is given in Conjecture 2.4 and remains open, and even more general conjectures are possible.

(91) Review of H. S. Wilf, *Generatingfunctionology*, *American Mathematical Monthly* **97** (1990), 864–866.

Herb Wilf and I both regretted that we never had a joint paper. We actually wrote a paper together on Stern sequences before discovering the result was already known. At least I had the opportunity to review Herb's beautiful treatment of generating functions.

(92) Some combinatorial aspects of the spectra of normally distributed random matrices (with P. Hanlon and J. Stembridge), *Contemporary Mathematics* **158** (1992), 151–174.

Colin Mallows brought to my attention a problem arising in statistics concerning the expectation of matrix functions $f(AUBU^t)$, where A and B are fixed, and the expectation is over all U in a classical compact group such as the unitary group $U(n)$. The expectation is symmetric in the eigenvalues of A and in the eigenvalues of B, so symmetric function theory comes into play. I mentioned this problem to Phil Hanlon and John Stembridge, resulting in the present paper.

(93) Some combinatorial properties of Schubert polynomials (with S. Billey and W. Jockusch), *J. Algebraic Combinatorics* **2** (1993), 345–374.

Once I learned that the symmetric function F_w of [56] was a limit of Schubert polynomials, it was natural to ask whether there was a formula for the Schubert polynomials themselves analogous to that for F_w. There turned out to be a remarkably elegant conjecture, but I was stuck on finding a proof. I mentioned this to Sara Billey, who was visiting M.I.T. at the time, and she got together with Willliam Jockusch, an M.I.T. graduate student of Jim Propp's, to prove the conjecture. I added some additional combinatorial material on Schubert polynomials, including the result that the reduced decompositions of a permutation w contain no "braid relations" $s_i s_{i+1} s_i$ or $s_{i+1} s_i s_{i+1}$ if and only if w is 321-avoiding. This observation began the theory of fully commutative elements of Coxeter groups.

(94) Schubert polynomials and the nilCoxeter algebra (with S. Fomin), *Advances in Math.* **103** (1994), 196–207.

Sergey Fomin showed me his beautiful proofs of the combinatorial formula for the Schubert polynomials $\mathfrak{S}_w$ given in [93] and of a formula of Macdonald for $\mathfrak{S}_w(1,1,1,\dots)$. My contribution to our joint paper was a proof of a formula for $\mathfrak{S}_w(1,q,q^2,\dots)$ conjectured by Macdonald. The proof is an extension of the nilCoxeter techniques that Fomin used to prove the $q=1$ case.

(95) Sets of vectors with many orthogonal pairs (with Z. Furedi), *Graphs and Combinatorics* **8** (1992), 391–394.

Erdős raised the problem of determining the most number $\alpha^{(d)}(k)$ of vectors in $\mathbb{R}^d$ such that any $k+1$ contain an orthogonal pair. Zoltan Furedi and I obtained some results on this problem, though the exact value of $\alpha^{(d)}(k)$ remains open.

(96) Flag f-vectors and the cd-index, *Math. Zeitschrift* **216** (1994), 483–499.

On a sabbatical visit to the Institut Mittag-Leffler in 1992 I decided to think about the cd-index $\Phi_P(c,d)$ of an Eulerian poset P. The cd-index is a noncommutative polynomial in c and d, first defined by Jonathan Fine (when P is the face lattice of a convex polytope), that efficiently encodes the flag f-vector (or flag h-vector) of P. A central question is when the coefficients of $\Phi_P(c,d)$ are nonnegative. I conjectured that this is true whenever P is a Gorenstein* poset. Using a shelling argument I proved this conjecture when P is the face lattice of a convex polytope. The full conjecture was proved by Kalle Karu in 2006.

(97) The general theory of convolutional codes (with R. J. McEliece), *The Telecommunications and Data Acquisition Progress Report 42-113, January–March 1993* (E. C. Posner, ed.), National Aeronautics and Space Administration, Jet Propulsion Laboratory (May 15, 1993), pp. 89–98.

Bob McEliece, one of my colleagues when I worked at JPL, contacted me about a survey paper he was writing on convolutional codes. He had some questions about a generating function called the *Hilbert series* of the code. I was able to answer these questions and thus became a coauthor.

(98) Flag-symmetric and locally rank-symmetric partially ordered sets, *Electronic J. Combinatorics* **3**, R6 (1996), 22 pp.; reprinted in *The Foata Festschrift* (J. Désarménien, A. Kerber, and V. Strehl, eds.), Imprimerie Louis-Jean, Gap, 1996, pp. 165–186.

Richard Ehrenborg had defined a quasisymmetric function $F_P(x)$ that encodes the flag f-vector (or flag h-vector) of a graded poset P. I thought that it would be interesting to investigate when $F_P(x)$ was actually a symmetric function, and then when this holds to expand $F_P(x)$ in terms of various symmetric function bases. In order to bring skew Schur functions into the picture, I also develop a theory of relative posets and relative lexicographic shellability.

(99) A survey of Eulerian posets, in *Polytopes: Abstract, Convex, and Computational* (T. Bisztriczky, P. McMullen, R. Schneider, A. I. Weiss, eds.), NATO ASI Series C, vol. 440, Kluwer Academic Publishers, Dordrecht/Boston/London, 1994, pp. 301–333.

It is rather surprising how much can be said about this simply-defined class of posets.

(100) A symmetric function generalization of the chromatic polynomial of a graph, *Advances in Math.* **111** (1995), 166–194.

This paper arose when I realized that the results and conjectures of [87] could be reformulated in terms of a symmetric function generalization X_G of the chromatic polynomial of a graph G. In this setting, the main conjecture of [87] states that if G is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-avoiding poset, then X_G is e-positive (a polynomial in the elementary symmetric functions with nonnegative coefficients). The most important work on this conjecture to date was done by Mathieu Guay-Paquet in 2013. He showed that it suffices to prove the conjecture for incomparability graphs of posets that avoid both $\mathbf{3}+\mathbf{1}$ and $\mathbf{2}+\mathbf{2}$, known as *unit interval graphs*.

Once I defined the chromatic symmetric function X_G, it was natural to see how to expand it in terms of various symmetric function bases and to relate it to some other symmetric function concepts. I consider these results to be rather routine except for Theorem 3.3, which concerns expressing X_G in terms of elementary symmetric functions. The result is a generalization of the main result $\mathfrak{o}(G) = (-1)^p\chi_G(-1)$ of [18]. The proof of Theorem 3.3 is rather indirect and uses the theory of quasisymmetric functions. It would be interesting to find a nicer proof.

(101) Graph colorings and related symmetric functions: ideas and applications, *Discrete Math.* **193** (1998), 267–286.

The somewhat awkward title of this paper is explained by taking a look at its title page. It is a sequel to [100] and includes generalizations of the chromatic symmetric function to Tutte polynomials and to hypergraphs. These generalizations seem less interesting than the chromatic symmetric functions themselves and have received little attention

(102) Lê numbers of arrangements and matroid identities (with D. Massey, R. E. Simion, D. L. Vertigan, D. J. A. Welsh, and G. M. Ziegler), *J. Combinatorial Theory (B)* **70** (1997), 118–133.

I made a contribution to the original version and so became a coauthor.

(103) Polygon dissections and standard Young tableaux, *J. Combinatorial Theory (A)* **76** (1996), 175–177.

A result first completely proved by Cayley gives a simple formula for the number $f(n,d)$ of ways to draw d noncrossing diagonals in a convex $(n+2)$-gon. This result seems more difficult to prove than a number of similar results. Kathy O'Hara and Andrei Zelevinsky observed that $f(n,d)$ is in fact the number of standard Young tableaux of shape $(d+1, d+1, 1^{n-1-d})$. Naturally a bijective proof is wanted, and I give a simple one in the present paper. Thus the formula for $f(n,d)$ becomes a special case of the hook-length formula.

(104) A matrix for counting paths in acyclic digraphs, *J. Combinatorial Theory (A)* **74** (1996), 169–172.

The main result of this paper is not difficult to prove and in fact follows from a more general result of Ian Goulden and David Jackson, but no one had stated it explicitly before.

(105) Hyperplane arrangements, interval orders, and trees, *Proc. Nat. Acad. Sci.* **93** (1996), 2620–2625.

A survey of some new results on the characteristic polynomial and number of regions of certain hyperplane arrangements related to the braid arrangement. Proofs and further aspects appear in [107] and [109]. My favorite result in the present paper is Theorem 3.1 on interval orders with generic interval lengths. However, unlike some of the other results in the paper, Theorem 3.1 has not spawned any applications or generalizations.

(106) Hipparchus, Plutarch, Schröder, and Hough, *American Math. Monthly* **104** (1997), 344–350.

Around 1900 years ago Plutarch made a famous statement to the effect that Hipparchus had shown that 103,049 compound statements can be made from ten simple statements. No one had made any sense of this statement (and indeed, there was little assurance that the number 103,049 had been transmitted correctly through the ages) until David Hough, a graduate student at George Washington University, observed that 103,049 is the number of plane trees with ten endpoints! Hough was reluctant to write up this astonishing discovery, so the honor of doing so fell on me. It is the only paper I have published in the *American Mathematical Monthly.* When I had completed the paper I sent a copy to Wilbur Knorr, perhaps the greatest living historian of ancient Greek mathematics. He sent back a very nice letter with many interesting comments. His main criticism of the paper was its title, which he thought was too frivolous. I did not realize until later that at the time he sent his letter, he was in a late stage of terminal melanoma. A much more detailed and scholarly account of Hough's discovery was later given by Fabio Acerbi in 2003, followed by some further work by Susanne Bobzien in 2011.

(107) Hyperplane arrangements, parking functions, and tree inversions, in *Mathematical Essays in Honor of Gian-Carlo Rota* (B. Sagan and R. Stanley, eds.), Birkhäuser, Boston/Basel/Berlin, 1998, pp. 359–375.

One of the results mentioned in [105] is an ingenious method to label the regions of the Shi arrangement $\mathcal{S}_n$ conjectured by Igor Pak. Pak's labeling by parking functions makes it easy to describe the distance enumerator of $\mathcal{S}_n$ (with respect to a natural base region) in terms of tree inversions. In the present paper I prove a generalization of Pak's conjecture to the extended Shi arrangement $\mathcal{S}_n^k$. Originally I did the case $k = 1$ (the usual Shi arrangement), and Catherine Huafei Yan (my academic sister) extended the proof to $k = 2$. After seeing the $k = 2$ proof it was evident how to handle the case of general k. Catherine went on to write several papers on parking functions and is a leading expert on this topic. The present paper appears in a volume dedicated to my thesis adviser Gian-Carlo Rota on the occasion of his 64th birthday. I was the main organizer of a conference at M.I.T. for Rota's 64th birthday. It was a huge amount of work but well worth the effort, especially since Rota unexpectedly passed away two years later.

(108) A q-deformation of a trivial symmetric group action (with P. Hanlon), *Trans. Amer. Math. Soc.* **350** (1998), 4445–4459.

Let $\Gamma_n(q) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} w \in \mathbb{C}\mathfrak{S}_n$, where $q \in \mathbb{C}$, $\mathbb{C}\mathfrak{S}_n$ denotes the group algebra of the symmetric group $\mathfrak{S}_n$, and $\mathrm{inv}(w)$ denotes the number of inversions of w. Alexander Varchenko raised a question equivalent to determining the $\mathfrak{S}_n$-module structure of the kernel of $\Gamma_n(q)$ acting on $\mathbb{C}\mathfrak{S}_n$ by left multiplication, for those values of q for which $\Gamma_n(q)$ is singular (known to occur if and only if $q^{j(j-1)} = 1$ for some $2 \leq j \leq n$). It was a lot of fun to work with Phil Hanlon on this problem. We completely solved it for q a primitive $n(n-1)$st root of unity, and we determined the determinant of $\Gamma_n(q)$ (for any q) acting on the isotypic components of $\mathbb{C}\mathfrak{S}_n$. The key to the proof is a remarkable factorization theorem for $\Gamma_n(q)$ due to Don Zagier, and the idea of looking at the action of $\Gamma_n(q)$ on certain virtual $\mathfrak{S}_n$-modules. I believe that this paper and [92] are the only papers I wrote with someone who later became a university president. (David Roselle is another combinatorialist whom I know and whose work was of interest to me, and who later became a university president, in fact, of two different universities.)

(109) Deformations of Coxeter hyperplane arrangements (with A. Postnikov), *J. Combinatorial Theory (A)* **91** (2000), 544–597.

A greatly expanded version, including proofs, of [105]. Alexander Postnikov's (large) contribution became his doctoral thesis. One cute result is that the characteristic polynomial of certain arrangements satisfy a "Riemann hypothesis," i.e., their zeros have the same real part. I had a pleasant fantasy that the limit of some of these polynomials was a suitable modification of the Riemann zeta function, so that the actual Riemann hypothesis would be a nice corollary to our work. Alex showed, however, that the limiting function is (essentially) $\sin z$. Thus we don't have a proof of the Riemann hypothesis, but we do have a new proof that the complex zeros of $\sin z$ are real!

(110) Parking functions and noncrossing partitions, *Electronic J. Combinatorics* **4**, R20 (1997), 14 pp.

The lattice $P = \mathrm{NC}_n$ of noncrossing partitions of an n-set has the property that the quasisymmetric function F_P of [98] is symmetric. It turns out that F_P is a "parking function symmetric function" previously considered by Haiman. Moreover, the number of maximal chains of NC_n is n^{n-2} (first shown by Kreweras), which is also the number of parking functions of length $n-1$. This motivated me to find a bijection between the maximal chains of NC_n and parking functions with the additional desirable property of giving an EL-labeling of NC_n.

(111) Flag-symmetry of the poset of shuffles and a local action of the symmetric group (with R. Simion), *Discrete Math.* **204** (1999), 369–396.

I discuss how this paper arose in [125]. The present paper is my only "real" joint paper with Rodica Simion. (I don't consider [102] to be a true collaboration.) Shuffle posets W_{MN} (first defined by Curtis Greene) are another example where the quasisymmetric function F_P of [98] is a symmetric function, as first noticed by Rodica. It turns out to be a very interesting example. Moreover, the semigroup of multiplicative functions naturally associated with W_{MN} (in the limit $M, N \to \infty$) has a very curious description (Theorem 5.2). I have often wondered whether this semigroup occurs in other contexts. The proof of Theorem 5.2 is a good example of the efficacy of the notation (or some variation of it), espoused by Adriano Garsia and Donald Knuth among others, $\chi(\mathcal{P}) = 1$ if the proposition $\mathcal{P}$ is true, and $\chi(\mathcal{P}) = 0$ if $\mathcal{P}$ is false.

(112) A combinatorial miscellany (with A. Björner), *L'Enseignement Math.*, Monograph no. 42, 2010.

This long survey paper of various topics in enumerative, algebraic, and topological combinatorics began around 1994 when Cambridge University Press asked Anders Björner and me to contribute to a book entitled *New Directions in Mathematics*. The book project fizzled, and about 60% of a draft of our paper lay around for ten years or so until Tatiana Nagnibeda-Smirnova approached us about publishing it in the *L'Enseignement Math.* monograph series. I remember that one issue that occupied a lot of our time was the choice of title, until I got the idea of borrowing from Littlewood's *A Mathematician's Miscellany.*

(113) *Enumerative Combinatorics*, vol. 2, Cambridge University Press, New York/Cambridge, 1999, xii + 581 pages.

This took about thirteen years of work after EC1 appeared. Of course I was doing other things during this time.

(114) Spanning trees and a conjecture of Kontsevich, *Ann. Combinatorics* **2** (1998), 351–363.

Maxim Kontsevich mentioned in a lecture attended by Jeff Lagarias (but not by me) a conjecture related to the number $f_G(q)$ of $\mathbb{F}_q$-rational points on a variety associated to the spanning trees of a graph G. In particular, $f_G(q)$ should be a universal polynomial in q, i.e., a polynomial in q independent from the characteristic p of the field $\mathbb{F}_q$. After looking at Jeff's notes, I realized that the conjecture is true for the complete graph K_n (and some variants) because by the Matrix-Tree theorem it reduces to counting $n \times n$ symmetric invertible matrices over $\mathbb{F}_q$, which was well-known to be a universal polynomial. When I sent an email to Kontsevich saying that I could prove his conjecture for K_n, he became very excited and gave me a phone call, because he thought that K_n was the most complicated graph. If the conjecture were true in that case, then it must be true for all G. However, when I explained the proof he became more subdued, since the situation for K_n is actually quite special. In fact, in the present paper I give a heuristic argument based on some computations of John Stembridge why Kontsevich's conjecture might be false. Indeed, Pralash Belkale and Patrick Brosnan showed in 2000 that Kontsevich's conjecture is false, and in 2009 Oliver Schnetz gave the first explicit counterexamples.

(115) Domino tilings with barriers (with J. Propp), *J. Combinatorial Theory (A)* **87** (1999), 347–356.

An application of symmetric function theory to the enumeration of domino tilings.

(116) Positivity problems and conjectures in algebraic combinatorics, in *Mathematics: Frontiers and Perspectives* (V. Arnold, M. Atiyah, P. Lax, and B. Mazur, eds.), American Mathematical Society, Providence, RI, 2000, pp. 295–319.

It was quite an honor to be invited to contribute to a book celebrating the state of mathematics at the end of the millennium, and edited by Arnold, Atiyah, Lax, and Mazur.

(117) A polytope related to empirical distributions, plane trees, parking functions, and the associahedron (with J. Pitman), *Discrete Comput. Geom.* **27** (2002), 603–634.

Jim Pitman asked me if one could say more about a polytope he defined that was associated with uniform order statistics and empirical distributions. It turned out to have lots of interesting properties and connections with other combinatorial topics. It is now known as the "Pitman-Stanley polytope." In 2005 Alex Postnikov developed a theory of a large class of polytopes called *generalized permutohedra*, which includes the Pitman-Stanley polytope.

(118) A generalized riffle shuffle and quasisymmetric functions, *Ann. Combinatorics* **5** (2001), 479–491.

After seeing the work of Dave Bayer and Persi Diaconis on riffle shuffles, I realized that their work could be elegantly described and generalized using quasisymmetric functions. In fact, it seems quite remarkable how these two previously studied concepts are so intimately related.

(119) A note on the symmetric powers of the standard representation of S_n (with D. Savitt), *Electronic J. Combinatorics* **7**, R6 (2000), 8 pp.

In 1999 David Savitt, a graduate student at Harvard, sent me an email concerning the dimension $D(n)$ of the space spanned by the characters of the symmetric powers of the defining n-dimensional representation of $\mathfrak{S}_n$. My first reaction is that this space should contain all the characters, but he showed me why this was false and in fact found an upper bound asymptotic to $\frac{1}{2}n^2$. My contribution was to give a lower bound that was also asymptotic to $\frac{1}{2}n^2$. The problem of finding a "nice" formula or generating function for $D(n)$ remains open.

(120) Rodica Simion, January 18, 1955 – January 7, 2000, *Pi Mu Epsilon Journal* **11** (2000), 83–86.

I really enjoyed working with Rodica Simion. Her enthusiasm and love of mathematics was quite infectious, and I was glad to have the opportunity to write this short description of her career. Rereading it just now reminds me of the great loss to combinatorics caused by her untimely death.

(121) Recent progress in algebraic combinatorics, *Bull. Amer. Math. Soc.* **40** (2003), 55–68.

In 2000 the American Mathematical Society held a conference at UCLA on "Mathematical Challenges of the 21st Century," at which I had the honor of being a speaker. (Eight of the 30 speakers were Fields Medalists.) Some of these talks including my own were published in an issue of the AMS *Bulletin*.

(122) On the enumeration of skew Young tableaux, *Advances in Applied Math.* **30** (2003), 283–294.

Brendan McKay, Jennifer Morse, and Herb Wilf considered the asymptotic behavior of the number $N(n;\alpha)$ of standard Young tableaux with n squares that contain a fixed SYT T of shape α. (Clearly this number depends only on n and the shape α of T.) Using the theory of symmetric functions I give an explicit formula for $N(n;\alpha)$ and then consider the asymptotics of the number $f^{\lambda/\alpha}$ of skew SYT of shape λ/α for fixed α. There are two different formulas depending on how we let λ become large. Andrei Okounkov saw an early version of the manuscript and contributed some interesting historical background.

(123) The rank and minimal border strip decompositions of a skew partition, *J. Combinatorial Theory (A)* **100** (2002), 349–375.

The *rank* of an integer partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is the largest r for which $\lambda_r \geq r$. It is not so clear how to generalize this definition to skew shapes. In 2000 Maxim Nazarov and Vitaly Tarasov gave such a definition, which I found very interesting. I give four equivalent definitions of this concept of $\mathrm{rank}(\lambda/\mu)$ and conjecture a fifth. This conjecture was proved in 2005 by William Yongchuan Chen and Arthur Libo Yang. The different characterizations of rank motivated a theory of border strip decompositions of skew shapes, which is included in the present paper. This theory involves a curious symmetric function obtained from a Schur function by expanding in terms of power sums and keeping only the terms of lowest degree with respect to the grading $\deg p_i = 1$ for all $i \geq 1$. My student Peter Clifford and I further looked at these symmetric functions, which we call *bottom Schur functions*, in [132]. Peter Clifford also developed a theory of rank for shifted shapes which is included in his 2003 Ph.D. thesis.

(124) Irreducible symmetric group characters of rectangular shape, *Sém. Lotharingien de Combinatoire* (electronic) **50** (2003), B50d.

Let λ be a partition of the rectangular shape $p \times q$. I use symmetric functions to obtain a new formula for the value χ^λ_ν of the irreducible character χ^λ of $\mathfrak{S}_n$ evaluated at a permutation of cycle type $\nu = (\mu, 1^{pq-k})$, where μ is a fixed partition of k. The formula is a polynomial in p and q whose coefficients are described explicitly.

I then show that one can extend the formula to arbitrary λ. We obtain a polynomial F_λ in parameters $p_1, \dots, p_m$ and $q_1, \dots, q_m$ now called the "Stanley coordinates" of λ. However, I was unable to give a combinatorial interpretation of the coefficients of F_λ when λ is not a rectangle. Joseph Katriel showed that the sum of the coefficients of F_λ is $(k+m-1)_k := (k+m-1)(k+m-2)\cdots m$. Thus the coefficients of F_λ are a refinement of the number of injective functions $[k] \to [k+m-1]$, so I searched unsuccessfully for this refinement.

Three years later I was preparing a lecture for an undergraduate course on combinatorics on the basic formula $x(x+1)\cdots(x+n-1) = \sum_{k=1}^n c(n,k)x^k$, where $c(n,k)$ is the number of permutations in $\mathfrak{S}_n$ with k cycles. I suddenly had the insight that for a combinatorial description of F_λ we should regard $(k+m-1)_k$ as counting permutations $w \in \mathfrak{S}_k$ whose cycles are "colored" with the integers $1, 2, \ldots, k + m - 1$. The rectangular case suggested that we needed to define some kind of product of an ordinary permutation with a colored permutation that produced another colored permutation. With this insight, it didn't take long to find an explicit conjecture. I was unable to prove it, so I posted it on the math arXiv [143]. A brilliant proof was found by Valentin Féray shortly after [143] appeared. I was happy to see Féray, Philippe Biane, Piotr Sniady, and others find interesting applications to character asymptotics and related topics.

(125) Rodica Simion and shuffle posets, *Advances in Applied Math.* **28** (2002), 282–284.

I had another opportunity to write about the work of Rodica for this special issue in her honor. My final sentence in this paper is still true after 15 years: "It is a great loss for all of combinatorics, as well as for me personally, that Rodica is no longer here to discover beautiful new mathematics and to inspire other researchers to do the same."

(126) Some remarks on sign-balanced and maj-balanced posets, *Advances in Applied Math.* **34** (2005), 880–902.

Frank Ruskey was the first person to consider the enumeration of linear extensions w of posets P according to their sign, i.e., whether w is an odd or even permutation. Dennis White obtained some beautiful results when P is a product of two chains. In the present paper I look at a number of further aspects. I especially like Conjecture 3.6, which remains wide open. In my thesis I use the theory of P-partitions to give a proof that the number of linear extension of certain posets P is even. Although this is a quite minor result, I had always been curious about proving it combinatorially. Such a proof dropped out essentially by accident in the present paper (Corollaries 2.2 and 2.4). In fact, the posets P have the same number of odd linear extensions as even linear extensions.

(127) A map on the space of rational functions (with G. Boros, J. Little, V. Moll, and E. Mosteig), *Rocky Mountain J. Math.* **35** (2005), 1861–1880.

Victor Moll had an interesting paper in the *AMS Notices* on the evaluation of certain integrals. It included a conjecture related to Eulerian polynomials that I was able to prove, so I was added as a coauthor to the present paper.

(128) The mathematical knight (with N. Elkies), *Math. Intelligencer* **25**, no. 1 (Winter 2003), 22–34.

Noam Elkies and I both have an interest in chess problems, though Noam is much more of an expert than me. Twenty years ago or so we decided to write a book on chess and mathematics. It is unclear whether this book will ever see the light of day. However, we did write a paper for the *Intelligencer* on some mathematical aspects of the chess knight as a sort of trailer for our book.

(129) Recent developments in algebraic combinatorics, *Israel J. Math.* **143** (2004), 317–340.

A continuation of [121].

(130) A super-class walk on upper-triangular matrices (with E. Arias-Castro and P. Diaconis), *J. Algebra* **278** (2004), 739–765.

I was happy to have a joint paper with Persi Diaconis after many stimulating mathematical discussions with him. The paper concerns a random walk on the group G_n of upper-triangular $n \times n$ matrices over a finite field. The usual character theory techniques cannot be used because G_n has a “wild” (hence intractable) representation theory. However, an elegant supercharacter theory had been developed by “merging” certain sets of irreducible characters into a single supercharacter. Thus the supercharacter theory could be applied to random walks. As an interesting sidelight, in 2010 there was a workshop at the American Institute of Mathematics on developing a connection between the supercharacters of G_n and combinatorial Hopf algebras which resulted in a paper authored by all 28 workshop participants (of which I was not one).

(131) Properties of some character tables related to the symmetric groups (with C. Bessenrodt and J. Olsson), *J. Algebraic Combinatorics*, **21** (2005), 163–177; `math.CO/0403110`.

I noticed that Hall-Littlewood functions could be used to give a simpler proof of a formula of Jørn Olsson for the determinant of certain submatrices of the character table of $\mathfrak{S}_n$. As a result I became a coauthor of the present paper.

(132) Bottom Schur functions (with P. Clifford), *Electronic J. Combinatorics* **11**(1) (2004), R67.

A continuation of [123]. The present paper has a couple of results that I find rather curious and amusing. The first is that the dimension of the space of bottom Schur functions of (usual) degree n is equal to the number of partitions of n whose parts differ by at least 2. Could there be a connection with the Rogers-Ramanujan identities? The second result says that if we define certain normalized power sums $\tilde{p}_\lambda$ and monomial symmetric functions $\tilde{m}_\lambda$, and if the bottom Schur function $\hat{s}_\lambda$ has the expansion $\sum_\mu c_\mu \tilde{p}_\mu$, then $\sum_\mu c_\mu p_\mu = \sum_\mu c_\mu \tilde{m}_\mu$. I should mention that the results of this paper are included in the 2003 Ph.D. thesis of Peter Clifford.

(133) Coefficients and roots of Ehrhart polynomials (with M. Beck, J. De Loera, M. Develin, and J. Pfeifle), *Contemp. Math.* **374** (2005), 15–36.

I made a small contribution to the original manuscript and so became a coauthor.

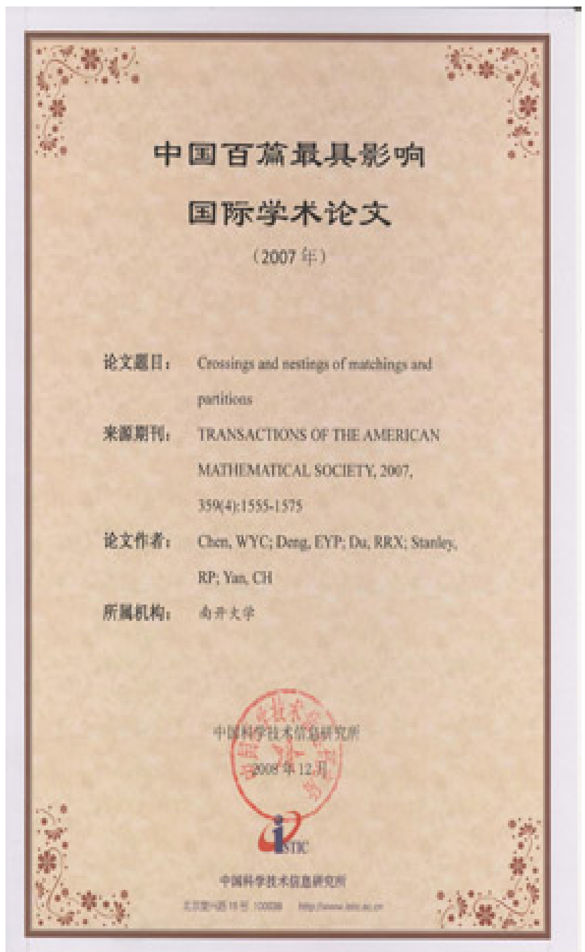
(134) Tilings (with F. Ardila), *Math. Intelligencer* **32** (2010), 32–43.

German translation by Günter Ziegler entitled “Pflasterungen” in *Math. Semesterber.* **53** (2006), 17–43. Spanish translation by Carolina Benedetti entitled “Teselaciones” in Gaceta de la Real Sociedad Matemática Española **14** (2011), 463–490.

Based on a Clay Public Lecture given at a Park City Mathematics Institute program on geometric combinatorics in 2004. Federico Ardila did a great job of writing the text from my notes.

(135) Crossings and nestings of matchings and partitions (with William Y. C. Chen, Eva Y. P. Deng, Rosena R. X. Du, and Catherine H. Yan), *Trans. Amer. Math. Soc.* **359** (2007), 1555–1575.

This paper is another example of the wishful thinking proof technique. When visiting Nankai University in 2004 Rosena Ruoxia Du explained to me some work she had done with William Yongchuan Chen and Eva Yuping Deng on crossings and nestings of matchings. There were many open questions. I knew that there was a bijection between oscillating tableaux and matchings, so I wondered (without any real basis) whether this connection might be useful for crossings and nestings. It turned out to be the perfect tool and led to some further generalizations. Catherine Huafei Yan then joined the list of authors and made some significant contributions. The paper won an award in China for the "Top 100 most cited Chinese papers published in international journals" in 2007. Christian Reidys and others later gave some applications to biology.



中国百篇最具影响

国际学术论文

（2007年）

论文题目： Crossings and nestings of matchings and partitions

来源期刊： TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, 2007, 359(4):1555-1575

论文作者： Chen, WYC; Deng, EYP; Du, RRX; Stanley, RP; Yan, CH

所属机构： 南开大学

中国科学技术信息研究所

2008年12月

ISTIC

中国科学技术信息研究所

(136) Ordering events in Minkowski space, *Advances in Applied Math.* **37** (2006), 514–525.

Mark I. Heiligman from the National Security Agency mentioned to me a connection between sequences of events in Minkowski spacetime and the theory of hyperplane arrangements. It is well-known in the special theory of relativity that the order of two events can depend on the reference frame of the observer. If we have k events in $(n+1)$-dimensional

Minkowski space (n space dimensions and one time dimension), what is the most possible number of different orders in which they can occur? This question can be answered precisely by counting the number of regions in a certain arrangement. For instance, when $n = 3$ (corresponding to the space in which we live), the most number of different orders is

$$\frac{1}{48}(k^6 - 7k^5 + 23k^4 - 37k^3 + 48k^2 - 28k + 48),$$

not what anyone would have guessed! For instance, four events can occur in any order, but five events can occur in at most 96 orders.

(137) Chains in the Bruhat order (with A. Postnikov), *J. Algebraic Combinatorics* **29** (2009), 133–174.

I was always fond of the enumeration of saturated chains from $\hat{0}$ (the identity element of $\mathfrak{S}_n$) to w in the weak (Bruhat) order on $\mathfrak{S}_n$ (the main topic of [56]) and of a certain weighted enumeration of such chains due to Macdonald. This weighted enumeration is equivalent to the Schubert polynomial evaluation $\mathfrak{S}_w(1, 1, 1, \dots)$ discussed after [94]. It seems natural to ask for something similar for the (strong) Bruhat order on $\mathfrak{S}_n$. In fact, around 1958 Claude Chevalley gave a formula of this nature for the degree of a Schubert variety X_w of type A, while John Stembridge in 2002 independently looked more at the formula when $w = w_0$, the element of longest length in $\mathfrak{S}_n$. I saw how Stembridge's work could be extended to some other elements w. Alex Postnikov greatly expanded my results, and we became coauthors. Some work remains to be done in this area, such as our Conjecture 16.1.

(138) Queue problems revisited, *Suomen Tehtäväniekat* **59**, no. 4 (2005), 193–203.

The most interesting feature of this paper is the obscure journal in which it appears, the journal of the Finnish Chess Problem Society. There is a special type of chess problem called a "queue problem" in which the object is to find the number of ways to achieve some stipulation in a specified number of moves. All solutions have the same set of moves, but the rules of chess force some moves to occur before others. Thus the total number of solutions is the number of linear extensions of a poset, a subject of great interest to me since my Ph.D. thesis. Most of the previous queue problems were published by Finnish chess problem composers, so it made sense for me to contribute my own ideas to *Suomen Tehtäväniekat.* In one of the problems, the number of solutions is the Catalan number $C_{17} = 129644790$ and for another the Euler number $E_7 = 272$ (the number of alternating permutations in $\mathfrak{S}_7$). In 2006 Antti Karttunen extended this idea to $E_9 = 7936$ solutions. Both the C_{17} and E_9 problems appear in [70].

(139) The descent set and connectivity set of a permutation, *J. Integer Sequences* (electronic) **8** (2005), article 05.3.8.

This paper arose from preparing an exercise for EC1 on connected permutations, that is, permutations $a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ such that $\{a_1, \dots, a_i\} \neq \{1, \dots, i\}$ for $1 \leq i \leq n - 1$.

(140) Longest alternating subsequences of permutations, *Michigan Math. J.* **57** (2008), 675–687.

There has been a lot work on the longest increasing subsequence of a permutation (see [141]). For instance, a famous result of Logan-Shepp and Vershik-Kerov states that the expected length $E(n)$ of the longest increasing subsequence of a random permutation $w \in \mathfrak{S}_n$ satisfies $E(n) \sim 2\sqrt{n}$. An even more spectacular result of Baik-Deift-Johansson asserts that the length of the longest increasing subsequence, when suitable normalized, has the Tracy-Widom distribution as a limiting distribution. I decided to see if anything could be said about the longest *alternating* subsequence $b_1 b_2 \cdots b_k$ of a permutation $w \in \mathfrak{S}_n$, i.e., $b_1 > b_2 < b_3 > b_4 < \cdots$. Surprisingly some very explicit results can be obtained with much less work than for increasing subsequences. For instance, the expected length of the longest alternating subsequence of $w \in \mathfrak{S}_n$ is exactly $(4n+1)/6$ for $n \geq 2$. I wondered whether a limiting distribution existed. Maybe it was a new distribution that would then be called the "Stanley distribution." Alas, several persons showed that the limiting distribution was just a Gaussian distribution. It seems unlikely that mathematicians are going to change the name of the Gaussian (or normal) distribution to the Stanley distribution.

Although this paper has nothing to do with the work of Mel Hochster, I was nevertheless pleased that it appeared in a volume of *Michigan Math. J.* in honor of his 65th birthday. During my "commutative algebra phase" Mel was extremely helpful to me and generous with his time and ideas on numerous occasions.

(141) Increasing and decreasing subsequences and their variants, *Proc. Internat. Cong. Math. (Madrid, 2006)*, vol. 1, American Mathematical Society, Providence, RI, 2007, pp. 545–579.

A survey for a plenary talk at the Madrid ICM. My talk was around 9:00 a.m. Since people eat dinner in Madrid around 10:00 p.m. at the earliest, I was very grateful to anyone who managed to show up for my talk.

(142) Alternating permutations and symmetric functions, *J. Combinatorial Theory (A)* **114** (2007), 436–460.

An early pioneer in the combinatorial aspects of symmetric functions was Herbert Owen Foulkes, whom I met once at Oberwolfach in 1973 (my first overseas trip). I was especially intrigued by his result that for a certain skew shape (a "ribbon staircase") all the values of the corresponding character of $\mathfrak{S}_n$ were either 0 or $\pm E_k$, where E_k is an Euler number. I wondered whether there could be some combinatorial applications to alternating permutations. In the present paper I show how to do this using umbral techniques. In general I am somewhat skeptical about the value of umbral calculus, but here it is used in an essential way. In particular, I obtain some simple explicit formulas, such as for the number of alternating permutations in $\mathfrak{S}_n$ that are also cycles, which I don't know how to obtain without this representation theoretic and umbral machinery. I am sorry that I did not obtain these results early enough to show to Foulkes, who died in 1977.

(143) A conjectured combinatorial interpretation of the normalized irreducible character values of the symmetric group, preprint; `arXiv:math/0606467`.

See the discussion under [124].

(144) An introduction to hyperplane arrangements, in *Geometric Combinatorics* (E. Miller, V. Reiner, and B. Sturmfels, eds.), IAS/Park City Mathematics Series, vol. 13, American Mathematical Society, Providence, RI, 2007, pp. 389–496.

These notes are based on a lecture series given at the Park City Mathematics Institute in 2004. They are suitable for a one-semester course on the combinatorics of hyperplane arrangements. At one time I was thinking of expanding them into a more substantial book. The next chapter would be on Tutte polynomials. Unfortunately it does not seem likely that this expanded version will ever be written.

(145) Pairs of noncrossing free Dyck paths and noncrossing partitions (with W. Y. C. Chen, S. X. M. Pang, and Ellen X. Y. Qu), *Discrete Math.* **309** (2009), 2834–2838.

I made a contribution to the original paper so became a coauthor.

(146) Promotion and evacuation, *Electronic J. Combinatorics*, volume 15(2) (2008-2009), R9.

I have been interested in promotion and evacuation since hearing Marcel-Paul Schützenberger give a talk on this topic around 1972. By 2008 I felt that I knew enough about it to write a survey paper. I did not expect that the part of the paper to generate the most interest was a side remark at the end of Section 2. It showed that promotion on the disjoint union of two chains is equivalent to another operation (now called *rowmotion*) on the direct product of these chains. Such persons as Jessica Striker, Nathan Williams, Jim Propp and Tom Roby have greatly extended this remark.

(147) Some combinatorial properties of hooks, contents, and parts of partitions, *Ramanujan J.* **23** (2010), 91–105.

Some generalizations of the fundamental formula $\sum_{\lambda\vdash n}(f^\lambda)^2 = n!$ motivated by a conjecture of Guoniu Han. Shortly thereafter Grigori Olshanski gave two further proofs that were more conceptual, as well as a generalization to Jack symmetric functions. A further generalization was later given by Sho Matsumoto and Jonathan Novak.

(148) Some Hecke algebra products and corresponding random walks (with Rosena R. X. Du), *J. Algebraic Combinatorics* **31** (2010), 159–168.

This papers considers the expansion of a certain simple product in the Hecke algebra $\mathcal{H}_n(q)$ of $\mathfrak{S}_n$. An interpretation is given in terms of a strange random walk on $\mathcal{H}_n(q)$. This collaboration with Rosena Ruoxia Du arose during her visit to M.I.T. during the academic year 2007–08.

(149) Two enumerative results on cycles of permutations, *European J. Combinatorics* **32** (2011), 937–943.

This paper consists of two unrelated parts. The first part is a proof of an intriguing conjecture of Miklós Bóna on the probability that 1 and 2 are in the same cycle of a product of two random n-cycles in $\mathfrak{S}_n$. This work is greatly generalized in [155]. The second part deals with the number of cycles of a product of two n-cycles and includes a proof and some additional aspects of a 1995 result of Don Zagier.

(150) A survey of alternating permutations, *Contemporary Mathematics* **531** (2010), 165–196.

A survey arising from a talk at a combinatorics conference in Tehran, Iran, in 2009.

(151) Formulae for Askey-Wilson moments and enumeration of staircase tableaux (with S. Corteel, D. Stanton, and L. Williams), *Trans. Amer. Math. Soc.*, to appear.

A further instance of making a contribution to the original manuscript and being added as a coauthor. The staircase tableaux of this paper are used to describe the stationary distribution of the asymmetric exclusion process, a topic that has received a lot of recent attention.

(152) An equivalence relation on the symmetric group and multiplicity-free flag h-vectors, *J. Combinatorics* **3** (2012), 277–298.

Steven Linton, Jim Propp, Tom Roby, and Julian West posted a paper on a certain class of equivalence relations on $\mathfrak{S}_n$. I saw that one could also analyze a related equivalence relation, namely, two permutations u, v in $\mathfrak{S}_n$ are equivalent if v can be obtained from u by successively interchanging two adjacent terms that differ by exactly one. The combinatorics is fairly routine except for an umbral argument (Theorem 2.5) dealing with an extension to permutations of the multiset $\{1^k, \dots, n^k\}$. One of the $\mathfrak{S}_n$-equivalence classes can be identified with the number of linear extensions of a poset. This observation led to the second topic of the paper, concerning finite distributive lattices whose flag h-vectors takes on only the values 0 and 1.

(153) Orientations, lattice polytopes, and group arrangements II: Modular and integral flow polynomials of graphs (with Beifang Chen), *Graphs and Combinatorics* **28** (2012), 751–779.

Beifang Chen has done a lot of interesting work on integral polytopes and Ehrhart theory, so I was happy to have an opportunity to collaborate with him on the present paper.

(154) Two remarks on skew tableaux, *Electronic J. Combinatorics* **18**(2) (2011–12), P16.

This paper has two independent parts. The first part was inspired by a paper of Yuliy Baryshnikov and Dan Romik on the number of standard Young tableaux of certain skew shapes. I give a more elementary approach that includes some of the results of Baryshnikov and Romik. The second part concerns the evaluation of skew Schur functions at $x_i = i^{-2k}$ for $k = 1, 2, 3$. Curiously, the method breaks down for $k \geq 4$.

(155) Separation probabilities for products of permutations (with Olivier Bernardi, Rosena R. X. Du, and Alejandro H. Morales), *Combinatorics, Probability and Computing* **23** (2014), 201–222.

This paper started as some work of Rosena Du and me which generalizes the results in [149] related to Bóna's conjecture on the product of two cycles. Then Olivier Bernardi and Alejandro Morales came along and developed a combinatorial approach that greatly generalized my work with Rosena.

(156) On the rank function of a differential poset (with F. Zanello), *Electronic J. Combinatorics* **19**(2) (2012), P13.

There are many fundamental open problems concerning the structure of differential posets (originally defined in [77]), such as the least possible number of elements of rank n in an r-differential poset. Here Fabrizio Zanello and I obtain a lower bound $f_r(n)$ such that $\log f_r(n) \sim 2\sqrt{rn}$, while the conjectured answer $g_r(n)$ satisfies $\log g_r(n) \sim \pi\sqrt{2rn/3}$. Some additional results about the rank function of differential posets are also obtained. This work was done when Fabrizio was visiting M.I.T. during the calendar year 2011.

(157) Counting conjugacy classes of elements of finite order in Lie groups (with Tamar Friedmann), *Europ. J. Combinatorics*, **36** (2014), 86-96.

Tamar Friedmann, whom I had never heard of before, walked into my office one day with a combinatorics question arising from "the need to find a formula for the number of certain vacua in the quantum moduli space of M-theory compactifications on manifolds of G_2 holonomy" (whatever that means). I was able to steer her in the right direction for solving this combinatorics question, so we became joint authors of the present paper (the math version) and the next (the physics version).

(158) The string landscape: on formulas for counting vacua (with Tamar Friedmann), *Nuclear Physics B* **869** (2013), 74–88.

The physics version of the previous paper.

(159) *Algebraic Combinatorics: Walks, Trees, Tableaux and More*, Springer, 2013.

An undergraduate textbook on algebraic combinatorics arising from teaching a course on this subject several times at M.I.T. and once at Harvard (in 2000). Three of the students in my Harvard course went on to become researchers in algebraic/enumerative combinatorics: Gregg Musiker, David Speyer, and Bridget Tenner.

(160) Valid orderings of real hyperplane arrangements, *Discrete Comput. Geom.*, to appear.

I had realized for a long time that the line shellings of a convex polytope $\mathcal{P}$, where the shelling line passes through a fixed point p in the interior of $\mathcal{P}$, corresponded to the regions of a hyperplane arrangement $\mathcal{A}$. I decided to look at this more carefully around 2012 and found that the arrangement $\mathcal{A}$ was related to the rather intractable concept of the Dilworth completion of a matroid, leading to the present paper. A referee pointed out that some of the paper had been previously obtained in a dual form by my former student Paul Edelman, which naturally I discuss in the final version.

(161) Smith normal form of a multivariate matrix associated with partitions (with Christine Bessenrodt), *Journal of Algebraic Combinatorics* (online), 2014.

For a long time I have been fond of a result of Leonard Carlitz, David Roselle, and Richard Scoville, arising from a question of Elwyn Berlekamp, on certain determinants that equal 1. (See Exercise 6.26 of [113].) There is a q-analogue for which the determinants are a power of an indeterminate q. More recently I have been interested in Smith normal form (SNF), which for a square matrix A over a PID (and sometimes more general rings) gives a canonical factorization of $\det A$. Thus it was natural to consider the SNF of the q-analogues of the Berlekamp matrices. It turned out that the matrix can be vastly generalized and that the SNF exists and can be computed explicitly over a polynomial ring $\mathbb{Z}[x_{ij}]$ in certain variables x_{ij}. After I sent a draft to Christine Bessenrodt she made some useful further contributions.

(162) How the Upper Bound Conjecture was proved, *Ann. Combinatorics* **18** (2014), 533–539.

People often ask me how I got interested in combinatorics and how I got certain ideas. I put a lot of this information into the present paper. I think that it also gives some insight for nonmathematicians and beginning mathematicians into the nature of mathematical research.

(163) A formula for the specialization of skew Schur functions (with Xiaomei Chen), *Ann. Combinatorics*, to appear.

This paper arose from a desire to generalize a formula of Sergei Kerov and of Adriano Garsia and Mark Haiman for the Laurent polynomial (a priori just a rational function) $s_{\lambda/1}(1,q,q^2,\dots)/(1-q)s_\lambda(1,q,q^2,\dots)$. Our main result is also a q-analogue of a formula of Andrei Okounkov and Grigori Olshanski for $f^{\lambda/\mu}/f^\lambda$. This work was done during the visit of Xiaomei Chen to M.I.T. for the 2012-2013 academic year.

(164) The Catalan case of Armstrong's conjecture on simultaneous core partitions (with Fabrizio Zanello), *SIAM J. Discrete Math.* **29** (2015), 658–666.

Drew Armstrong made a fascinating conjecture about the average size of a partition that is simultaneously a p-core and a q-core, where p and q are relatively prime positive integers. Fabrizio Zanello (visiting M.I.T. for the 2013 calendar year) proved this conjecture for the case $q=p+1$. Our technique was generalized in 2014 by Amol Aggarwal to the case $q=mp+1$. Finally in 2015 Paul Johnson proved the entire conjecture using a completely different method.

(165) A distributive lattice connected with arithmetic progressions of length three (with Fu Liu), *Ramanujan J.* **36** (2015), 203–226.

Perhaps the most interesting aspect of this paper is that it arose from an article in the *New York Times* Numberplay blog. The editor of this blog asked Ron Graham to contribute an open problem that wasn't already well-known and that would be comprehensible to readers of the blog. Graham's problem was then solved by Noam Elkies. Elkies' proof led him to a conjecture in combinatorial number theory, which is what was proved by Fu Liu and me. It is interesting that the proof involves such unexpected tools as the structure of finite distributive lattices and the enumeration of semistandard Young tableaux. It also involves the wishful thinking technique: we needed to count the elements of a certain distributive lattice L_n. I could think of only one "interesting" distributive lattice M_n whose number of elements was the same as that conjectured for L_n. The nicest situation would be that L_n and M_n were isomorphic, and indeed this turned out to be the case.

(166) The Smith normal form of a matrix associated with Young's lattice (with Tommy Wuxing Cai), *Proc. Amer. Math. Soc.*, to appear.

Alex Miller and Vic Reiner came up with an intriguing conjecture about the Smith normal form of the operator DU associated with a differential poset P. In the present paper Tommy Wuxing Cai (visiting M.I.T. during the 2013–2014 academic year) and I prove this conjecture in the special case where P is Young's lattice. (I am somewhat skeptical of the truth of the conjecture in general.) In this case DU is equivalent to the operator $\frac{\partial}{\partial p_1}p_1$ operating on integral homogeneous symmetric functions of degree n, where p_1 is a power sum symmetric function. We also give a conjecture for the operators $k\frac{\partial}{\partial p_k}p_k$ for $k \geq 2$ which I think is quite interesting. This conjecture was proved in 2015 by Zipei Nie, an M.I.T. undergraduate.

(167) Supersolvability and freeness for ψ-graphical arrangements (with Lili Mu), *Discrete Comput. Geom.*, to appear.

A sequel to [160], written when Lili Mu was visiting M.I.T. during the 2013–2014 academic year.

(168) Unimodality of partitions with distinct parts inside Ferrers shapes (with Fabrizio Zanello), *Europ. J. Math.*, to appear.

A collaboration arising from Fabrizio's visit to MIT during the calendar year 2013. We split this work into the present paper and the next.

(169) Some asymptotic results on q-binomial coefficients (with Fabrizio Zanello), *Ann. Combinatorics*, to appear..

An investigation of the asymptotic behavior of the coefficients of the q-binomial coefficient $\binom{a+k}{k}$ for fixed k.

(170) *Catalan Numbers*, Cambridge University Press, New York/Cambridge, 2015.

I was aware since the 1960's of the fascinating sequence of Catalan numbers. When I started teaching enumerative combinatorics I also began collecting different combinatorial interpretations of these numbers. By the time [113] was written I could include 66 combinatorial interpretations, as well as many additional properties and extensions. Since then I maintained a web page of further interpretations and properties, assisted by many contributors. Finally I decided to wrap up my work on Catalan numbers by gathering it into the present monograph. It contains 214 combinatorial interpretations, 68 additional problems, come basic background information, and an appendix written by Igor Pak on the history of Catalan numbers.

One unpublished paper seems worthy of mention: An analogue of Young's lattice for compositions (with Anders Björner). After we wrote this paper Sergey Fomin pointed out that the poset we consider is just a special case of a class of posets due to Björner known as *subword orders*. Therefore we decided not to publish our paper, although it contains some interesting information that is particular to the composition poset, rather than to subword orders in general.

Department of Mathematics, University of Miami, Coral Gables, FL 33124
E-mail address: rstan@math.mit.edu