A SURVEY OF EULERIAN POSETS

RICHARD P. STANLEY*
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139

Abstract. An *Eulerian poset* is a finite graded poset with $\hat{0}$ and $\hat{1}$ such that every interval of length at least one has the same number of elements of odd rank as of even rank. For instance, the face lattice of a convex polytope is Eulerian. We survey some numerical and polynomial invariants associated with an Eulerian poset P. The *flag f-vector* counts the number of chains of P whose elements have specified ranks. A convenient way to represent the flag f-vector is by a noncommutative polynomial $\Phi_P(c,d)$ called the *cd-index* of P. The problem of characterizing the flag f-vector of certain classes of Eulerian posets, notably those which are Cohen-Macaulay, is best approached in the context of the cd-index. For the special class of *simplicial* Eulerian posets (which include face lattices of simplicial polytopes and triangulations of spheres), much more can be said about the flag f-vector. A high point of this subject is the *g-theorem*, which characterizes the f-vectors of simplicial convex polytopes. In Section 4 we discuss the concept of the h-vector of a lower Eulerian poset and its connection with intersection homology theory. The notion of h-vector leads naturally to the theory of *acceptable functions* on a lower Eulerian poset and their connection with subdivisions and the Ehrhart polynomial.

Key words: Eulerian poset, f-vector, h-vector, flag f-vector, flag h-vector, cd-index, simplicial poset, polytope, subdivision, local h-vector, acceptable function, Ehrhart polynomial

1. Preliminaries.

In this paper we will survey some of the fascinating properties of a class of posets (partially ordered sets) called *Eulerian posets.* We will be concerned almost exclusively with certain numerical and polynomial invariants associated with Eulerian posets. For results of a more structural nature, see e.g. [54]. Basic poset notation and terminology may be found in [65]. All posets, CW complexes, simplicial complexes, etc., considered in this paper are always assumed to be *finite.* Let P be a finite graded poset of rank $n+1$ with $\hat{0}$ and $\hat{1}$. (Often the letter d is used in place of our n. We use n to avoid confusion with the variable d of the cd-index, discussed in the next section.) Let ρ denote the rank function and μ the Möbius function of P. Thus $\rho(\hat{0}) = 0$ and $\rho(\hat{1}) = n+1$. If $s \leq t$ in P then we write $\rho(s,t) = \rho(t) - \rho(s)$, the rank (length) of the interval $[s,t]$. We say that P is *Eulerian* if $\mu(s,t) = (-1)^{\rho(s,t)}$ for all $s \leq t$ in P. Equivalently [65, Exer. 3.69(a)], P is Eulerian if and only if every interval of rank at least one contains as many elements of even rank as of odd rank, i.e.,

$$\sum_{u\in[s,t]} (-1)^{\rho(u)} = 0, \text{ if } s < t \text{ in } P.$$

* Partially supported by NSF grant #DMS-9206374.

T. Bisztriczky et al. (eds.), POLYTOPES: Abstract, Convex and Computational, 301–333.

The prototypical example of an Eulerian poset is the face lattice $P_{\mathcal{P}}$ of a convex polytope $\mathcal{P}$. If $\dim(\mathcal{P}) = n$, then $P_{\mathcal{P}}$ has rank $n+1$. For instance, if $\mathcal{P}$ is an n-simplex, then $P_{\mathcal{P}}$ is isomorphic to the boolean algebra B_{n+1}. A more general example of an Eulerian poset is the following. Recall that a (finite) *regular CW complex* Γ consists of a finite collection of disjoint open cells σ_i in a Euclidean space such that each σ_i is homeomorphic to an open ball $\mathbb{B}^{n_i}$ of some dimension n_i, and such that the boundary $\partial\sigma_i$ is homeomorphic to a sphere $\mathbb{S}^{n_i-1}$ of dimension $n_i - 1$. From this one can show that each cell boundary $\partial\sigma_i$ is a union of cells of Γ. By convention the empty set $\emptyset$ is also a cell of Γ, unless $\Gamma = \emptyset$. We will henceforth identify Γ with its *face poset*, whose elements are the cells of Γ, partially ordered by inclusion of their closures. The *dimension* $\dim \Gamma$ of Γ is the maximum dimension of any cell. The *body* $|\Gamma|$ is defined by

$$|\Gamma| = \bigcup_{\sigma\in\Gamma} \sigma.$$

An equivalent definition of a regular CW complex is due to Björner [14]. An (abstract) *simplicial complex* Δ is a collection of sets F (called *faces*) such that if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. We sometimes identify Δ with its *face poset*, i.e., the set of faces of Δ partially ordered by inclusion. A nonempty poset is then a simplicial complex if and only if it is a meet-semilattice (and hence has a $\hat{0}$ since all posets considered here are finite), and every interval $[\hat{0}, t]$ is a boolean algebra. We assume familiarity with the notion of the *geometric realization* $|\Delta|$ of an abstract simplicial complex Δ. For us, $|\Delta|$ is only considered to be a topological space. Thus, for instance, if $|\Delta|$ is a sphere then we say that Δ is a *triangulation of a sphere.* Given any poset P, define the *order complex* $\Delta(P)$ of P to be the set of all chains of P, regarded as an abstract simplicial complex. We can now state Björner's result: A (finite) poset Γ with $\hat{0}$ is a regular CW-complex if and only if for all $t > \hat{0}$ the geometric realization $|\Delta(\hat{0}, t)|$ of the order complex $\Delta(\hat{0}, t)$ of the open interval $(\hat{0}, t)$ is a sphere. Note that a simplicial complex is a special case of a regular CW complex, since if $[\hat{0}, t]$ is a boolean algebra then $|\Delta(\hat{0}, t)|$ is a sphere.

It is now an elementary topological result that a regular CW sphere (i.e., a regular CW complex whose body is a sphere), with a $\hat{1}$ adjoined, is Eulerian. By slight abuse of notation, we will call such posets (after the $\hat{1}$ has been adjoined) also regular CW spheres. Such posets need not even be lattices, as shown by Figure 1. An Eulerian poset need not be a regular CW complex. For example, let P be the poset of Figure 2. Note that P is a regular CW complex but not a sphere. Let C_2 be a two-element chain. Then $P \times C_2$ is Eulerian but not a regular CW complex.

A fundamental combinatorial invariant of a graded poset P of rank $n+1$ is the number p_i of elements of rank i. We also write $f_i = p_{i-1}$ and call the vector $f(P) = (f_0, f_1, \ldots, f_{d-1})$ the *f-vector* of P. Note that $f_{-1} = 1$ unless P is empty. Though as we shall see the f-vector of an Eulerian poset (or indeed of any graded poset) has many interesting properties, often it is too crude an invariant to be useful. We need to count not just elements of a given rank, but rather chains (or *flags*) of element of specified ranks. To this end, if $n \in \mathbb{N}$ (the set of nonnegative integers), write $[n] = \{1, 2, \ldots, n\}$, so in particular $[0] = \emptyset$. Also write $S = \{a_1, a_2, \ldots, a_k\}_<$ to denote that $S = \{a_1, a_2, \ldots, a_k\} \subset \mathbb{R}$ and $a_1 < a_2 < \cdots < a_k$. If now $S =$

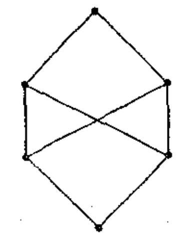

Fig. 1. An Eulerian nonlattice

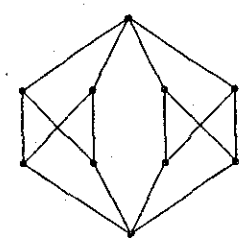

Fig. 2. A nonspherical Eulerian poset

$\{a_1, \ldots, a_k\}_< \subseteq [n]$, then define the *rank-selected subposet* P_S of P by

$$P_S = \{\hat{0}, \hat{1}\} \cup \{t \in P : \rho(t) \in S\}.$$

Now define $\alpha_P(S)$ to be the number of maximal chains of P_S. Equivalently, $\alpha_P(S)$ is the number of *S-flags* of P, i.e., flags (= chains) $\hat{0} < t_1 < \cdots < t_k < \hat{1}$ such that $S = \{\rho(t_1), \ldots, \rho(t_k)\}$. Thus for instance $\alpha_P(\emptyset) = 1$, $\alpha_P(\{i\})$ (abbreviated $\alpha_P(i)$) is the number $p_i = f_{i-1}$ of elements of P of rank i, and $\alpha_P([n])$ is the number of maximal chains of P. The function α_P is called the *flag f-vector* of P.

It is often the case that it is not the flag f-vector itself that is most natural to use, but rather a certain linear transformation of it defined as follows. For $S \subseteq [n]$ let

$$\beta_P(S) = \sum_{T\subseteq S} (-1)^{|S-T|} \alpha_P(T). \tag{1}$$

Equivalently (by the Principle of Inclusion-Exclusion),

$$\alpha_P(S) = \sum_{T\subseteq S} \beta_P(T). \tag{2}$$

The function β_P is called the *flag h-vector* of P. This function was first defined for distributive lattices in [56][57], and subsequently extended to other posets. It is an immediate consequence of "Philip Hall's theorem" [52, §3, Prop. 6] [65, Prop. 3.8.5] in the theory of Möbius functions that

$$\beta_P(S) = (-1)^{\#S+1}\mu_S(\hat{0},\hat{1}), \tag{3}$$

where μ_S denotes the Möbius function of P_S (see [65, (34) on p. 131]).

We briefly explain the reason for the terminology "flag h-vector." Let Δ be any simplicial complex of dimension $n-1$, with f_i i-dimensional faces. The *h-vector* $(h_0, h_1, \ldots, h_n)$ of Δ is defined by

$$\sum_{i=0}^{n} f_{i-1}(x-1)^{n-i} = \sum_{i=0}^{n} h_i x^{n-i}. \tag{4}$$

The h-vector has long been known to be of fundamental importance; see for instance [62] and Section 3 below. If P is a poset with $\hat{0}$ and $\hat{1}$, define the *reduced* order complex $\tilde{\Delta}(P)$ of P to be the ordinary order complex $\Delta(P-\{\hat{0},\hat{1}\})$ of $P-\{\hat{0},\hat{1}\}$. If now P is graded of rank $n+1$, then it is easy to see that

$$f_{i-1}(\tilde{\Delta}(P)) = \sum_{\substack{S\subseteq[n]\\ \#S=i}} \alpha_P(S)$$

$$h_i(\tilde{\Delta}(P)) = \sum_{\substack{S\subseteq[n]\\ \#S=i}} \beta_P(S).$$

Thus the flag f-vector is a refinement, in a natural way, of the usual f-vector; and in exactly the same way the flag h-vector is a refinement of the usual h-vector.

We now discuss an important class of Eulerian posets. A poset P with $\hat{0}$ and $\hat{1}$ is said to be *Cohen-Macaulay* (over a fixed ground field K) if for every $s<t$ in P we have

$$\dim \tilde{H}_i(\Delta(s,t);K) = 0 \text{ if } i < \dim\Delta(s,t),$$

where $\tilde{H}_i(\Delta(s,t);K)$ denotes the ith reduced simplicial homology group (over K) of the order complex $\Delta(s,t)$ of the open interval (s,t). (The coefficient field K is fixed once and for all and is often suppressed from the notation.) In other words, every open interval has all its (reduced) homology only in the top possible dimension. It is easy to show from this definition that every Cohen-Macaulay poset is graded.

If f is a function whose domain consists of the set of intervals $[s,t]$ of a poset P, then we often abbreviate $f([s,t])$ as f_{st}. Let μ denote the Möbius function of P. Recall that if $s<t$ in P, then Philip Hall's theorem is equivalent to the formula

$$\mu_{st} = \tilde{\chi}(\Delta(s,t)),$$

where $\tilde{\chi}$ denotes the reduced Euler characteristic. It follows that if P is Cohen-Macaulay, then for any $s<t$ in P we have

$$\mu_{st} = (-1)^m \dim \tilde{H}_m(\Delta(s,t)), \tag{5}$$

where $m = \dim\Delta(s,t)$. (Because P is graded, we in fact have $m = \rho(s,t)-2$.) In particular, if P is Cohen-Macaulay and Eulerian then

$$\dim \tilde{H}_m(\Delta(s,t)) = 1, \text{ for all } s<t \text{ in } P.$$

A poset P which is both Cohen-Macaulay and Eulerian is called a *Gorenstein** poset.

The following theorem summarizes two basic results in the theory of Cohen-Macaulay posets. For further information and references, see [16].

1.1 Theorem. *(a) Cohen-Macaulayness is a topological property, i.e., the question of whether a poset P (say with $\hat{0}$ and $\hat{1}$) is Cohen-Macaulay (over a fixed field K) depends only on the geometric realization $|\tilde{\Delta}(P)|$. In particular, P is Cohen-Macaulay if $|\tilde{\Delta}(P)|$ is a sphere.*

(b) If a poset P is Cohen-Macaulay, then so is any rank-selected subposet P_S.

1.2 Corollary. *If P is Cohen-Macaulay then $\beta_P(S) \geq 0$ for all S.*

Proof. Immediate from Theorem 1.1(b) and equations (3) and (5). □

A more general notion than Cohen-Macaulay poset that we will need later is a *Cohen-Macaulay simplicial complex* (over a field K). This is a simplicial complex Δ such that for every $F\in\Delta$ (including $F=\emptyset$), we have

$$\dim_K \tilde{H}_i(\mathrm{lk}_\Delta F;K) = 0, \text{ if } i < \dim(\mathrm{lk}_\Delta F).$$

Here $\tilde{H}_i(\mathrm{lk}_\Delta F;K)$ denotes the ith reduced homology group of $\mathrm{lk}_\Delta F$ with coefficient field K, and $\mathrm{lk}_\Delta F$ is the *link* of F in Δ, defined by

$$\mathrm{lk}_\Delta F = \{G\in\Delta : G\cap F=\emptyset \text{ and } G\cup F\in\Delta\}.$$

In particular, $\mathrm{lk}_\Delta\emptyset = \Delta$. Examples of Cohen-Macaulay simplicial complexes include triangulations of spheres and balls, and the independent sets of a matroid. It can be shown that a simplicial complex Δ is Cohen-Macaulay as a simplicial complex if and only it is Cohen-Macaulay as a poset (i.e., its face poset is a Cohen-Macaulay poset). Moreover, using commutative algebra techniques one can show that the h-vector of a Cohen-Macaulay complex (short for Cohen-Macaulay simplicial complex) is nonnegative. In fact, Cohen-Macaulay complexes seem to be the most general "natural" class of simplicial complexes whose h-vector is nonnegative.

2. The *cd*-Index.

In this section we will discuss what can be said about the flag f-vector (or flag h-vector) of a general Eulerian poset. First let us consider *linear* conditions on the flag f-vector. For instance, it is an immediate consequence of the definition of an Eulerian poset (say of rank $n+1$) that

$$f_0 - f_1 + f_2 - \cdots + (-1)^{n-1} f_{n-1} = 1 + (-1)^n.$$

This is a linear condition on the flag f-vector (since $f_i = \alpha(i+1)$). However, there are many additional linear conditions. We will mention a couple of special cases before moving to the general result.

Let P be a poset with $\hat{0}$ and $\hat{1}$. The *zeta polynomial* $Z(P,r)$ is defined for $r\in\mathbb{P}$ to be the number of multichains $\hat{0}=t_0\leq t_1\leq\cdots\leq t_r=\hat{1}$ of length r in P. It is easily seen that $Z(P,r)$ is a polynomial function of r whose degree is the rank of P. In particular, since $Z(P,r)$ is a polynomial it is defined for all integers (or even complex numbers) r. For these and other basic properties of the zeta polynomial see [65, Ch. 3.11]. For an Eulerian poset P of rank $n+1$, it is not hard to show [65, Prop. 3.14.1] that

$$Z(P,r) = (-1)^{n+1} Z(P,-r). \qquad (6)$$

For instance, the face-lattices of the dodecahedron and icosahedron have zeta polynomial $5r^4 - 4r^2$. Equation (6) is equivalent to certain linear conditions on the flag f-vector of P, which turn out to be given by

$$h_i(\tilde{\Delta}(P)) = h_{n-i}(\tilde{\Delta}(P)).$$

In fact, there holds the much stronger result

$$\beta_P(S) = \beta_P(\bar{S}), \qquad (7)$$

for all $S \subseteq [n]$, where $\bar{S} = [n] - S$. See e.g. [65, Cor. 3.14.6] for a proof.

Equation (7) yields 2^{n-1} independent linear relations satisfied by the flag f-vector of an Eulerian poset. But there are still others; for instance, every element of rank two in an Eulerian poset covers exactly two elements, from which it follows that $\alpha(1,2) = 2\alpha(2)$. This is a linear relation independent from (7). A complete set of linear relations was found by Bayer and Billera [6]. An elegant formulation of their result due to Fine (see [7]) will now be discussed. (Another proof appears in [70].) The idea is to work with certain noncommutative generating functions for the flag f and h-vectors. Fix a graded poset P of rank $n+1$ with $\hat{0}$ and $\hat{1}$. If $S \subseteq [n]$, then define a noncommutative monomial $u_S = u_1 u_2 \cdots u_n$ in the variables a and b by

$$u_i = \begin{cases} a, & \text{if } i \notin S \\ b, & \text{if } i \in S. \end{cases}$$

For instance, if $n = 6$ and $S = \{2, 6\}$, then $u_S = abaaab$. Let

$$\Upsilon_P(a,b) = \sum_{S \subseteq [n]} \alpha_P(S) u_S \qquad (8)$$

$$\Psi_P(a,b) = \sum_{S \subseteq [n]} \beta_P(S) u_S.$$

It is an immediate consequence of (1) or (2) that

$$\Psi_P(a,b) = \Upsilon_P(a-b,b) \qquad (9)$$

$$\Upsilon_P(a,b) = \Psi_P(a+b,b).$$

For instance, if P is the boolean algebra B_3, then

$$\Upsilon_P(a,b) = a^2 + 3ba + 3ab + 6b^2$$

$$\Psi_P(a,b) = a^2 + 2ba + 2ab + b^2. \qquad (10)$$

The result of Bayer and Billera, as formulated by Fine, is the following.

2.1 Theorem. *Let P be an Eulerian poset. Then $\Psi_P(a,b)$ can be written as a polynomial $\Phi_P(c,d)$ in $c = a+b$ and $d = ab+ba$.*

The polynomial $\Phi_P(c,d)$ is called the *cd-index* of P. The cd-index of an Eulerian poset is unique since $a+b$ and $ab+ba$ are algebraically independent (as noncommutative polynomials) over any field K. For instance, from (10) we see that $\Phi_{B_3}(c,d) = c^2 + d$. If we define $\deg(c) = 1$ and $\deg(d) = 2$, then clearly $\Phi_P(c,d)$ is homogeneous of degree n with integer coefficients. The cd-index is a very compact way of presenting the flag f-vector of an Eulerian poset. For instance, if P is the Bruhat order of the symmetric group S_4 (as defined e.g. in [18]), then P is a fairly complicated poset of rank 6 with 24 elements and 168 maximal chains, yet

$$\Phi_P(c,d) = c^5 + dc^3 + 2cdc^2 + 2c^2dc + c^3d + 2d^2c + dcd + 2cd^2.$$

Note that since $a+b$ and $ab+ba$ are invariant under interchanging a and b, it follows from Proposition 2.1 that for an Eulerian poset P we have $\Psi_P(a,b) = \Psi_P(b,a)$. This formula is equivalent to (7). This symmetry condition is not enough to guarantee the existence of the cd-index. Indeed, a noncommutative polynomial $\Psi(a,b)$ satisfies $\Psi(a,b) = \Psi(b,a)$ if and only if $\Psi(a,b)$ is a polynomial (necessarily unique) in the variables $a+b, a^2+b^2, a^3+b^3, \ldots$. On the other hand, $\Psi(a,b) = \Phi(a+b, ab+ba)$ for some polynomial Φ if and only if $\Psi(a,b)$ is a polynomial just in the variables $a+b$ and $a^2+b^2 = (a+b)^2 - (ab+ba)$.

The existence of the cd-index imposes certain linear relations on the flag f-vector of an Eulerian poset, and it is not hard to show that there are no other relations. Indeed, Bayer and Billera [6] prove the stronger result that there are no additional relations even among flag f-vectors of face-lattices of convex polytopes. For arbitrary Eulerian posets the proof is easier and appears in [70, Prop. 1.5].

Even the cd-index $U_m(c,d)$ of the boolean algebra B_m is a subtle object. It turns out to be identical (as shown by Purtill [50]) to a polynomial studied earlier by Foata and Schützenberger [29], called a (noncommutative) *André polynomial*. Several related combinatorial interpretations of $U_m(c,d)$ were given by Foata and Schützenberger. We state here a slightly simpler interpretation due to S. Sundaram. We say that a sequence $\rho = \rho_1\rho_2\cdots\rho_m$ of integers *has no double descents* if there does not exist i for which $\rho_{i-1} > \rho_i > \rho_{i+1}$. A *simsun permutation* (named after Rodica Simion and Sheila Sundaram; see [71]) is a permutation π of $[m]$ such that for any $j \geq 0$, the word $\rho_1\rho_2\cdots\rho_{m-j}$ obtained from $\pi = \pi_1\pi_2\cdots\pi_m$ by removing $n, n-1, \ldots, n-j+1$ has no double descents. (In particular, the case $j=0$ states that π has no double descents.) The number of simsun permutations of $[m]$ is the *Euler number* E_m, defined by

$$\sum_{m \geq 0} E_m \frac{x^m}{m!} = \tan x + \sec x.$$

E_m is also the number of *alternating permutations* of $[m]$, as defined e.g. in [65, pp. 148–149]. If π has no double descents, then define U_π to be the noncommutative monomial obtained from π by replacing each factor $\pi_i\pi_{i+1}$ with a d if $\pi_i > \pi_{i+1}$, and replacing each remaining term π_k with a c.

2.2 Proposition. *We have*

$$U_m(c,d) = \sum U_\pi,$$

summed over all simsun permutations of $[m]$.

For instance, the sixteen simsun permutations π of [4], together with the monomials U_π, are given by

1234	$cccc$	1324	cdc	3412	cdc	2143	dd
1243	ccd	1423	cdc	2134	dcc	3142	dd
1342	ccd	2314	cdc	3124	dcc	4132	dd
2341	ccd	2413	cdc	4123	dcc	4231	dd

Hence $U_4(c,d) = c^4 + 3c^2d + 5cdc + 3dc^3 + 4d^2$. One can also give a formula for the (noncommutative) exponential generating function $\sum_{m\geq 0} U_m(c,d)x^m/m!$; see [70, Cor. 1.9] for details.

Note that Proposition 2.2 gives a combinatorial interpretation to the coefficients of $U_m(c,d)$, thereby showing that they are nonnegative. We can ask whether there are similar results for other Eulerian posets. In general the cd-index of an Eulerian poset need not be nonnegative (i.e., have nonnegative coefficients). It is easy to see [70, Prop. 1.6] that for an Eulerian poset P, the numbers $\beta_P(S)$ are sums of certain coefficients of $\Phi_P(c,d)$. Hence a necessary condition for $\Phi_P(c,d)$ to be nonnegative (denoted $\Phi_P(c,d) \geq 0$) is that each $\beta_P(S)$ is nonnegative. This observation, together with Corollary 1.2, suggests the following conjecture, which is one of the most intriguing open problems in the theory of Eulerian posets.

2.3 Conjecture. *Let P be a Gorenstein* poset. Then $\Phi_P(c,d) \geq 0$.*

Conjecture 2.3 was first formulated by Fine for face lattices of convex polytopes, then for regular CW spheres by Bayer and Klapper [7, Conj. 5] and extended to its present form in [70, Conj. 2.1]. To prove Conjecture 2.3, presumably one would have to give either a combinatorial or algebraic (e.g., the dimension of a vector space) interpretation of the coefficients of $\Phi_P(c,d)$. It is easy to see that the sum $\Phi_P(1,1)$ of all the coefficients is given by $\Phi_P(c,d) = \beta_P(1,3,5,\dots) = \beta_P(2,4,6,\dots)$. Hence when $\beta_P(1,3,5,\dots)$ has a natural combinatorial or algebraic interpretation, one can try to find a "refinement" into the coefficients of the cd-index. Only for very special posets called *lexicographically shellable* is a combinatorial interpretation known for $\beta_P(1,3,5,\dots)$ (or more generally for any $\beta_P(S)$). For certain of these, most notably face lattices of *simplicial* polytopes, Purtill [50] was able to interpret the cd-index combinatorially and thereby prove nonnegativity. However, even though the face lattice of *any* convex polytope is lexicographically shellable, Purtill's methods have not yet been extended to them. Note that equations (3) and (5) give an interpretation of $\beta_P(S)$ as the dimension of a certain vector space (homology group). Thus a more promising line of attack on Conjecture 2.3 would be to find a refinement of the homology of $\tilde{\Delta}(P)$. The face ring techniques discussed in the next section can be used to interpret $\beta_P(S)$ as the dimension of a certain homogeneous part of a graded algebra [59], so one could also try to impose further structure (such as a finer gradation or filtration) on this algebra. Despite the plethora of techniques involving face rings which are now available, no progress has been made on the problem of finding a suitable refinement of the gradation.

Despite the lack of progress discussed in the previous paragraph, nevertheless some positive results related to Conjecture 2.3 have been obtained. One such result appears in the next section (Theorem 3.6). An additional result is given by the following theorem [70, Thm. 2.6].

2.4 Theorem. *Let P be the face poset of an S-shellable regular CW sphere Γ, with a $\hat{1}$ adjoined. Then $\Phi_P(c,d) \geq 0$.*

We will not give the rather technical definition of "S-shellability" here. It says roughly that there is a "nice" ordering of the facets of Γ. The recursive nature of the definition allows one to prove Theorem 2.4 by induction. The definition of S-shellability differs slightly from the more standard notion [14, §4] of shellability of a regular CW complex. However, the Bruggesser-Mani line shellings of a polytope [20] turn out to be S-shellable. Hence as a corollary to Theorem 2.2 we get a proof of the original conjecture of Fine that polytopes have nonnegative cd-index.

Conjecture 2.3 (and the weaker Theorem 2.4) give linear inequalities satisfied by the flag f-vector of certain Eulerian posets. It is natural to ask whether there are any additional inequalities independent from these. The answer is given by the next result, which explains the importance of Conjecture 2.3.

2.5 Theorem. *Conjecture 2.3, if true, gives all linear inequalities satisfied by flag f-vectors of Gorenstein* posets. Moreover, Theorem 2.4 gives (without any conjectures) all the linear inequalities satisfied by flag f-vectors of S-shellable regular CW spheres.*

Proof. (sketch) For any Eulerian poset P, the coefficient of c^n in $\Phi_P(c,d)$ is equal to one. Hence it suffices to show that for any cd-word $w \neq c^n$ of degree n (where deg $c = 1$ and deg $d = 2$), we can find an S-shellable CW sphere whose face poset with a $\hat{1}$ adjoined has cd-index for which the coefficient of w is an arbitrary factor larger than any of the other coefficients. Suppose $w = w_1 \cdots w_k$, where $w_i = c$ or d. Fix $m \geq 3$. Define posets T_i for $1 \leq i \leq k$ by $T_i = B_2$ (the boolean algebra of rank two) if $w_i = c$, and T_i is the face lattice of a convex m-gon if $w_i = d$. Given two Eulerian posets P and Q, define their *join* $P*Q$ to be the poset obtained by removing the $\hat{1}$ of P and $\hat{0}$ of Q, and putting every element of $Q - \hat{0}$ above every element of $P - \hat{1}$. It is easy to see that $P * Q$ is Eulerian. Now define $P_{w,m} = T_1 * T_2 * \cdots * T_k$. It is easy to show that $P_{w,m}$ is S-shellable (and a regular CW sphere), and that

$$\Phi_{P_{w,m}}(c,d) = w(c, c^2+d),$$

i.e., substitute $c^2 + d$ for d in w. The coefficient of w in $w(c,c^2+d)$ is m^t, where t is the number of d's in w. The coefficient of any other cd-word in $w(c,c^2+d)$ is m^s for some $s < t$. Hence the proof follows by letting $m \to \infty$. □

Unfortunately the posets $P_{w,m}$ are not face lattices of polytopes (indeed, they are not even lattices), so Theorem 2.5 does not (necessarily) give all the linear inequalities satisfied by flag f-vectors of convex polytopes. In fact, one can show that flag f-vectors of polytopes satisfy linear inequalities not implied by the nonnegativity of the cd-index. An example of such an inequality, due to Kalai [37, Thm. 7.1], is

$$3f_2(\mathcal{P}) \geq 2f_1(\mathcal{P}) + 2f_3(\mathcal{P}),$$

valid for every 5-polytope $\mathcal{P}$ (where $f(\mathcal{P}) = (f_0, f_1, f_2, f_3, f_4)$ is the usual f-vector of $\mathcal{P}$).

In general, in trying to understand f-vectors, flag f-vectors, etc., of certain posets, geometric complexes, etc., there are three successively stronger levels of results: (a) linear equalities, (b) linear inequalities, (c) complete characterization. For flag f-vectors of Gorenstein* posets, we have just discussed the first two levels. Namely, (a) is completely solved, while (b) has a conjectured solution which has been proved to be necessary in some significant special cases. At present very little progress has been made toward a complete characterization, so this remains an interesting area of research. In the next section we will see some further examples of this hierarchy of results, including a difficult complete characterization (Theorem 3.9).

The next result, proved in [59, Cor. 4.5], does give some nonlinear information about flag f-vectors. However, it is a rather weak result when applied to Gorenstein* posets, since it holds for a much wider class of objects ("completely balanced Cohen-Macaulay simplicial complexes").

2.6 Theorem. *Let P be a Gorenstein* poset of rank $n+1$. Then there exists a simplicial complex Δ of dimension $n-1$ on a vertex set V and a map $\eta : V \to [n]$ satisfying the following conditions:*

(a) The restriction of η to any face F of Δ is injective.

(b) For any $S \subseteq [n]$, the number of faces $F \in \Delta$ for which $\eta(F) = S$ is equal to $\beta_P(S)$.

A numerical form of Theorem 2.6 can be gleaned from the results of [15] and [30]. (The statement in [59] after Corollary 4.5 that the converse to Corollary 4.5 is false is incorrect.)

Conjecture 2.3 and Theorems 2.4 and 2.5 suggest looking also at *inhomogeneous* linear equalities and inequalities satisfied by the coefficients of the cd-index, but in general they give only trivial additional information. For instance, the only inhomogeneous linear equality or inequality satisfied by flag f-vectors of Gorenstein* posets independent of Conjecture 2.3 is that the coefficient of c^n in $\Phi_P(c,d)$ is equal to one. However, additional hypotheses lead to more interesting possibilities. For instance, suppose that one particular coefficient of $\Phi_P(c,d)$ is specified. What can be said about the minimum value of the other coefficients? This question has yet to be investigated. We also venture the following conjecture.

2.7 Conjecture. *Let P be a Gorenstein* lattice of rank $n+1$. Then $\Phi_P(c,d) \geq U_{n+1}(c,d)$. In other words, among all Gorenstein* lattices of rank $n+1$, the boolean algebra B_{n+1} minimizes all the coefficients of the cd-index.*

Some of the inequalities implied by Conjecture 2.3 have the particularly simple form $\beta_P(T) \leq \beta_P(S)$. These are summarized by the next result. To this end, given $S \subseteq [n]$ define $\omega(S) \subseteq [n-1]$ by the condition $i \in \omega(S)$ if and only if exactly one of i and $i+1$ belongs to S. For instance, if $n=8$ and $S=\{2,4,5,8\}$ then $\omega(S)=\{1,2,3,5,7\}$. If $n \geq 9$ then $\omega(S) = \{1,2,3,5,7,8\}$.

2.8 Proposition. *Let P be Eulerian with $\Phi_P(c,d) \geq 0$. Then $\beta_P(T) \leq \beta_P(S)$ whenever $\omega(T) \leq \omega(S)$. Moreover, if S and T are subsets of $[n]$ such that $\beta_P(T) \leq \beta_P(S)$ for every Gorenstein* poset (or even face poset of an S-shellable regular CW complex, with a $\hat{1}$ adjoined) P of rank $n+1$, then $\omega(T) \subseteq \omega(S)$.*

2.9 Corollary. *If $\Phi_P(c,d) \geq 0$ as above, then $\beta_P(S)$ is maximized for $S = \{1,3,5,\ldots\} \cap [n]$ and $S = \{2,4,6,\ldots\} \cap [n]$ (and possibly other values of S, depending on the poset P).*

There is an application of the cd-index to an intriguing conjecture of Charney

and Davis [24, Conj. D]. The conjecture is a kind of combinatorial analogue of a well-known conjecture of Hopf that the Euler characteristic of a $2n$-dimensional closed Riemannian manifold M of nonpositive sectional curvature satisfies $(-1)^n \chi(M) \geq 0$. To state the Charney-Davis conjecture, define a *flag complex* to be a simplicial complex for which every minimal set of vertices which do not form a face (sometimes called a "missing face") has two elements. Charney and Davis conjecture that the h-vector $(h_0, \ldots, h_{2m})$ of a $(2m-1)$-dimensional Gorenstein* flag complex satisfies

$$(-1)^m (h_0 - h_1 + h_2 - \cdots + h_{2m}) \geq 0.$$

Now the order complex of any poset is a flag complex. When specialized to Gorenstein* posets P, it was observed by E. Babson that the Charney-Davis conjecture is equivalent to the statement that the coefficient of d^m in $\Phi_P(c,d)$ is nonnegative. Hence the Charney-Davis conjecture for the special case of order complexes follows from Conjecture 2.3. Moreover, it follows from Theorem 2.4 that the Charney-Davis conjecture holds for face-lattices of convex polytopes (or even S-shellable regular CW spheres), as noted by Charney and Davis in Section 7 of their paper.

3. Simplicial Eulerian Posets.

In this section we discuss some Eulerian posets for which much more can be said about the flag f-vector. We define a poset P with $\hat{0}$ to be *simplicial* if the interval $[\hat{0}, t]$ is a boolean algebra for all $t \in P$. For instance, the face poset P of a simplicial complex Δ is simplicial. One should think of an arbitrary simplicial poset as a kind of "generalized simplicial complex" in which the intersection of two faces can be any subcomplex of their boundaries, and not just a single face. By slight abuse of terminology, we say that an *Eulerian* poset P is simplicial if $P - \{\hat{1}\}$ is simplicial (as defined above). Thus if Δ is a triangulation of a sphere (regarded as a poset), then $\Delta \cup \{\hat{1}\}$ is Eulerian and simplicial.

Suppose that P is simplicial and Eulerian of rank $n+1$, with f_i elements of rank $i+1$. If t is an element of P of rank $m < n+1$, then since the interval $[\hat{0}, t]$ is a boolean algebra we see that the number of chains $\hat{0} = t_0 < t_1 < \cdots < t_k = t$ such that $\rho(t_i) = a_i$ is given by the multinomial coefficient $\binom{m}{a_1, a_2 - a_1, \ldots, m - a_{k-1}}$. Hence if $S = \{a_1, a_2, \ldots, a_k\}$, then there follows

$$\alpha_P(S) = \binom{m}{a_1, a_2 - a_1, \ldots, m - a_{k-1}} f_{m-1}.$$

As a consequence, the flag f-vector is completely determined by the f-vector. Thus we may restrict our attention just to the f-vector when dealing with the flag f-vector of simplicial (Eulerian) posets. Just as we defined the h-vector of a simplicial complex in (4), so we can define the h-vector $(h_0, h_1, \ldots h_n)$ of any simplicial Eulerian poset P of rank $n+1$. More precisely, let $\bar{P} = P - \{\hat{1}\}$, and define $h(\bar{P}) = (h_0, h_1, \ldots, h_n)$ by

$$\sum_{i=0}^{n} f_{i-1}(x-1)^{n-i} = \sum_{i=0}^{n} h_i x^{n-i}, \tag{11}$$

where $\tilde{P}$ has f_{i-1} elements of rank i. The reason we define $(h_0, h_1, \ldots, h_n)$ to be the h-vector of $\tilde{P}$ rather than P is that in the next section we will define the h-vector of a class of posets (called *lower Eulerian*) which includes both P and $\tilde{P}$, such that the definition agrees with (11) for the case $\tilde{P}$. (Of course we could define $h(\tilde{P})$ for *any* Eulerian poset P by (11). In fact, one could define the h-vector of any *graded* poset P by (11), but this turns out to be a not very useful definition when P is not simplicial. The "correct" definition of $h(\tilde{P})$ for an arbitrary Eulerian poset P is quite subtle and will be discussed in the next section. There is no "correct" definition known of the h-vector of an arbitrary (graded) poset.)

Our main concern here will be with the following classes of simplicial Eulerian posets: (a) Gorenstein*, (b) face lattices of simplicial polytopes, and (c) Gorenstein* lattices. For the first case, the f-vector (or h-vector) has been completely characterized for, loosely speaking, 3/4 of the posets. More precisely, we have the following result [67, §4].

3.1 Theorem. *Let* $\mathbf{h} = (h_0, h_1, \ldots, h_n) \in \mathbb{Z}^{n+1}$. *Suppose that either* n *is odd, or that* n *is even and* $h_{n/2}$ *is even. Then the following two conditions are equivalent.*
(a) $\mathbf{h}$ *is the* h*-vector of a simplicial Gorenstein* poset of rank* $n+1$.
(b) $h_0=1$, $h_i \geq 0$, *and* $h_i = h_{n-i}$ *for all* i.

The first step in the proof of Theorem 3.1 is the following result.

3.2 Theorem. *Let* P *be a Gorenstein* simplicial poset (or more generally a Cohen-Macaulay simplicial poset) with* h*-vector* $(h_0, h_1, \ldots, h_n)$. *Then* $h_i \geq 0$ *for all* i.

Proof (sketch). The idea is to associate a commutative ring A_P with P and use techniques from commutative algebra. Hence we first review some concepts from commutative algebra. Let K be an arbitrary ground field. A *finitely-generated graded* K*-algebra* is a commutative ring R which is an algebra over the field K, together with a vector space direct sum decomposition $R = R_0 \oplus R_1 \oplus \cdots$ satisfying the conditions:

- $R_0 = K$ (so that the identity element 1 of K is the identity element of R)
- $R_i R_j \subseteq R_{i+j}$. We then say that elements of R_i are *homogeneous* of degree i.
- R is finitely-generated as a K-algebra.

Henceforth we will call a finitely-generated graded K-algebra R simply a *graded algebra*. If the graded algebra R is generated as a K-algebra by elements of degree one, then we say that R is *standard*. If R is integral over the subalgebra $K[R_1]$ of R generated by elements of degree one (equivalently, R is a finitely-generated $K[R_1]$-module), then we say that R is *semistandard*. Clearly standard graded algebras are semistandard.

The *Krull dimension* $\dim R$ of a graded K-algebra R is the maximum number of elements of R which are algebraically independent over K. (See e.g. [1, Thm. 11.14] for some equivalent definitions of Krull dimension.) The *Noether normalization lemma* asserts that if $d = \dim R$, then we can find d homogeneous elements $\theta_1, \ldots, \theta_d$ of R of positive degree such that the quotient ring $R/(\theta_1, \ldots, \theta_d)$ has Krull dimension zero (equivalently, is a finite-dimensional vector space). The elements $\theta_1, \ldots, \theta_d$ are called a *homogeneous system of parameters* (h.s.o.p.). If each θ_i has degree one, then we call $\theta_1, \ldots, \theta_d$ a *linear system of parameters* (l.s.o.p.). If the ground field K is infinite, then the existence of an l.s.o.p. is equivalent to R being semistandard.

The hypothesis that R is finitely-generated guarantees that each homogeneous component R_i is a finite-dimensional vector space. The *Hilbert function* $H(R, \cdot)$ is defined for nonnegative integers i by $H(R,i) = \dim_K(R_i)$. It is the most natural invariant which measures the "size" of a graded algebra R. The *Hilbert series* of R is the formal power series

$$F(R,x) = \sum_{i \geq 0} H(R,i)x^i.$$

A well-known result of Hilbert (see e.g. [58, eqn. (1)]) asserts that when R is semistandard, then

$$F(R,x) = \frac{P(x)}{(1-x)^d}, \tag{12}$$

where $P(x) \in \mathbb{Z}[x]$, $P(1) \neq 0$, and $d = \dim R$. It follows from standard results about generating functions (e.g., [65, Prop. 4.2.2 and Cor. 4.3.1]) that $H(R,i)$ is a polynomial in i of degree $d-1$ for sufficiently large i. This polynomial is called the *Hilbert polynomial* of R. The Hilbert polynomial agrees with the Hilbert function for all $i \geq 0$ if and only if $\deg P(x) < d$. If $P(x) = h_0 + h_1 x + \cdots + h_s x^s$ for some $s \geq 0$, then we write $h(R) = (h_0, h_1, \ldots, h_s)$ and call $h(R)$ the *h-vector* of R. Similarly $P(x)$ is called the *h-polynomial* of R. The h-vector and h-polynomial are only defined when R is semistandard. Note that trailing 0's in the h-vector are irrelevant, e.g, $(1,2,1)$ and $(1,2,1,0,0)$ are considered to be the same h-vector.

From now on let us assume that R is semistandard and K is infinite, so R has an l.s.o.p. We now come to the crucial concept of a *Cohen-Macaulay ring*. We will be content with an "operational" definition of what it means for R to be Cohen-Macaulay. It is not difficult to show that this definition is equivalent to the more customary definition involving regular sequences. For further information see e.g. [21][36][43][58][62][63]. Namely, we say that R is a Cohen-Macaulay ring if for some (equivalently, every) l.s.o.p. $\theta_1, \ldots, \theta_d$, the quotient ring $S = S_0 \oplus S_1 \oplus \cdots := R/(\theta_1, \ldots, \theta_d)$ (where the grading is inherited from R) satisfies $H(S,i) = h_i(R)$ for all i. Equivalently (by (12)) the Hilbert series $F(S,x)$ of S is given by

$$F(S,x) = (1-x)^d F(R,x). \tag{13}$$

It is an immediate consequence of the above definition that $h(R) \geq 0$ if R is Cohen-Macaulay.

Example. Let $R = K[u,v]/(uv)$ with $\deg u = \deg v = 1$. Let $\theta_1 = u+v$. Then θ_1 is an l.s.o.p. for R, and $S := R/\theta_1 R$ has a K-basis $\{1, u\}$. Hence $H(S,x) = 1 + x$. Moreover, a K-basis for R is given by $\{1, u, v, u^2, v^2, \ldots\}$, so

$$F(R,x) = 1 + \frac{2x}{1-x} = \frac{1+x}{1-x}.$$

Hence $F(S,x) = (1-x)F(R,x)$, so R is Cohen-Macaulay. On the other hand, we leave it to the reader to verify that the ring $K[u,v,w]/(uv,uw)$ is not Cohen-Macaulay.

We are now ready to discuss the ring A_P which we associate with a simplicial poset P. First let us consider the case when P is a meet-semilattice, so that P is the face lattice of a simplicial complex Δ. In this case A_P turns out to be the *face ring* (also called the *Stanley-Reisner ring*) $K[\Delta]$ of Δ, where K is an arbitrary (infinite) ground field as before. Let us review the definition of $K[\Delta]$. Let the vertices of Δ be $x_1, \ldots, x_m$, and let $K[x_1, \ldots, x_m]$ denote the polynomial ring over K in the variables (indeterminates) $x_1, \ldots, x_m$. Set $K[\Delta] = K[x_1, \ldots, x_m]/I_\Delta$, where I_Δ is the ideal generated by all monomials (which we may assume are squarefree) $x_{i_1} x_{i_2} \cdots x_{i_k}$ such that $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \notin \Delta$. We wish to extend this definition to A_P, for any simplicial poset P. Suppose that the elements of P are $\hat{0} = y_0, y_1, \ldots, y_p$. Define I_P to be the ideal of the polynomial ring $K[y_0, y_1, \ldots, y_p]$ generated by the following elements:

(R_1) $y_i y_j$, if y_i and y_j have no common upper bound in P.

(R_2) $y_i y_j - (y_i \wedge y_j)(\sum_z z)$, where z ranges over all *minimal* upper bounds of y_i and y_j, if y_i and y_j have a common upper bound in P.

(R_3) $y_0 - 1$.

NOTE. It is clear that the greatest lower bound $y_i \wedge y_j$ exists whenever y_i and y_j have an upper bound z in P, since the interval $[\hat{0}, z]$ is a boolean algebra (and therefore a lattice). Thus (R_2) is well-defined.

It is easy to see that when P is the face poset of a simplicial complex Δ then $A_P \cong K[\Delta]$, the face ring of Δ. Hence A_P is a generalization of the face ring. We can give the polynomial ring $K[y_0, y_1, \ldots, y_p]$ the structure of a graded K-algebra by defining the degree of $y \in P$ to be its rank $\rho(y)$. The relations (R_1)–(R_3) are homogeneous with respect to this grading, so we obtain a grading $A_P = (A_P)_0 \oplus (A_P)_1 \oplus \cdots$ of A_P. An important property of A_P [67, Lemma 3.9] is that it is semistandard (with respect to the grading just defined).

A simple computation [67, Prop. 3.8] shows that the Hilbert series of A_P is given by

$$F(A_P, x) = \frac{h_0 + h_1 x + \cdots + h_n x^n}{(1-x)^n}, \tag{14}$$

where $(h_0, h_1, \ldots, h_n)$ is the h-vector of P. In other words $h(P) = h(A_P)$. This is the means by which the h-vector enters into the structure of A_P.

A famous theorem of G. Reisner [21, Cor. 5.3.9][36, Ch. VI][51][62, Cor. 4.2] characterizes those simplicial complexes Δ for which the face ring $K[\Delta]$ is Cohen-Macaulay.

3.3 Theorem (Reisner's theorem). *The face ring $K[\Delta]$ of a simplicial complex Δ is Cohen-Macaulay if and only if Δ is a Cohen-Macaulay simplicial complex (over K). In particular, the face ring $K[\Delta(P)]$ of the order complex of a poset P is Cohen-Macaulay if and only if P is a Cohen-Macaulay poset.*

Theorem 3.3 of course explains the terminology "Cohen-Macaulay complex" and "Cohen-Macaulay poset." We will assume the validity of Theorem 3.3 in this survey. A detailed proof involves concepts from homological algebra which require some work for the uninitiated to absorb. The extension of Reisner's theorem to simplicial posets P is given by the next result [67, Cor. 3.5].

3.4 Theorem. *Let P be a simplicial poset. Then A_P is Cohen-Macaulay if and only P is Cohen-Macaulay.*

Proof (brief sketch). The only part to concern us here is the "if" part. By Reisner's theorem the face ring $K[\Delta(P)]$ of the order complex of P is Cohen-Macaulay if (and only if) P is Cohen-Macaulay. One then shows that A_P is an "algebra with straightening law" (ASL) on P. This means that A_P in a sense is "nicely approximated" by $K[\Delta(P)]$. In particular, the property that $K[\Delta(P)]$ is Cohen-Macaulay is transferred to A_P, so A_P is Cohen-Macaulay when P is Cohen-Macaulay. □

Note that the proof of Theorem 3.2 is now immediate. If P is Cohen-Macaulay then A_P is a Cohen-Macaulay ring, and hence by (13) the h-vector of P is nonnegative. □

Theorem 3.2 can be used to prove Conjecture 2.3 for Eulerian simplicial posets (generalizing the special case of simplicial polytopes due to Purtill [50], so now we have two generalizations of this result, viz., Theorems 2.4 and 3.6). We simply state the relevant result [70, Thm. 3.3] here.

3.5 Theorem. *Let n be a positive integer. Then there exist homogeneous polynomials $\check{\Phi}_1, \ldots, \check{\Phi}_{n-1}$ of degree n in the variables c, d (with $\deg c = 1$, $\deg d = 2$) with nonnegative coefficients such that for every simplicial Eulerian poset P of rank $n+1$ with h-vector $(h_0, h_1, \ldots, h_n)$, we have*

$$\Phi_P(c, d) = \sum_{i=0}^{n-1} h_i \check{\Phi}_i(c, d).$$

Putting together Theorems 3.2 and 3.5 yields the following result.

3.6 Theorem. *Let P be a Gorenstein* simplicial poset. Then $\Phi_P(c, d) \geq 0$.*

The next ingredient we need for our 3/4-characterization of f-vectors of Gorenstein* simplicial posets is the following result, known as the *Dehn-Sommerville equations* for Eulerian simplicial posets.

3.7 Theorem. *Let P be a simplicial Eulerian poset of rank $n+1$ with h-vector $(h_0, h_1, \ldots, h_n)$. Then $h_i = h_{n-i}$ for all i.*

Theorem 3.7 goes back to Dehn, who proved it in 1905 for simplicial 4-polytopes and 5-polytopes and suggested that it be extended to higher dimensions. Such an extension was carried out by Sommerville in 1927, who proved it for arbitrary simplicial polytopes. A generalization to what is essentially the special case of Theorem 3.7 when P is a meet semilattice (i.e., the face poset of a simplicial complex) was proved by Klee in 1964. Numerous generalizations and extensions have been subsequently given. A result equivalent to Theorem 3.7 itself appears in [65, Thm. 3.14.9]. A very general version of the Dehn-Sommerville equations may be found in [68, Lemma 6.4].

Theorems 3.2 and 3.7 show that (a) implies (b) in Theorem 3.1. To show the converse, it is necessary to construct simplicial Gorenstein* posets with suitable h-vectors. This is a fairly easy inductive construction whose details appear in [67, Lemmas 4.1 and 4.2]. With this the proof of Theorem 3.1 is complete. □

It is an intriguing problem to fill in the gap inherent in Theorem 3.1, i.e., to characterize h-vectors of simplicial Gorenstein* posets P of odd rank $n+1$ for which $h_{n/2}$ is also odd. Given that P has odd rank, the condition that $h_{n/2}$ is odd is equivalent to the condition that P has an odd number of facets (elements of rank n). This is because $h_i = h_{n-i}$ and $h_0 + h_1 + \cdots + h_n = f_{n-1}$, the number of facets. We collect together from [67] and [27, Ch. 2] some partial results related to this problem.

3.8 Proposition.

(a) Suppose $h_0 = 1$, $h_i = h_{n-i}$, and $h_i > 0$ for all i. Then there exists a simplicial Gorenstein poset P with $h(P) = (h_0, h_1, \ldots, h_n)$.*

(b) There is no Eulerian simplicial poset P with an odd number of facets (equivalently, P has odd rank $n+1$ and $h_{n/2}$ is odd) and $h_1(P) = 0$.

(c) There is no Eulerian simplicial poset P of rank $n+1$ with an odd number of facets such that

$$h_0(P) = h_1(P) = \cdots = h_i(P) = 1, \quad h_{i+1}(P) = 0,$$

where $i \geq 1$ and $i+1 < n$.

(d) There is no Eulerian simplicial poset P of rank $n+1$ with an odd number of facets such that $\sum_{i=0}^{n} h_i(P) \leq n$.

(e) There is no Eulerian simplicial poset P with an odd number of facets such that $h_1(P) = 2$, $h_2(P) = 0$.

(f) There is no Gorenstein simplicial poset P with an odd number of facets such that $h_1(P) = 3$, $h_2(P) = 0$.*

The following question in particular remains open concerning simplicial Gorenstein* posets P. If P has odd rank $n+1$ and $h_i = 0$ for some $1 \leq i \leq n-1$, then is $h_{n/2}$ even? If the answer is positive then Theorem 3.1 and Proposition 3.8(a) would give a complete solution to the problem of characterizing h-vectors of simplicial Gorenstein* posets. Let us finally remark that the problem of characterizing h-vectors of simplicial Gorenstein* posets bears some similarities to the problem of characterizing the Betti numbers of compact manifolds (see [23, Thm. 2.2]).

We now consider one of the most interesting classes of simplicial Gorenstein* posets, viz., the face lattices $L_{\mathcal{P}}$ of simplicial convex polytopes $\mathcal{P}$ (i.e., convex polytopes all of whose facets are simplices). For general information about convex polytopes see e.g. [19][35][47][73]. According to Theorems 3.2 and 3.7 the h-vector of $\mathcal{P}$ satisfies $h_0 = 1$, $h_i \geq 0$, and $h_i = h_{n-i}$, where $n = \dim \mathcal{P}$. In 1971 McMullen and Walkup [48] published the conjecture that the h-vector was *unimodal*, i.e., weakly increases and then decreases. In view of the Dehn-Sommerville equations ($h_i = h_{n-i}$) this is equivalent to the inequalities

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor n/2 \rfloor}. \tag{15}$$

The inequalites (15) were called the *Generalized Lower Bound Conjecture* (GLBC), because they implied an earlier *Lower Bound Conjecture* (proved by Barnette [2][3] in 1971 and 1973 by geometric reasoning). McMullen and Walkup [48] showed that the GLBC, if true, gave the all the homogeneous linear inequalities satisfied by the f-vector of a simplicial convex polytope (except for $h_0 \geq 0$, which is trivial since $h_0 = 1$). Also, the only additional *inhomogeneous* linear equality or inequality is given by the trivial $h_0 = 1$.

In 1970 in East Lansing, Michigan, McMullen made an inspired conjecture, based on very little evidence. Namely, he conjectured a complete characterization of the h-vector of a simplicial polytope. This conjecture was published in 1971 [44] and became known as *McMullen's g-conjecture* (because of the use of the notation g_{i-1} to denote the crucial quantity $h_i - h_{i-1}$; it is now customary to use g_i for this quantity). It was later proved by Billera-Lee and Stanley (as discussed below), and so is now called the *g-theorem*. To state this result, we use the fact (easily proved by induction) that given positive integers h and i, there is a unique way to write

$$h = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}, \tag{16}$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$. For a nice discussion of the significance of this representation, see [34, Section 8]. Now define

$$h^{\langle i \rangle} = \binom{n_i + 1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j + 1}{j+1},$$

and set $0^{\langle i \rangle} = 0$. The number $h^{\langle i \rangle}$ is sometimes called the *ith upper pseudopower of h*. Call a vector $(h_0, h_1, \ldots, h_d) \in \mathbb{Z}^{d+1}$ an *M-vector* (after F. S. Macaulay, for a reason soon to be made clear) if $h_0 = 1$ and $0 \leq h_{i+1} \leq h_i^{\langle i \rangle}$ for $1 \leq i \leq d-1$. We can now state the remarkable g-theorem first conjectured by McMullen.

3.9 The g-Theorem. *A vector $(h_0, \ldots, h_d) \in \mathbb{Z}^{d+1}$ is the h-vector of some simplicial d-polytope $\mathcal{P}$ if and only if $h_i = h_{d-i}$ and $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an M-vector.*

The sufficiency of McMullen's g-theorem (i.e., the "if" part of the theorem) was first proved by Billera and Lee in 1979 (see [12] and [13]) and the necessity soon afterwards by this writer. The proof of Billera and Lee involved a clever inductive construction. The proof of necessity, on the other hand, required deep techniques from algebraic geometry. The first step is an algebraic interpretation of M-vectors.

3.10 Theorem. *A vector $(k_0, k_1, \ldots, k_s) \in \mathbb{Z}^{s+1}$ is an M-vector if and only if there exists a standard graded algebra $R = R_0 \oplus R_1 \oplus \cdots \oplus R_s$ satisfying $H(R, i) = h_i$ for $0 \leq i \leq s$.*

The difficult part of the proof of Theorem 3.10 is due to F. S. Macaulay [40]. His intricate argument has subsequently been simplified [55][72][25], though it is still not an easy result. The purely numerical form of the theorem as stated here first appeared in [58].

To prove the necessity of the g-theorem (i.e., the "only if" part), it follows from Theorem 3.10 that we need to find a standard graded algebra R whose Hilbert function satisfies $H(R, i) = h_i - h_{i-1}$ for $1 \leq i \leq \lfloor n/2 \rfloor$, where $(h_0, h_1, \ldots, h_n)$ is the h-vector of the simplicial polytope $\mathcal{P}$. Reisner's theorem implies that the face ring $K[\mathcal{P}]$ of the boundary complex of $\mathcal{P}$ is Cohen-Macaulay. Hence if $\theta_1, \ldots, \theta_n$ is any l.s.o.p., then the quotient ring $S = K[\mathcal{P}]/(\theta_1, \ldots, \theta_n)$ satisfies $H(S, i) = h_i$. This is not what we want, but it is close. Suppose we could find an element $\omega \in S_1$ which is a non-zerodivisor on S_i for $i < \lfloor n/2 \rfloor$. Let $R = S/\omega S$. Then the condition on ω implies that for $i < \lfloor n/2 \rfloor$,

$$\begin{aligned} \dim_K R_i &= \dim_K S_i - \dim_K \omega S_{i-1} \\ &= \dim_K S_i - \dim_K S_{i-1} \\ &= h_i - h_{i-1}. \end{aligned}$$

Hence R is the desired ring, and the necessity of the g-theorem would be proved.

To show the existence of the critical element ω (called a *Lefschetz element*), assume without loss of generality that the simplicial n-polytope $\mathcal{P}$ is embedded in

$\mathbb{R}^n$ so that the origin is in the interior and the vertices have rational coordinates. (Not every convex polytope is *rational*, i.e., can be embedded in $\mathbb{R}^n$ so that the vertices have rational coordinates. However, all simplicial polytopes are rational since small perturbations of their vertices do not change their combinatorial type. For further information, see [35, p. 95] [17, Fig. 3.4.1][73, Example 6.21].) It is then possible to associate with $\mathcal{P}$ an algebraic variety $X_{\mathcal{P}}$, which is a special case of a *toric variety*. In general there is an intimate connection between convex polytopes and related geometric objects on the one hand, and toric varieties on the other. For further information on these fascinating connections, see for example [26][28][31][49]. The connection between $\mathcal{P}$ and $X_{\mathcal{P}}$ which is of concern to us here is the following: The (singular) cohomology ring (say with real coefficients) $H^*(X_{\mathcal{P}};\mathbb{R})$ of $X_{\mathcal{P}}$ is isomorphic to $\mathbb{R}[\mathcal{P}]/(\theta_1,\ldots,\theta_n)$, where $\theta_1,\ldots,\theta_n$ is a certain l.s.o.p. of $\mathbb{R}[\mathcal{P}]$ (which depends on how $\mathcal{P}$ is embedded in $\mathbb{R}^n$).

The varieties $X_{\mathcal{P}}$ are projective and have very simple singularities (called "finite quotient singularities"). From this it can be shown that they satisfy the *hard Lefschetz theorem* (so called because Lefschetz stated two theorems about the cohomology of algebraic varieties, one of which is much more difficult to prove than the other). The hard Lefschetz theorem implies the existence of the desired Lefschetz element ω, completing the proof of the necessity of the g-theorem.

For a more detailed exposition of the above proof, see [10][28][31][49][64]. The reference [64] contains in particular some historical comments on the proof (p. 221). Let us just add here an additional historical remark which does not appear in the previous references. The crucial and most difficult result used in the proof is the hard Lefschetz theorem for the varieties $X_{\mathcal{P}}$. In the original published proof [61] of the necessity of the g-theorem, a proof of the hard Lefschetz theorem for $X_{\mathcal{P}}$ was attributed to Steenbrink in 1976. However, it turned out that the proof of Steenbrink was incorrect. The first correct proof was not given until 1983 by M. Saito [53], using the deep machinery of filtered $\mathcal{D}$-modules. Fortunately I did not become aware of the incorrectness of Steenbrink's proof until after Saito had published his correct proof!

After the g-theorem was proved by the methods indicated above, it became a challenge to try to simplify the proof. The use of ring theory seems inevitable because of the simple algebraic interpretation of M-vectors. One would like, however, to show the existence of a Lefschetz element ω without using the theory of toric varieties. Recently exactly such a proof was found by McMullen [46], using his important concept of the *polytope algebra* [45]. McMullen's proof is a high point in the geometric tradition founded by Minkowski. The polytope algebra is a fascinating object which will undoubtedly receive intense further study.

A natural question at this point is the following: Does the g-theorem hold for a wider class of geometric objects than simplicial polytopes? Three classes of objects to consider in this context are (a) PL spheres (which may be defined as triangulations of spheres such that the link of every vertex is also a sphere), (b) triangulations of spheres, and (c) Gorenstein* lattices. Each of the three classes is more general than the preceding. For all three classes it remains open whether the g-theorem is valid. A promising approach toward proving the g-theorem for case (b) (and possibly even (c)) was found by Kalai, based on his theory of algebraic shifting and some work on embeddability of simplicial complexes due to Sarkaria. However, this approach has not yet led to a proof of the g-theorem for spheres. The problem of extending the g-theorem to spheres is perhaps the most outstanding open problem in the subject of Eulerian posets. About the only significant generalization of the g-theorem to date is an extension of the GLBC to triangulations of $(n-2)$-dimensional spheres which are subcomplexes of boundary complexes of simplicial n-polytopes. This result was first proved by Kalai [38, §8] using algebraic shifting, and was given a somewhat simpler proof based on face rings in [69, Cor. 2.4].

4. The h-Vector of an Eulerian Poset.

In the previous section we defined the h-vector of a simplicial Eulerian poset P using the simple formula (11). (CAVEAT: Do not confuse this h-vector with the h-vector of the order complex of P.) We mentioned that extending this definition in the obvious way to an arbitrary Eulerian poset does not seem interesting. Instead there is a subtle generalization motivated by topology and algebraic geometry. Recall from our discussion of McMullen's g-theorem that if $\mathcal{P}$ is a simplicial n-polytope (embedded in $\mathbb{R}^n$ with rational vertices and origin in the interior), then the cohomology ring $H^*(X_{\mathcal{P}};\mathbb{R})$ of the toric variety $X_{\mathcal{P}}$ is isomorphic to $S := \mathbb{R}[\mathcal{P}]/(\theta_1,\ldots,\theta_n)$ for a certain l.s.o.p $\theta_1,\ldots,\theta_n$. The usual grading of $H^*(X_{\mathcal{P}};\mathbb{R})$ is such that $H^{2i}(X_{\mathcal{P}};\mathbb{R})$ corresponds to S_i (so that $H^{2i+1}(X_{\mathcal{P}};\mathbb{R})=0$ for all i). Thus

$$\dim_{\mathbb{R}} H^{2i}(X_{\mathcal{P}};\mathbb{R}) = h_i(\mathcal{P}). \tag{17}$$

Now a toric variety $X_{\mathcal{P}}$ can be defined for *any* rational polytope $\mathcal{P}$. (It need not be simplicial.) We could try to define an h-vector for $\mathcal{P}$ *via* equation (17). Unfortunately this naive definition is not satisfactory. For one thing, it is not determined solely by the combinatorial type of $\mathcal{P}$ but rather depends on how $\mathcal{P}$ is embedded into $\mathbb{R}^n$. There is, however, a homology theory which is better behaved, viz., the *intersection homology* (or cohomology) theory of Goresky and MacPherson (see e.g. [32][33][42]). Let $IH^i(X_{\mathcal{P}};\mathbb{R})$ denote the i-th real middle-perversity intersection cohomology group of the toric variety $X_{\mathcal{P}}$, and let

$$h_i = h_i(\mathcal{P}) = \dim_{\mathbb{R}} IH^{2i}(X_{\mathcal{P}};\mathbb{R}).$$

One can show the following facts:

(IH$_1$) The odd-degree groups $IH^{2i+1}(X_{\mathcal{P}};\mathbb{R})$ are all 0.

(IH$_2$) h_i depends only on the combinatorial type of the rational polytope $\mathcal{P}$, and not on its embedding.

(IH$_3$) $h_i = h_{n-i}$ for all i (where $n=\dim\mathcal{P}$), and $h_i = 0$ for $i > n$. This result is a consequence of *Poincaré duality* for intersection homology.

(IH$_4$) $1 = h_0 \le h_1 \le \cdots \le h_{\lfloor n/2\rfloor}$. This is a consequence of the difficult *hard Lefschetz theorem* for intersection homology due to Beilinson, Bernstein and Deligne [8] (see [42, §1.6.2]).

Since $h_i(\mathcal{P})$ depends only on the combinatorial type of $\mathcal{P}$, we would like a combinatorial formula or rule for computing $h_i(\mathcal{P})$. Such a result was given independently by Khovanskiĭ and MacPherson (and perhaps others), and was first published in [66]. This rule turns out to make sense for any Eulerian poset P, so we use it to *define* the h-vector of P (more precisely, of $P - \{\hat{1}\}$). The definition depends on the following result.

4.1 Theorem. *Let P be an Eulerian poset of rank $n+1$ with rank function ρ. Then there exists a unique function $h : P \to \mathbb{Z}[x]$ satisfying the conditions (where we write h_t for the value of h at the point $t \in P$):*

(a) For all $t > \hat{0}$ in P we have

$$\deg h_t(x) \leq \lfloor \frac{1}{2}(\rho(t) - 1) \rfloor.$$

(b) For all $t \in P$, we have

$$\sum_{s \leq t} h_s(x)(x-1)^{\rho(s,t)} = x^{\rho(t)} h_t(1/x). \tag{18}$$

For instance, when the interval $[\hat{0}, t]$ is a boolean algebra, then the binomial theorem implies that $h_t(x) = 1$.

If P is Eulerian, then the polynomial $h(P,x) := h_{\hat{1}}(x)$ is called the *h-polynomial* of P or *g-polynomial* of $P - \{\hat{1}\}$. We want, however, to define the h-polynomial of $P - \{\hat{1}\}$. More generally, define a (finite) poset P to be *lower Eulerian* if P has a $\hat{0}$ and every interval $[\hat{0}, t]$ is Eulerian. Note that for such a poset, the polynomial $h_t(x)$ of Theorem 4.1 is still defined for each $t \in P$, viz., $h_t(x) = h([\hat{0}, t], x)$. Thus for any lower Eulerian poset P of rank n, we can define the *h-polynomial* of P by the formula

$$x^n h(P, 1/x) = \sum_{t \in P} h_t(x)(x-1)^{n-\rho(t)}. \tag{19}$$

Moreover, if $h_P(x) = h_0 + h_1 x + \cdots + h_n x^n$, then we define $(h_0, h_1, \ldots, h_n)$ to be the *h-vector* of P.

The definition of the h-vector of a lower Eulerian poset is not easy to understand, and many of its properties remain mysterious. Let us point out a few salient facts concerning the above definition.

- Suppose that P is simplicial (and hence lower Eulerian) of rank n. Since $h(B_k, x) = 1$ for any boolean algebra B_k, equation (19) reduces to

$$\begin{aligned} x^n h(P, 1/x) &= \sum_{t \in P} (x-1)^{n-\rho(t)} \\ &= \sum_{i=0}^{n} f_{i-1}(x-1)^{n-i}, \end{aligned}$$

where P has f_{i-1} elements of rank i. Comparing with (11) shows that the the two definitions we have given of the h-vector (or h-polynomial) of P agree.

- We mentioned that the original motivation for the definition of $h(P,x)$ came from the intersection homology of toric varieties. Specifically, if $\mathcal{P}$ is a *rational* n-dimensional polytope, then let P be its face poset with $\hat{1}$ removed. Thus P is lower Eulerian. It is then the case that $h_i(P) = \dim_{\mathbb{R}} IH^{2i}(X_{\mathcal{P}}; \mathbb{R})$. In particular, $h_i(P) \geq 0$. Note that (IH$_3$) gives $h_i = h_{n-i}$ (a generalization of the Dehn-Sommerville equations), while (IH$_4$) yields $1 = h_0 \leq h_1 \leq \cdots \leq h_{\lfloor n/2 \rfloor}$.

- It is natural to ask to what extent the three results $h_i = h_{n-i}$, $h_i \geq 0$, and $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor n/2 \rfloor}$ hold for lower Eulerian posets more general than face posets of rational polytopes (with $\hat{1}$ removed). So far as the Dehn-Sommerville equations are concerned, the following result can be obtained by a formal computation [66, Thm. 2.4] [65, Thm. 3.14.9][68, Lemma 6.4].

4.2 Theorem. *Let P be an Eulerian poset of rank $n+1$ with $\hat{1}$ removed (so P has rank n). Then $h_i(P) = h_{n-i}(P)$ for all i.*

- Suppose that P is Eulerian of rank $n+1$, and let $\tilde{P} = P - \{\hat{1}\}$. Let $h(\tilde{P}, x) = h_0 + h_1 x + \cdots + h_n x^n$. Thus by Theorem 4.2 we have $h_i = h_{n-i}$. It is a simple consequence of the definition of $h(P,x)$ that

$$h_i(P) = \begin{cases} h_i(\tilde{P}) - h_{i-1}(\tilde{P}), & \text{if } i \leq \lfloor n/2 \rfloor \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

In particular, knowing either of $h(P,x)$ or $h(\tilde{P},x)$ determines the other.

- Let us consider the question of the nonnegativity and unimodality of the h-vector of an arbitrary lower Eulerian poset. The unimodality question seems natural only in the presence of the Dehn-Sommerville equations $h_i = h_{n-i}$. The most general "natural" class of posets with this property consists of (by Theorem 4.2) Eulerian posets with $\hat{1}$ removed. Now note that by (20), we have $h(P,x) \geq 0$ if and only if $h(\tilde{P},x)$ is unimodal. Hence the unimodality question is subsumed by the nonnegativity question. Moreover, since the most general "natural" class of simplicial complexes with nonnegative h-vector are the Cohen-Macaulay ones, we should restrict our attention to Cohen-Macaulay lower Eulerian posets.

It is false that $h_i(P) \geq 0$ for all Cohen-Macaulay lower Eulerian posets. For instance, the Cohen-Macaulay Eulerian poset P given by Figure 1 satisfies $h(P,x) = 1 - x$, since

$$(x-1)^3 + 2(x-1)^2 + 2(x-1) + (1-x) = x^3(1 - \frac{1}{x}).$$

Moreover, the Cohen-Macaulay lower Eulerian poset P (which is an Eulerian poset with $\hat{1}$ removed) given by Figure 3 satisfies

$$h(P,x) = x^3 h(P,1/x) = (x-1)^3 + 2(x-1)^2 + 2(x-1) + 2(1-x) = x^3 - x^2 - x + 1.$$

Although not all Cohen-Macaulay lower Eulerian posets have nonnegative h-vector, we do have the following conjecture [66, Conj. 4.2(b)].

4.3 Conjecture. *Let P be a Cohen-Macaulay lower Eulerian meet-semilattice. Then $h_i(P) \geq 0$ for all i.*

Conjecture 4.3 is open even for face lattices of *nonrational* polytopes, with or without $\hat{1}$ removed. Conjecture 4.3 is one of the outstanding open problems in the theory of Eulerian posets.

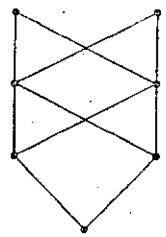

Fig. 3. A lower Eulerian poset with nonpositive h-vector

- It is not difficult to show that for any lower Eulerian poset P, the h-polynomial $h(P,x)$ can be computed from the flag f-vector of P [5, Prop. 6.1]. In fact, if P is Eulerian then each $h_i(P)$ or $h_i(\tilde{P})$ is a *unique* linear combination of the coefficients of the cd-index $\Phi_P(c,d)$. More precisely, for each n there is a unique linear transformation φ from the space of all cd-polynomials of degree n (with $\deg c = 1$ and $\deg d = 2$ as usual) with rational coefficients to the space $\mathbb{Q}^{n+1}$ which for every Eulerian poset P of rank $n+1$ satisfies $\varphi(\Phi_P(c,d)) = h(\tilde{P})$. However, this expression for $h_i(P)$ or $h_i(\tilde{P})$ in terms of the cd-index seems quite messy and not of much use. For example, writing $[w]$ for the coefficient of the cd-word w in $\Phi_P(c,d)$, we have:

$$\begin{aligned} h_0(P) &= [c^n] \\ h_1(P) &= -(n-2)[c^n] + [dc^{n-2}] \\ h_2(P) &= \frac{1}{2}(n-1)(n-4)[c^n] - (n-4)[dc^{n-2}] - [c^2dc^{n-4}] + [d^2c^{n-4}]. \end{aligned}$$

5. Acceptable Functions on a Lower Eulerian Poset.

The condition (18) satisfied by the h-vector of an Eulerian (or even lower Eulerian) poset is quite interesting for its own sake. Thus given a lower Eulerian poset P, define a function $f : P \to \mathbb{Q}[x]$ (other fields could be used instead of $\mathbb{Q}$, but there is no reason to do so here) to be *acceptable* if for all $t \in P$ we have

$$\sum_{s \le t} f_s(x)(x-1)^{\rho(s,t)} = x^{\rho(t)} f_t(1/x). \tag{21}$$

Note that if f is acceptable then $\deg f_t \le \rho(t)$ for all $t \in P$, since otherwise the right-hand side of (21) would not be a polynomial. Let $\mathcal{A}(P)$ denote the set of all acceptable functions $f : P \to \mathbb{Q}[x]$. Clearly $\mathcal{A}(P)$ is a vector space over $\mathbb{Q}$; moreover, since $\deg f_t \le \rho(t)$ it follows that

$$\dim \mathcal{A}(P) \le \sum_{t \in P} (\rho(t) + 1).$$

The next theorem gives a more precise result.

5.1 Theorem. *Suppose P is lower Eulerian. Then*

$$\dim \mathcal{A}(P) = \sum_{t \in P} \lfloor \frac{1}{2}\rho(t) + 1 \rfloor.$$

Proof (sketch). Let $t \in P$. Suppose that we have defined f_s for all $s < t$ so that equation (21) holds (with t replaced by s). We claim that the set of all polynomials f_t satisfying (21) is a vector space of dimension $\lfloor \frac{1}{2}\rho(t) + 1 \rfloor$, from which the proof will follow. The key point is that the polynomial $k_t(x)$ given by

$$k_t(x) = \sum_{s<t} f_s(x)(x-1)^{\rho(s,t)}$$

is *antisymmetric*, i.e., $k_t(1/x) = -k_t(x)$. This is proved by a formal argument, using the fact that $\mu(u,v) = (-1)^{\rho(u,v)}$ in a lower Eulerian poset P. For complete details (in a more general setting), see [68, Lemma 6.4]. Now from (21) we have

$$k_t(x) = x^{\rho(t)} f_t(1/x) - f_t(x). \tag{22}$$

Since both $k_t(x)$ and the right-hand side of (22) are antisymmetric, we are free to choose the first $\lfloor \frac{1}{2}\rho(t)+1 \rfloor$ coefficients of f_t arbitrarily, and the remaining coefficients are then uniquely determined. From this the proof follows. (Note that if $k_t(x)$ were not antisymmetric, then there would be no way to choose $f_t(x)$, so it would not be possible to build up f "greedily" as we have done.) □

Theorem 5.1 describes the structure of the space of all acceptable functions on P. We would now like to give some interesting examples of acceptable functions, in addition to the h-polynomial. We will give two classes of examples, the first dealing with subdivisions and the second with Ehrhart polynomials.

The definition of a subdivision of an Eulerian poset is motivated by the usual notion in topology of a subdivision of a simplicial complex (or of more general geometric objects). A *geometric simplex* is the convex hull in some Euclidean space $\mathbb{R}^m$ of a set of affinely independent points. A *geometric simplicial complex* is a collection Σ of finitely many geometric simplices contained in $\mathbb{R}^m$ such that (a) if $\alpha \in \Sigma$ and τ is a face of α, then $\tau \in \Sigma$, and (b) if $\alpha, \tau \in \Sigma$, then $\alpha \cap \tau$ is a common face (possibly empty) of both α and τ. A *geometric simplicial subdivision* (or *geometric triangulation*, often called just a *triangulation*) of Σ is a geometric simplicial complex Σ' such that

$$\bigcup_{\alpha \in \Sigma} \alpha = \bigcup_{\alpha' \in \Sigma'} \alpha',$$

and such that every face of Σ' is contained in a face of Σ. Now let Δ and Δ' be the face posets of Σ and Σ', respectively. We may regard Δ and Δ' as *abstract* simplicial complexes. How can we abstract the notion that Σ' is a subdivision of Σ in the most general useful way to define what it means for Δ' to be a subdivision of Δ? There turn out to be many possible definitions, depending on how much of the geometric essence of the notion of subdivision we wish to preserve. In all cases, we will have a map $\sigma : \Sigma' \to \Sigma$ satisfying certain properties. One should think of $\sigma(t)$ as the smallest face of Δ which contains t (called the *carrier* of t). Thus the motivation for the following definition should be clear. We say that Δ' (or more precisely, the pair (Δ', σ)) is a *topological subdivision* of Δ if (a) for every face F of Δ, the subset $\sigma^{-1}(2^F)$ of Δ' (where 2^F denotes the subcomplex of Δ consisting of F and all its faces) is a subcomplex of Δ' whose geometric realization $|\sigma^{-1}(2^F)|$ is homeomorphic to a ball of dimension $\#F - 1$, and (b) $\sigma^{-1}(F)$ consists of the interior faces of the ball $\sigma^{-1}(2^F)$. We have called this type of subdivision *topological* because it depends on the topological notion of *ball* to define it. As such it is a difficult definition to work with, because there is no useful combinatorial or algebraic characterization of simplicial complexes whose geometric realization is a ball. In fact, it is known that the question of deciding whether a simplicial complex is a ball is undecidable! What are the properties of balls that are essential here? This depends on what kind of applications we have in mind. If one is only interested in showing that certain functions are acceptable, then the key property we need is the following result. For a proof of a more general result, see [62, Ch. II, Cor. 7.2]. A result equivalent to a generalization of Theorem 5.2 to manifolds with boundary was earlier proved by Macdonald [41, Thm. 2.1]. Let Δ be a simplicial complex whose geometric realization is an $(n-1)$-dimensional ball. Recall that the h-polynomial $h(\Delta, x) = h_0 + h_1 x + \cdots + h_n x^n$ of Δ is defined by

$$x^n h(\Delta, 1/x) = \sum_{F \in \Delta} (x-1)^{n-\#F}.$$

Define the *interior h-polynomial* $h(\mathrm{int}(\Delta), x)$ by

$$x^n h(\mathrm{int}(\Delta), 1/x) = \sum_{F \in \mathrm{int}(\Delta)} (x-1)^{n-\#F},$$

where $\mathrm{int}(\Delta)$ denotes the set of interior faces (faces not contained in the boundary) of Δ.

5.2 Theorem. *With Δ as above, we have*

$$h(\mathrm{int}(\Delta), x) = x^n h(\Delta, 1/x).$$

Now let P and P' be lower Eulerian posets, and let $\sigma : P' \to P$ be a map satisfying:

(S_1) For all $t \in P'$, $\rho(t) \le \rho(\sigma(t))$ (where ρ denotes the rank function of P).

(S_2) For every $u \in P$, let $P_{\le u} = \{v \in P : v \le u\}$ (the principal order ideal generated by u). Then $P'_u := \sigma^{-1}(P_{\le u})$ is an order ideal of P', i.e., if $t \in P'$ and $s \le t$ then $s \in P'$.

Let $h(P', x)$ denote the h-polynomial of P', as defined by equation (19). Define the *interior* $\mathrm{int}(P'_u)$ of the poset P'_u by $\mathrm{int}(P'_u) = \sigma^{-1}(u)$. Finally define the h-polynomial $h(\mathrm{int}(P'_u), x)$ by

$$x^{\rho(u)} h(\mathrm{int}(P'_u), 1/x) = \sum_{t \in \mathrm{int}(P'_u)} h_t(x)(x-1)^{\rho(u)-\rho(t)}.$$

Guided by Theorem 5.2, we now say that P' (or more accurately, the pair (P', σ)) is a *formal subdivision* of P if (S_1) and (S_2) are satisfied, and if the following condition is also satisfied:

(S_3) For all $u \in P$ we have

$$x^{\rho(u)} h(P'_u, 1/x) = h(\mathrm{int}(P'_u), x). \tag{23}$$

In particular, topological subdivisions of simplicial complexes (or even of regular CW complexes, with an obvious extension of the definition of topological subdivision) are formal subdivisions. The main theorem on formal subdivisions is the following [68, Thm. 7.5].

5.3 Theorem. *Let $\sigma : P' \to P$ be a formal subdivision of the lower Eulerian poset P. Let $f : P \to \mathbb{Z}[x]$ be defined by $f_u(x) = h(P'_u, x)$. Then f is acceptable.*

Theorem 5.3 shows that every formal subdivision of an Eulerian poset P "induces" an acceptable function f on P. For topological triangulations Δ' of a simplicial complex Δ, the value f_t of f on a face u of Δ is just the usual h-polynomial of the restriction of Δ' to Δ. For a formal subdivision $\sigma : P' \to P$ of an *Eulerian* (rather than lower Eulerian) poset P, we can define a highly interesting polynomial $\ell(\sigma, x)$, called the *local h-polynomial* of σ, by the formula

$$\ell(\sigma, x) = \sum_{u \in P} (-1)^{\rho(u,\hat{1})} h(P'_u, x) h([u, \hat{1}]^*, x). \tag{24}$$

Here $[u, \hat{1}]^*$ denotes the dual of the interval $[u, \hat{1}]$. One fundamental property of $\ell(\sigma, x)$ is the following symmetry result [68, Cor. 7.7], which is proved as a formal consequence of equation (23).

5.4 Theorem. *Let $\sigma : P' \to P$ be a formal subdivision of the Eulerian poset P of rank $n+1$. Then*

$$x^n \ell(\sigma, 1/x) = \ell(\sigma, x). \tag{25}$$

In the more intuitive case of simplicial complexes, P is a boolean algebra (the face lattice of a simplex 2^V), so $h([u, \hat{1}]^*, x) = 1$ for all $u \in P$. Moreover P' is just a simplicial complex Γ, and we often write $\ell_V(\Gamma, x)$ for $\ell(\sigma, x)$ (though $\ell_V(\Gamma, x)$ actually depends on σ [68, Ex. 2.3(e)]). Equation (24) takes the simple form

$$\ell_V(\Gamma, x) = \sum_{u \in P} (-1)^{\rho(u,\hat{1})} h(P'_u, x), \tag{26}$$

i.e., the alternating sum of the h-polynomials of the restrictions of Γ to the faces of the simplex 2^V.

We do not have the space here to go into a detailed discussion of the local h-polynomial $\ell_V(\Gamma, x)$, but let us mention a few highlights. We need to introduce two further types of subdivisions. Define a topological subdivision $\sigma : \Delta' \to \Delta$, where Δ' and Δ are simplicial complexes, to be *quasi-geometric* if no face t of Δ' has the following property: There is a face F of Δ such that $\dim(F) < \dim(t)$ and every vertex x of t satisfies $\sigma(x) \le F$. In other words, no face t of Δ' has all its vertices on a face of Δ of lower dimension than t.

Note that geometric subdivisions (more accurately, triangulations of abstract simplicial complexes which can be geometrically realized by a geometric triangulation) are quasi-geometric, since the vertices of a geometric simplex are affinely independent. It can be shown that there exist quasi-geometric subdivisions which are not geometric [68, p. 814]. The simplest example of a topological subdivision which is not quasi-geometric is the following. Let Δ be a triangle with vertices a, b, c. Add a new vertex d on the line ab and let Δ' have facets abc and abd. This subdivision is not quasi-geometric since the two-dimensional face abd of Δ' has all its vertices on the one-dimension face ab of Δ.

The last type of subdivision which we consider (though there are still others of interest, such as homological subdivisions and homotopical subdivisions) is the following. A geometric triangulation of a simplicial complex is called *regular* if (speaking somewhat informally) its restriction to each face (simplex) can be realized as the projection of a strictly convex polyhedral surface. For a more detailed definition, see [15][39][68, Def. 5.1]. An abstract triangulation of a simplicial complex is *regular* if it has a regular geometric realization. It can be shown that not every geometric subdivision of a geometric simplicial complex is regular [15, Fig. 1][39, Fig. 2(b)].

We now have all the definitions necessary to summarize some basic properties of the local h-polynomial $\ell_V(\Gamma, V)$ of a triangulation Γ of the simplex 2^V.

First, as a special case of the symmetry result (25), we have

$$x^n \ell_V(\Gamma, 1/x) = \ell_V(\Gamma, x), \tag{27}$$

where $\#V = n$.

Now let Δ be a pure (i.e., all maximal faces have the same dimension) $(n-1)$-dimensional simplicial complex, and let Δ' be a topological triangulation. Then the h-polynomial $h(\Delta', x)$ is given by the following result [68, Thm. 3.2].

5.5 Theorem. *With notation as above, we have*

$$h(\Delta', x) = \sum_{F \in \Delta} \ell_F(\Delta'_F, x) h(\mathrm{lk}_\Delta F, x), \tag{28}$$

where Δ'_F denotes the restriction of Δ' to the face F of Δ.

Equation (28) is a fundamental result for reducing questions about f-vectors of triangulations to properties of local h-vectors.

A much deeper result on local h-vectors than equation (27) and Theorem 5.5 is the following.

5.6 Theorem. *Let Γ be a quasi-geometric triangulation of the simplex 2^V. Then $\ell_V(\Gamma, x) \geq 0$ (i.e., every coefficient is nonnegative).*

Proof (brief sketch). The proof requires commutative and homological algebra. The idea is as follows. Let $K[\Gamma]$ be the face ring of Γ, and let $V = \{x_1, \ldots, x_n\}$. An l.s.o.p. $\theta_1, \ldots, \theta_n$ is called *special* if for each i, we have that θ_i is a linear combination of vertices of Γ whose carrier contains x_i. It is not hard to show that if Γ is a topological triangulation of 2^V, then a special l.s.o.p. exists if and only if Γ is quasi-geometric [68, Cor. 4.4]. Supposing that $\theta_1, \ldots, \theta_n$ is special, let Ω be the ideal of $K[\Gamma]$ generated by all monomials whose support does not lie on the boundary of Γ. Let $L_V[\Gamma]$ be the image of Ω in the quotient $K[\Gamma]/(\theta_1, \ldots, \theta_n)$. Then $L_V[\Gamma]$ is a graded $K[\Gamma]$-module, say $L_V[\Gamma] = L_0 \oplus L_1 \oplus \cdots$. One shows by a homological argument [68, Thm. 4.6] that the Hilbert series $F(L_V[\Gamma], x) := \sum_i (\dim_K L_i) x^i$ is just $\ell_V(\Gamma, x)$, and the proof follows. □

The conclusion of Theorem 5.6 fails for topological triangulations. For example, let $V = \{a, b, c, d\}$. Let Γ be the triangulation of 2^V obtained by adding a vertex e inside face abc, and letting the facets of Γ be $abcd$ and $abce$. Then $\ell_V(\Gamma, x) = -x^2$. This example was found by Clara Chan.

The next result gives an important application of Theorem 5.6 (in fact, the original *raison d'etre* for the creation of local h-vectors).

5.7 Corollary. *Let Δ be a Cohen-Macaulay simplicial complex and Δ' a quasi-geometric triangulation. Then $h(\Delta') \geq h(\Delta)$. In other words, the h-vector of a Cohen-Macaulay complex increases under quasi-geometric triangulation.*

Proof. Consider equation (28). By Theorem 5.6 each polynomial $\ell_F(\Delta'_F, x)$ is nonnegative. By definition of a Cohen-Macaulay complex Δ, each link $\mathrm{lk}_\Delta F$ is Cohen-Macaulay. We mentioned in Section 1 the result that Cohen-Macaulay complexes have nonnegative h-vectors. Hence each polynomial $h(\mathrm{lk}_\Delta F, x)$ appearing in (28) is nonnegative. Moreover, the term in the sum on the right-hand side of (28) indexed by $F = \emptyset$ is just $h(\Delta, x)$. From this the proof is immediate. □

Corollary 5.7 is false if Δ is not assumed to be Cohen-Macaulay (but is assumed to be pure) [68, Ex. 4.12]. It is not known whether the h-vector of a Cohen-Macaulay complex increases under *topological* triangulation. This is an intriguing open problem in the theory of Cohen-Macaulay complexes (due to G. Kalai and this writer), which we state as an explicit conjecture.

5.8 Conjecture. *Let Δ' be a topological triangulation of a Cohen-Macaulay simplicial complex. Then*

$$h(\Delta') \geq h(\Delta).$$

We next consider the situation for regular triangulations.

5.9 Theorem. *Let Γ be a regular triangulation of the simplex 2^V. Then $\ell_V(\Gamma, x)$ is unimodal, i.e. (using Theorem 5.4), if $\ell_V(\Gamma, x) = \ell_0 + \ell_1 x + \cdots + \ell_n x^n$, then $\ell_0 \leq l_1 \leq \cdots \leq \ell_{\lfloor n/2 \rfloor}$.*

The proof of Theorem 5.9 is very deep; it requires the hard Lefschetz theorem for the decomposition theorem of intersection homology (though the proof is quite easy if one is willing to accept all this machinery). For further details see [68, Thm. 5.2]. There is an application of Theorem 5.9 analogous to how Corollary 5.7 follows from Theorem 5.6. The easy proof is omitted.

5.10 Corollary. *Let Δ be a pure simplicial complex such that for every face $F \in \Delta$ the h-vector of $\mathrm{lk}_\Delta F$ is symmetric ($h_i = h_{d-\#F-i}$ for all i) and unimodal ($h_0 \leq h_1 \leq \cdots \leq h_{\lfloor \frac{1}{2}(d-\#F) \rfloor}$, assuming symmetry). Let Δ' be a regular triangulation of Δ. Then*

$$h_i(\Delta) - h_{i-1}(\Delta) \leq h_i(\Delta') - h_{i-1}(\Delta'), \quad 0 \leq i \leq \lfloor d/2 \rfloor.$$

Note that by Theorem 3.7 the boundary complex of a simplicial convex polytope satisfies the hypothesis of the above corollary. At the end of Section 3 some conjectured extensions are discussed. There is an intriguing conjectured generalization of Theorem 5.9, analogous to the conjectured generalization of the g-theorem for simplicial polytopes to the case of spheres.

5.11 Conjecture. *Theorem 5.9 continues to hold when Γ is a geometric (or even quasi-geometric) subdivision of 2^V.*

Note that the example $\ell_V(\Gamma, x) = -x^2$ of Clara Chan mentioned after Theorem 5.6 shows that Theorem 5.9 certainly cannot be extended to topological subdivisions.

There are many additional interesting open problems associated with local h-vectors. For instance, can one find a "nice" characterization of quasi-geometric (or geometric or regular) triangulations Γ of 2^V for which $\ell_V(\Gamma, x) = 0$? For further information see [68, pp. 821–823]. Can one characterize (or at least obtain significant new information on) the h-vectors of suitable classes of triangulations (e.g., topological, quasi-geometric, geometric, regular) Γ of 2^V? Even more strongly, can one characterize the functions f defined on subsets W of V by $f(W) = \ell_W(\Gamma_W, x)$, where Γ is a (suitably restricted) triangulation of 2^V? For some work on this problem, see [22].

We mentioned earlier that we would like to discuss two classes of acceptable functions. The first class, just discussed, deals with subdivisions. We now come to the second class, which is concerned with Ehrhart polynomials. There is by now a vast literature on Ehrhart polynomials, but we will be content with a few remarks which show the connection with acceptability. For an introduction to Ehrhart polynomials see [65, pp. 235–241] or [36, Part 2]. Our account below is taken from [68, Ex. 7.13].

Let L be a lattice in $\mathbb{R}^N$. (One can take $L = \mathbb{Z}^N$ without significant loss of generality.) Let $\mathcal{P}$ be an *L-polytope*, i.e., a convex polytope with vertices in L. (Much of what we say below can be generalized to L-polyhedral complexes, but for simplicity we only consider polytopes.) As usual we partially order the faces of $\mathcal{P}$ by inclusion; this makes $\mathcal{P}$ into an Eulerian poset. Given an integer $n > 0$ and a d-dimensional L-polytope $\mathcal{P} \subset \mathbb{R}^N$, define

$$i(\mathcal{P}, n) = \#(n\mathcal{P} \cap L),$$

where $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$. Also set $i(\mathcal{P}, 0) = 1$. It is known that $i(\mathcal{P}, n)$ is a polynomial function of n of degree d, called the *Ehrhart polynomial* of $\mathcal{P}$. It follows that if we set

$$\omega(\mathcal{P}, x) = (1-x)^{d+1} \sum_{n \geq 0} i(\mathcal{P}, n) x^n, \tag{29}$$

then $\omega(\mathcal{P}, x)$ is a polynomial in x of degree $\leq d$. Since every face F of $\mathcal{P}$ is also an L-polytope, we may regard ω as a function on $\mathcal{P}$, i.e., the value of ω at F is just $\omega(F, x)$. It is noted in [66, p. 201] (for the lattice $L = \mathbb{Z}^N$) that ω is an acceptable function on $\mathcal{P}$. This result is equivalent to Ehrhart's "law of reciprocity," which states that

$$(-1)^d i(\mathcal{P}, -n) = \#(\mathrm{int}(\mathcal{P}) \cap L),$$

where $\mathrm{int}(\mathcal{P})$ denotes the relative interior of $\mathcal{P}$.

Now assume that $\mathcal{S}$ is a d-dimensional L-simplex in $\mathbb{R}^N$. The "Ehrhart analogue" of the local h-vector of a triangulation of a simplex is given by $\ell(\omega, x)$. We will use the notation $\ell^*(\mathcal{S}, x)$ instead of $\ell(\omega, x)$, where the asterisk * indicates that we are working in the context of lattice points and Ehrhart polynomials. Thus we have explicitly the formula

$$\ell^*(\mathcal{S}, x) = \sum_F \omega(F, x)(-1)^{d-\dim(F)},$$

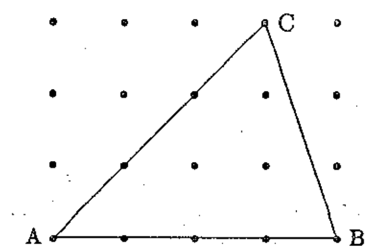

Fig. 4. A lattice triangle

where F runs over all faces of $\mathcal{S}$. Note a fundamental difference between ℓ and ℓ^*: $\ell_V(\Gamma, x)$ depends on the choice Γ of triangulation of 2^V, while $\ell^*(\mathcal{S}, x)$ depends only on $\mathcal{S}$ (but with the structure of an L-simplex, not just an abstract simplex). By Theorem 5.4 we have the symmetry formula

$$x^{d+1}\ell^*(\mathcal{S}, 1/x) = \ell^*(\mathcal{S}, x).$$

Moreover, Betke and McMullen [9] (using a different notation and viewpoint) give a geometric interpretation of the coefficients of $\ell^*(\mathcal{S}, x)$ which show that they are nonnegative (a considerably easier result than the analogous Theorem 5.6 for triangulations). From this and the lattice analogue of Theorem 5.5 it follows that for any L-polytope $\mathcal{P}$ the polynomial $\omega(\mathcal{P}, x)$ has nonnegative coefficients. Another proof of this fact based on shelling was given in [60]. An algebraic approach to ℓ^* analogous to the theory needed to prove Theorem 5.6 was given by Batyrev [4, Def. 9.1ff]. For further information and references, see [68, Ex. 7.13].

We conclude with an example illustrating the definition of $\ell^*(\mathcal{S}, x)$. Let $\mathcal{S} = ABC$ be the triangle with vertices $A = (0,0)$, $B = (4,0)$, and $C = (3,3)$, as shown in Figure 4. Then $\sum_{n \geq 0} i(ABC, n)x^n = (1 + 8x + 3x^2)(1-x)^{-3}$, so $\omega(ABC, x) = 1 + 8x + 3x^2$. Similarly $\omega(AB, x) = 1 + 3x$, $\omega(AC, x) = 1 + 2x$, $\omega(BC, x) = 1$, $\omega(A, x) = \omega(B, x) = \omega(C, x) = \omega(\emptyset, x) = 1$. Hence

$$\begin{aligned} \ell^*(ABC, x) &= 1 + 8x + 3x^2 - (1 + 3x) - (1 + 2x) - 1 + 1 + 1 + 1 - 1 \\ &= 3x + 3x^2. \end{aligned}$$

(For a lattice triangle $\mathcal{S}$ it is in fact not difficult to see that $\ell^*(\mathcal{S}, x) = kx + kx^2$, where k is the number of interior lattice points of $\mathcal{S}$. For higher dimensional simplices the situation is considerably more complicated.) Note that $x^3\ell^*(ABC, 1/x) = \ell^*(x)$, in accordance with Theorem 5.4.

References

1. Atiyah, M. F., and Macdonald, I.G.: 1969, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Massachusetts.
2. Barnette, D. W.: 1971, The minimum number of vertices of a simple polytope, *Israel J. Math.* 9, pp. 121–125.

3. Barnette, D. W.: 1973, A proof of the lower bound conjecture for convex polytopes, *Pacific J. Math.* 46, pp. 349–354.
4. Batyrev, V. V.: 1993, Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori, *Duke Mathematics J.* 69, pp. 349–409.
5. Bayer, M.: 1986, The generalized Dehn-Sommerville equations revisited, Technical Report, Northeastern University.
6. Bayer, M. M., and Billera, L. J.: 1985, Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Invent. math.* **79**, pp. 143–157.
7. Bayer, M. M., and Klapper, A.: 1991, A new index for polytopes, *Discrete Comput. Geom.* **6**, pp. 33–47.
8. Beilinson, A., Bernstein, J., and Deligne, P.: 1983, Faisceaux Pervers, *Astérisque* **100**.
9. Betke, U., and McMullen, P.: 1985, Lattice points in lattice polytopes, *Monatsh. Math.* **99**, pp. 253–265.
10. Billera, L. J.: 1983, Polyhedral theory and commutative algebra, *Mathematical Programming: The State of the Art* (A. Bachem, M. Grötschel, and B. Korte, eds.), Springer-Verlag, New York/Berlin, pp. 57–77.
11. Billera, L. J., Filliman, P., and Sturmfels, B.: 1990, Constructions and complexity of secondary polytopes, *Advances in Math.* **83**, pp. 155–179.
12. Billera, L. J., and Lee, C. W.: 1980, Sufficiency of McMullen's condition for f-vectors of simplicial polytopes, *Bull. Amer. Math. Soc.* **2**, pp. 181–185.
13. Billera, L. J., and Lee, C. W.: 1981, A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial polytopes, *J. Combinatorial Theory (A)* **31**, pp. 237–255.
14. Björner, A.: 1984, Posets, regular CW complexes and Bruhat order, *Europ. J. Combinatorics* **5**, pp. 7–16.
15. Björner, A., Frankl, P., and Stanley, R.: 1987, The number of faces of balanced Cohen-Macaulay complexes and a generalized Macaulay theorem, *Combinatorica* **7**, pp. 23–34.
16. Björner, A., Garsia, A., and Stanley, R.: 1982, An introduction to the theory of Cohen-Macaulay partially ordered sets, *Ordered Sets* (I. Rival, ed.), Reidel, Dordrecht/Boston/London, pp. 583–615.
17. Björner, A., Las Vergnas, M., Sturmfels, B., White, N., and Ziegler, G. M.: 1993, *Oriented Matroids*, Encyclopedia of Mathematics **46**, Cambridge University Press, Cambridge.
18. Björner, A., and Wachs, M.: 1982, Bruhat order of Coxeter groups and shellability, *Advances in Math.* **43**, pp. 87–100.
19. Brøndsted, A.: 1983, *An Introduction to Convex Polytopes*, Graduate Texts in Mathematics **90**, Springer-Verlag, New York/Berlin.
20. Bruggesser, M., and Mani, P.: 1971, Shellable decompostions of cells and spheres, *Math. Scand.* **29**, pp. 197–205.
21. Bruns, W., and Herzog, J.: 1993, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge.
22. Chan, C.: On subdivisions of simplicial complexes: Characterizing local h-vectors, preprint.

23. Chan, C., Jungreis, D., and Stong, R.: 1993, Buchsbaum and Eulerian complexes, preprint.
24. Charney, R., and Davis, M.: The Euler characteristic of a nonpositively curved, piecewise linear Euclidean manifold, preprint.
25. Clements, G., and Lindström, B.: 1969, A generalization of a combinatorial theorem of Macaulay, *J. Combinatorial Theory* **7**, pp. 230–238.
26. Danilov, V. I.: 1978, The geometry of toric varieties, *Russian Math. Surveys* **33**, pp. 97–154.
27. Duval, A. M.: 1991, *Simplicial posets: f-vectors and free resolutions*, Ph.D. thesis, M.I.T.
28. Ewald, G.: in preparation, *Combinatorial Convexity and Algebraic Geometry*, Graduate Texts in Mathematics, Springer-Verlag, New York/Berlin.
29. Foata, D., and Schützenberger, M.: 1973, Nombres d'Euler et permutations alternantes, *A Survey of Combinatorial Theory* (J. N. Srivastava, et al., eds.), North-Holland, Amsterdam, pp. 173–187.
30. Frankl, P., Furedi, Z., and Kalai, G.: 1988, Shadows of colored complexes, *Math. Scand.* **63**, pp. 169–178.
31. Fulton, W. F.: 1993, *Introduction to Toric Varieties*, Annals of Mathematics Studies, no. 131, Princeton University Press, Princeton, N.J.
32. Goresky, M., and MacPherson, R.: 1983, Intersection homology theory, *Topology* **19**, pp. 135–162.
33. Goresky, M., and MacPherson, R.: 1983, Intersection homology theory II, *Invent. math.* **71**, pp. 77–129.
34. Greene, C., and Kleitman, D. J.: 1978, Proof techniques in the theory of finite sets, *Studies in Combinatorics* (G.-C. Rota, ed.), Mathematical Association of America, pp. 22–79.
35. Grünbaum, B.: 1967, *Convex Polytopes*, John Wiley & Sons, New York, N.Y.
36. Hibi, T.: 1992, *Algebraic Combinatorics on Convex Polytopes*, Carslaw, Glebe, Australia.
37. Kalai, G.: 1988, A new basis for polytopes, *J. Combinatorial Theory (A)* **49**, pp. 191–209.
38. Kalai, G.: 1991, The diameter of graphs of convex polytopes and f-vector theory, *Applied Geometry and Discrete Mathematics — The Victor Klee Festschrift* (P. Gritzmann and B. Sturmfels, eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science **4**, Amer. Math. Soc., pp. 387–411.
39. Lee, C. W.: 1991, Regular triangulations of convex polytopes, *Applied Geometry and Discrete Mathematics — the Victor Klee Festschrift* (P. Gritzmann and B. Sturmfels, eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science **4**, Amer. Math. Soc., pp. 443–456.
40. Macaulay, F. S.: 1927, Some properties of enumeration in the theory of modular systems, *Proc. London Math. Soc.* **26**, pp. 531–555.
41. Macdonald, I. G.: 1971, Polynomials associated with finite cell complexes, *J. London Math. Soc. (2)* **4**, pp. 181–192.
42. MacPherson, R.: 1990, Intersection homology and perverse sheaves, preprint.
43. Matsumura, H.: 1986, *Commutative Ring Theory*, Cambridge University Press, Cambridge.

44. McMullen, P.: 1971, The numbers of faces of simplicial polytopes, *Israel J. Math.* **9**, pp. 559–570.
45. McMullen, P.: 1989, The polytope algebra, *Advances in Math.* **87**, pp. 76–130.
46. McMullen, P.: 1993, On simple polytopes, *Invent. math.* **113**, pp. 419–444.
47. McMullen, P., and Shephard, G. C.: 1971, *Convex Polytopes and the Upper Bound Conjecture*, Cambridge University Press, Cambridge.
48. McMullen, P., and Walkup, D.: 1971, A generalized lower-bound conjecture for simplicial polytopes, *Mathematika* **18**, pp. 264–273.
49. Oda, T.: 1988, *Convex Bodies and Algebraic Geometry (An Introduction to the Theory of Toric Varieties*, Springer-Verlag, Berlin/New York.
50. Purtill, M.: 1993, André permutations, lexicographic shellability and the *cd*-index of a convex polytope, *Trans. Amer. Math. Soc.* **338**, pp. 77–104.
51. Reisner, G.: 1976, Cohen-Macaulay quotients of polynomial rings, *Advances in Math.* **21**, pp. 30–49.
52. Rota, G.-C.: 1964, On the foundations of combinatorial theory I. Theory of Möbius functions, *Z. Wahrscheinlichkeitstheorie* **2**, pp. 340–368.
53. Saito, M.: 1983, Hodge filtrations on Gauss-Manin systems. II, *Proc. Japan Acad., Ser. A* **59**, pp. 37–40.
54. Santhi, V. K.: 1992, *Topics in Commutative Algebra*, Ph.D. thesis, Madurai Kamaraj University.
55. Sperner, E.: 1930, Über einen kombinatorischen Satz von Macaulay und seine Anwendung auf die Theorie der Polynomideale, *Abh. Math. Sem. Univ. Hamburg* **7**, pp. 149–163.
56. Stanley, R.: 1971, *Ordered Structures and Partitions*, Ph.D. thesis, Harvard University.
57. Stanley, R.: 1972, Ordered structures and partitions, *Mem. Amer. Math. Soc.*, no. 119.
58. Stanley, R.: 1978, Hilbert functions of graded algebras, *Advances in Math.* **28**, pp. 57–83.
59. Stanley, R.: 1979, Balanced Cohen-Macaulay complexes, *Trans. Amer. Math. Soc.* **249**, pp. 139–157.
60. Stanley, R.: 1980, Decompositions of rational convex polytopes, *Annals of Discrete Math.* **6**, pp. 333–342.
61. Stanley, R.: 1980, The number of faces of a simplicial convex polytope, *Advances in Math.* **35**, pp. 236–238.
62. Stanley, R.: 1983, *Combinatorics and Commutative Algebra*, Birkhäuser, Boston/Basel/Stuttgart.
63. Stanley, R.: 1984, An introduction to combinatorial commutative algebra, *Enumeration and Design* (D. M. Jackson and S. A. Vanstone, eds.), Academic Press, Toronto/Orlando, pp. 3–18.
64. Stanley, R.: 1985, The number of faces of simplicial polytopes and spheres, *Discrete Geometry and Convexity* (J. E. Goodman, et al., eds.), *Ann. New York Acad. Sci.*, vol. 440, pp. 212-223.
65. Stanley, R.: 1986, *Enumerative Combinatorics*, vol. 1, Wadsworth and Brooks/Cole, Pacific Grove, CA.

66. Stanley, R.: 1987, Generalized h-vectors, intersection cohomology of toric varieties, and related results, *Commutative Algebra and Combinatorics* (M. Nagata and H. Matsumura, eds.), *Advances Studies in Pure Mathematics* **11**, Kinokuniya, Tokyo, and North-Holland, Amsterdam/New York, pp. 187–213.
67. Stanley, R.: 1991, f-vectors and h-vectors of simplicial posets, *J. Pure Appl. Alg.* **71**, pp. 319–331.
68. Stanley, R.: 1992, Subdivisions and local h-vectors, *J. Amer. Math. Soc.* **5**, pp. 805–851.
69. Stanley, R.: 1993, A monotonicity property of h-vectors and h^*-vectors, *European J. Combinatorics* **14**, pp. 251–258.
70. Stanley, R.: to appear, Flag f-vectors and the *cd*-index, *Math. Z.*
71. Sundaram, S.: The homology of partitions with an even number of blocks, preprint.
72. Whipple, F.: 1928, On a theorem due to F. S. Macaulay, *J. London Math. Soc.* **8**, pp. 431–437.
73. Ziegler, G. M.: 1993, *Lectures on Polytopes*, Technical Report TR 93-6, Konrad-Zuse-Zentrum für Informationstechnik Berlin.