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Schubert Polynomials and the NilCoxeter Algebra

SERGEY FOMIN*

Theory of Algorithms Laboratory, Spiiran, Russia

AND

RICHARD P. STANLEY†

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Schubert polynomials were introduced and extensively developed by Lascoux and Schützenberger, after an earlier less combinatorial version had been considered by Bernstein, Gelfand and Gelfand and Demazure. We give a new development of the theory of Schubert polynomials based on formal computations in the algebra of operators $u_1, u_2, \ldots$ satisfying the relations $u_i^2 = 0$, $u_i u_j = u_j u_i$ if $|i-j| \geqslant 2$, and $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$. We call this algebra the *nilCoxeter algebra* of the symmetric group $\mathscr{S}_n$. Our development leads to simple proofs of many standard results, in particular, (a) symmetry of the "stable Schubert polynomials" F_w, (b) an explicit combinatorial formula for Schubert polynomials due to Billey, Jockusch and Stanley, (c) the "Cauchy formula" for Schubert polynomials, and (d) a formula of Macdonald for $\mathfrak{S}_w(1, 1, \ldots)$. Our main new result is a proof of a conjectured q-analogue of (d), due to Macdonald which gives a formula for $\mathfrak{S}_w(1, q, q^2, \ldots)$.

1. INTRODUCTION

Schubert polynomials were introduced and extensively developed by Lascoux and Schützenberger ([L-S$_1$, L-S$_2$], etc.), after an earlier less combinatorial version had been considered by Bernstein, Gelfand, and Gelfand [B-G-G] and Demazure [D]. A treatment of this work, with much additional material, appears in [M] and will be our main reference on Schubert polynomials. The Schubert polynomial indexed by the permutation w is denoted $\mathfrak{S}_w = \mathfrak{S}_w(x_1, x_2, \ldots)$. The theory of Schubert polynomials

* Partially supported by the Mittag-Leffler Institute. Present address: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139.

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is intimately related to the *divided difference operators* ∂_i (see Definition 3.4), which satisfy the "nilCoxeter relations"

$$\partial_i^2 = 0$$
$$\partial_i \partial_j = \partial_j \partial_i, \qquad |i-j| \geqslant 2$$
$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}.$$

We will give a treatment of Schubert polynomials based on formal elements u_i satisfying the above relations. More precisely, all our computations take place in the algebra $\mathcal{N}_n$ with generators $u_1, u_2, ..., u_{n-1}$. By elementary applications of the nilCoxeter relations we recover much of the known theory of Schubert polynomials, including (a) symmetry of the "stable Schubert polynomials" F_w (see comments after Lemma 2.1), (b) an explicit combinatorial formula for Schubert polynomials due to Billey, Jockusch, and Stanley (Theorem 2.2), (c) the "Cauchy formula" for Schubert polynomials (Corollary 4.4), and (d) a formula of Macdonald for $\mathfrak{S}_w(1, 1, ...)$ (Lemma 2.3). Our main new result is a q-analogue of (d) above, viz., a formula for $\mathfrak{S}_w(1, q, q^2, ...)$ originally conjectured by Macdonald [M, (6.11$_q$?)]. Our methods are similar to those in [F], where related techniques are used to derive basic properties of Schur functions, skew Schur functions, and the "shifted Schur functions" P_λ and Q_λ.

2. Main Results

Let $\mathcal{S}_n$ denote the symmetric group of permutations of $\{1, ..., n\}$. Let s_i, $i = 1, ..., n-1$, be adjacent transpositions, i.e., permutations interchanging i and $i+1$ and leaving all the other elements fixed. For any $w \in \mathcal{S}_n$ denote by $l(w)$ the *length* of w, i.e., the minimal p such that w can be represented as

$$w = s_{a_1} s_{a_2} \cdots s_{a_p}$$

for certain $a_1, a_2, ..., a_p$; such a sequence $a = (a_1, ..., a_p)$ is called a *reduced decomposition* (or reduced word) provided $p = l(w)$. The set of all reduced decompositions of w is denoted $R(w)$.

Let K be a field (or in fact any commutative ring). We assume K contains various indeterminates used later, viz., $x, x_1, x_2, ..., y, y_1, y_2, ..., q$. Define a K-algebra $\mathcal{N}_n$ with identity I, which we call the *nilCoxeter algebra* of the symmetric group $\mathcal{S}_n$, as follows: $\mathcal{N}_n$ has generators $u_1, u_2, ..., u_{n-1}$ and relations

$$u_i^2 = 0$$
$$u_i u_j = u_j u_i, \qquad |i-j| \geqslant 2 \tag{2.1}$$
$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}, \qquad 1 \leqslant i \leqslant n-2.$$

There are several "concrete" ways of looking at the algebra $\mathcal{N}_n$: (a) $\mathcal{N}_n$ is the K-algebra with basis consisting of $\mathcal{S}_n$ and multiplication $\circ$ given by

$$u \circ v = \begin{cases} uv, & \text{if } \ l(uv) = l(u) + l(v) \\ 0, & \text{otherwise,} \end{cases}$$

where $u, v \in \mathcal{S}_n$ and uv denotes ordinary product of permutations. (b) $\mathcal{N}_n$ is faithfully represented by the algebra generated by the divided difference operators $\partial_1, ..., \partial_{n-1}$ defined in Definition 3.4. (c) $\mathcal{N}_n$ is faithfully represented by the algebra of operators generated by $\varphi_1, ..., \varphi_{n-1}$: $K\mathcal{S}_n \to K\mathcal{S}_n$ defined by

$$\varphi_i(w) = \begin{cases} ws_i, & l(ws_i) = l(w) + 1 \\ 0, & \text{otherwise.} \end{cases}$$

If $(a_1, ..., a_p)$ is a reduced word, then we will identify a monomial $u_{a_1} u_{a_2} \cdots u_{a_j}$ in $\mathcal{N}_n$ with the permutation $w = s_{a_1} s_{a_2} \cdots s_{a_j} \in \mathcal{S}_n$. The relations (2.1) ensure that this notation is well-defined, and that $\mathcal{N}_n$ has the K-basis $\mathcal{S}_n$. Write $\langle f, w \rangle$ for the coefficient of $w \in \mathcal{S}_n$ in the element f of $\mathcal{N}_n$. For instance,

$$\langle ((1 + u_1)(1 + u_2))^m, 321 \rangle = 2\binom{m}{3} + 2\binom{m}{2}.$$

Throughout the paper $\mathbf{x} = (x_1, ..., x_{n-1})$, $\mathbf{y} = (y_1, ..., y_{n-1})$, $\mathbf{x}_\infty = (x_1, x_2, ...)$. Define

$$A_i(x) = A_{i,n}(x) = (I + xu_{n-1})(I + xu_{n-2}) \cdots (I + xu_i) \tag{2.2}$$

for $i = 1, ..., n-1$. Let

$$G(\mathbf{x}_\infty) = A_1(x_1)\, A_1(x_2) \cdots, \tag{2.3}$$

$$\mathfrak{S}(\mathbf{x}) = A_1(x_1) \cdots A_{n-1}(x_{n-1}). \tag{2.4}$$

Thus $A_i(x)$, $G(\mathbf{x}_\infty)$, $\mathfrak{S}(\mathbf{x}) \in \mathcal{N}_n$. Denote for $w \in \mathcal{S}_n$

$$G_w(\mathbf{x}_\infty) = \langle G, w \rangle$$

$$\mathfrak{S}_w(\mathbf{x}) = \langle \mathfrak{S}, w \rangle.$$

Direct inspection allows us to rewrite these formulae as

$$G_w(\mathbf{x}_\infty) = \sum_{(a_1, ..., a_p) \in R(w)} \sum_{\substack{b_1, ..., b_p \\ 1 \leqslant b_1 \leqslant \cdots \leqslant b_p \\ a_i < a_{i+1} \Rightarrow b_i < b_{i+1}}} x_{b_1} \cdots x_{b_p},$$

$$\mathfrak{S}_w(\mathbf{x}) = \sum_{(a_1, ..., a_p) \in R(w)} \sum_{\substack{b_1, ..., b_p \\ 1 \leqslant 1 \leqslant b_1 \leqslant \cdots \leqslant b_p \leqslant n-1 \\ a_i < a_{i+1} \Rightarrow b_i < b_{i+1} \\ b_i \leqslant a_i}} x_{b_1} \cdots x_{b_p}.$$

This definition of G_w appears in [S, (1); M, (7.18)] (where F_w in these two references is our $G_{w^{-1}}$), and that of $\mathfrak{S}_w$ in [B-J-S]. We will derive the main properties of these polynomials directly from the relations (2.1).

2.1. LEMMA. *$A_i(x)$ and $A_i(y)$ commute.*

This immediately implies that the F_w's and G_w's are symmetric functions in $x_1, x_2 \cdots$ (cf. [S, Theorem 2.1; M, p. 102]).

2.2. THEOREM. *$\mathfrak{S}_w(\mathbf{x})$ is a Schubert polynomial.*

(See [M] for (alternative) definitions of Schubert polynomials, and [B-J-S] for another proof of Theorem 2.2.)

2.3. LEMMA. *Suppose* $\operatorname{char} K = 0$. *Then* $\mathfrak{S}(1, ..., 1) = \exp(u_1 + 2u_2 + 3u_3 + \cdots)$.

This identity can be rewritten as

$$\mathfrak{S}_w(1, ..., 1) = \frac{1}{p!} \sum_{(a_1, ..., a_p) \in R(w)} a_1 \cdots a_p$$

(cf. [M, (6.11)]).

The following q-analogue of this formula was conjectured by Macdonald [M, (6.11$_q$?)].

2.4. THEOREM.

$$\mathfrak{S}_w(1, q, ..., q^{n-2}) = \frac{1}{[p]!} \sum_{(a_1, ..., a_p) \in R(w)} [a_1] \cdots [a_p]\, q^{\sum_{a_i \leqslant a_{i+1}} i},$$

where $[m]$ *denotes* $1 + q + \cdots + q^{m-1}$, *and* $[p]! = [1][2] \cdots [p]$.

3. PROOFS OF LEMMA 2.1 AND THEOREM 2.2

Define $h_i(x) = I + xu_i$. A straightforward computation provides a full list of relations between the $h_i(x)$'s.

3.1. LEMMA. (i) $h_i(x)\, h_j(y) = h_j(y)\, h_i(x)$, $|i-j| \geqslant 2$;

(ii) $h_i(x)\, h_i(y) = h_i(x+y)$, $h_i(0) = I$ (*so* $h_i(x)\, h_i(-x) = I$);

(iii) $h_i(x)\, h_{i+1}(x+y)\, h_i(y) = h_{i+1}(y)\, h_i(x+y)\, h_{i+1}(x)$.

Rewrite (2.2) as $A_i(x) = h_{n-1}(x)\, h_{n-2}(x) \cdots h_i(x)$. The following simple observations will be useful in the sequel:

$$A_i(x) = A_{i+1}(x)\, h_i(x);$$
$$A_{i+1}(x) = A_i(x)\, h_i(-x);$$
$$A_i(x)\, h_j(x) = h_j(x)\, A_i(x) \qquad \text{if} \quad i \geqslant j+2.$$

Proof of Lemma 2.1. Induction on $n-i$. For $i = n-1$ the fact is trivial. Assume

$$A_{i+1}(x)\, A_{i+1}(y) = A_{i+1}(y)\, A_{i+1}(x)$$

is given. Then

$$\begin{aligned} A_i(x)\, A_i(y) &= A_{i+1}(x)\, h_i(x)\, A_{i+2}(y)\, h_{i+1}(y)\, h_i(y) \\ &= A_{i+1}(x)\, A_{i+2}(y)\, h_i(x)\, h_{i+1}(y)\, h_i(y-x)\, h_i(x) \\ &= A_{i+1}(x)\, A_{i+2}(y)\, h_{i+1}(y-x)\, h_i(y)\, h_{i+1}(x)\, h_i(x) \\ &= A_{i+1}(y)\, A_{i+1}(x)\, h_{i+1}(-x)\, h_i(y)\, h_{i+1}(x)\, h_i(x) \\ &= A_{i+1}(y)\, A_{i+2}(x)\, h_i(y)\, h_{i+1}(x)\, h_i(x) \\ &= A_{i+1}(y)\, h_i(y)\, A_{i+2}(x)\, h_{i+1}(x)\, h_i(x) \\ &= A_i(y)\, A_i(x). \quad ▮ \end{aligned}$$

3.2. Lemma. $A_i(x)\, A_{i+1}(y)\, u_i = A_i(y)\, A_{i+1}(x)\, u_i$.

Proof. This identity means that the left-hand side is symmetric in x and y. Write

$$\begin{aligned} A_i(x)\, A_{i+1}(y)\, u_i &= A_{i+1}(x)(I + xu_i)\, A_{i+1}(y)\, u_i \\ &= A_{i+1}(x)\, A_{i+1}(y)\, u_i + xA_{i+1}(x)\, u_i A_{i+2}(y)(I + yu_{i+1})\, u_i \end{aligned}$$

and note that the first summand is symmetric by Lemma 2.1. Now

$$\begin{aligned} &xA_{i+1}(x)\, A_{i+2}(y)\, u_i(I + yu_{i+1})\, u_i \\ &\quad = xA_{i+1}(x)\, A_{i+2}(y)\; y(I + yu_{i+1})\, u_{i+1} u_i u_{i+1} \\ &\quad = xyA_{i+1}(x)\, A_{i+1}(y)\, u_{i+1} u_i u_{i+1} \end{aligned}$$

which is also symmetric. ▮

3.3. Lemma. $A_i(x)\, A_{i+1}(y) - A_i(y)\, A_{i+1}(x) = (x-y)\, A_i(x)\, A_{i+1}(y)\, u_i$.

Proof.

$$
\begin{aligned}
&A_i(x)\,A_{i+1}(y) + yA_i(x)\,A_{i+1}(y)\,u_i \\
&\quad = A_i(x)\,A_{i+1}(y)(I + yu_i) = A_i(x)\,A_i(y) = A_i(y)\,A_i(x) \quad \text{(Lemma 2.1)} \\
&\quad = A_i(y)\,A_{i+1}(x)(I + xu_i) = A_i(y)\,A_{i+1}(x) + xA_i(y)\,A_{i+1}(x)\,u_i \\
&\quad = A_i(y)\,A_{i+1}(x) + xA_i(x)\,A_{i+1}(y)\,u_i \qquad \text{(Lemma 3.2).} \ \blacksquare
\end{aligned}
$$

3.4. Definition (Schubert Polynomials [M]). Define the *divided difference operators* ∂_i acting in K (recall K is assumed to contain independent indeterminates $\mathbf{x}$) by

$$\partial_i f(\mathbf{x}) = \frac{f(\mathbf{x}) - f(T_i\mathbf{x})}{x_i - x_{i+1}},$$

where $T_i\mathbf{x} = (x_1, ..., x_{i-1}, x_{i+1}, x_i, x_{i+2}, ...)$; in other words, T_i interchanges x_i and x_{i+1}. Schubert polynomials can be defined recursively by

(i) $\mathfrak{S}_{w_0}(\mathbf{x}) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ for $w_0 = n, n-1, ..., 1$,

(ii) $\mathfrak{S}_w(\mathbf{x}) = \partial_i \mathfrak{S}_{ws_i}(\mathbf{x})$ whenever $w \in \mathcal{S}_n$, $l(ws_i) = l(w) + 1$.

(It is shown in [M] that $\mathfrak{S}_w(x)$ is well-defined by (i)–(ii).)

3.5. Lemma. $\partial_i \mathfrak{S}(\mathbf{x}) = \mathfrak{S}(\mathbf{x})\,u_i$.

Proof. Since

$$\partial_i \mathfrak{S}(\mathbf{x}) = A_1(x_1) \cdots \partial_i (A_i(x_i)\,A_{i+1}(x_{i+1}))\,A_{i+2}(x_{i+2}) \cdots$$

and

$$\mathfrak{S}(\mathbf{x})\,u_i = A_1(x_1) \cdots A_i(x_i)\,A_{i+1}(x_{i+1})\,u_i A_{i+2}(x_{i+2}) \cdots,$$

the statement of the lemma follows from

$$\partial_i (A_i(x_i)\,A_{i+1}(x_{i+1})) = A_i(x_i)\,A_{i+1}(x_{i+1})\,u_i.$$

This is exactly Lemma 3.3. $\blacksquare$

Proof of Theorem 2.2. We need to prove, in our terms, that

$$\langle \mathfrak{S}(\mathbf{x}), w_0 \rangle = x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \tag{3.1}$$

and

$$\partial_i \langle \mathfrak{S}(\mathbf{x}), ws_i \rangle = \langle \mathfrak{S}(\mathbf{x}), w \rangle \qquad \text{if} \quad l(ws_i) = l(w) + 1. \tag{3.2}$$

The equality (3.1) is almost obvious from (2.4) and (2.2): to get from I to w_0 one should take $x_i u_j$ from each factor in the expansion of $\mathfrak{S}(\mathbf{x})$; (3.2) follows from Lemma 3.5. ∎

Theorem 2.2 was originally proved in [B-J-S] by more complicated means. It can also be derived from some work of Lascoux and Schützenberger, as shown in [R-S, after Theorem 2].

Remark. There is a class of polynomials related to Schubert polynomials for which an analogue of Theorem 2.2 holds. Define a K-algebra $\mathscr{K}_n$ with identity I, called the *nilCoxeter-Knuth* or *nilplactic algebra* (first defined by Lascoux and Schützenberger in [L-S$_3$]), as follows: $\mathscr{K}_n$ has generators $v_1, v_2, ..., v_{n-1}$ and relations

$$v_1^2 = 0$$

$$v_k v_i v_j = v_i v_k v_j \quad \text{and} \quad v_j v_i v_k = v_j v_k v_i, \text{ if } i<j<k$$

$$v_k v_i v_k = v_k v_k v_i \quad \text{and} \quad v_k v_i v_i = v_i v_k v_i, \text{ if } k-i \geqslant 2$$

$$v_i v_{i+1} v_i = v_{i+1} v_i v_{i+1}, \qquad 1 \leqslant i \leqslant n-2.$$

Thus the nilCoxeter algebra $\mathscr{N}_n$ is a quotient algebra of $\mathscr{K}_n$. Pick a basis M for $\mathscr{K}_n$ which consists of monomials in the v_i's. Just as for the nilCoxeter algebra, let

$$B_i(x) = B_{i,n}(x) = (I + xv_{n-1})(I + xv_{n-2}) \cdots (I + xv_i),$$

for $i = 1, ..., n-1$; and let

$$\mathfrak{K}(x) = B_1(x_1) \cdots B_{n-1}(x_{n-1}).$$

Given $u \in M$, it follows from [R-S, Theorem 3] that $\langle \mathfrak{K}, u \rangle$ is a *key* polynomial K_α for a certain sequence $\alpha = (\alpha_1, \alpha_2, ...)$ of nonnegative integers α_i with finite sum. We refer the reader to [R-S] and the papers of Lascoux and Schützenberger cited there for the precise definition and additional properties of key polynomials.

4. Cauchy Formula and Double Schubert Polynomials

Define

$$\tilde{A}_i(y) = \tilde{A}_{i,n}(y) = h_i(y)\, h_{i+1}(y) \cdots h_{n-1}(y)$$

(cf. (2.2)). By analogy with (2.4), let

$$\tilde{\mathfrak{S}}(\mathbf{y}) = \tilde{A}_{n-1}(y_{n-1}) \cdots \tilde{A}_1(y_1).$$

4.1. LEMMA. *$A_i(x)$ and $\tilde{A}_i(y)$ commute.*

Proof. By Lemma 3.1(ii), we have $\tilde{A}_i(y) = A_i^{-1}(-y)$. Since $A_i(x)$ and $A_i(y)$ commute by Lemma 2.1, the proof follows. ∎

4.2. LEMMA. $\tilde{A}_{n-1}(y_{n-1}) \cdots \tilde{A}_i(y_i)\, A_i(x) = h_{n-1}(y_{n-1}+x) \cdots h_i(y_i+x)$ $\tilde{A}_{n-1}(y_{n-2}) \cdots \tilde{A}_{i+1}(y_i)$.

Proof. Use Lemma 4.1 to obtain

$$\begin{aligned}
&\tilde{A}_{n-1}(y_{n-1}) \cdots \tilde{A}_i(y_i)\, A_i(x)\\
&\quad = \tilde{A}_{n-1}(y_{n-1}) \cdots \tilde{A}_{i+1}(y_{i+1})\, A_i(x)\, \tilde{A}_i(y_i)\\
&\quad = \tilde{A}_{n-1}(y_{n-1}) \cdots \tilde{A}_{i+2}(y_{i+2})\, A_{i+1}(x)\, \tilde{A}_{i+1}(y_{i+1})\, h_i(x)\, \tilde{A}_i(y_i)\\
&\quad = \tilde{A}_{n-1}(y_{n-1}) \cdots \tilde{A}_{i+3}(y_{i+3})\, A_{i+2}(x)\\
&\qquad \times \tilde{A}_{i+2}(y_{i+2})\, h_{i+1}(x)\, \tilde{A}_{i+1}(y_{i+1})\, h_i(x)\, \tilde{A}_i(y_i)\\
&\quad = \cdots = h_{n-1}(x)\, \tilde{A}_{n-1}(y_{n-1})\, h_{n-2}(x)\, \tilde{A}_{n-2}(y_{n-2}) \cdots h_i(x)\, \tilde{A}_i(y_i)\\
&\quad = h_{n-1}(x+y_{n-1})\, h_{n-2}(x+y_{n-2})\, \tilde{A}_{n-1}(y_{n-2}) \cdots h_i(x+y_i)\, \tilde{A}_{i+1}(y_i)\\
&\quad = h_{n-1}(x+y_{n-1})\, h_{n-2}(x+y_{n-2}) \cdots\\
&\qquad h_i(x+y_i)\, \tilde{A}_{n-1}(y_{n-2}) \cdots \tilde{A}_{i+1}(y_i).
\end{aligned}$$ ∎

4.3. LEMMA.

$$\tilde{\mathfrak{S}}(\mathbf{y})\, \mathfrak{S}(\mathbf{x}) = \prod_{d=2-n}^{n-2} \prod_{\substack{i-j=d\\ i+j\leq n}} h_{i+j-1}(x_i + y_j), \tag{4.1}$$

where in the first product the factors corresponding to $d = 2-n, 3-n, \ldots, n-2$ are multiplied left-to-right. (Factors in the second product commute.)

Proof. Repeatedly apply Lemma 4.2 and then rearrange factors. ∎

4.4. COROLLARY ("Cauchy Formula" [M, (5.10)]).

$$\langle \tilde{\mathfrak{S}}(\mathbf{y})\, \mathfrak{S}(\mathbf{x}), w_0 \rangle = \prod_{i+j\leq n} (x_i + y_j).$$

(*Recall* $w_0 = (n, \ldots, 1)$.)

Proof. The right-hand side of (4.1) contains exactly $\binom{n}{2}$ factors. Thus to obtain w_0 from I one should take $(x_i + u_j)\, u_{i+j-1}$ from each factor. ▮

4.5. LEMMA. *Let $w \in \mathcal{S}_n$. The polynomial $\langle \tilde{\mathfrak{S}}(\mathbf{y})\, \mathfrak{S}(\mathbf{x}), w \rangle$ is the double Schubert polynomial $\mathfrak{S}_w(\mathbf{x}, -\mathbf{y})$ of Lascoux and Schützenberger (see Macdonald* [M, (6.1)]).

Proof. Use descending induction on $l(w)$. The basis ($w = w_0$) is exactly Corollary 4.4. The induction step follows immediately from Lemma 3.5. ▮

5. MACDONALD'S IDENTITIES

5.1. LEMMA (*cf.* [M, *p.* 89]). $\mathfrak{S}(y, ..., y)\, \mathfrak{S}(x, ..., x) = \mathfrak{S}(y + x, ..., y + x)$.

Proof. In the case $y_1 = \cdots = y_{n-1} = y$ one can check that

$$\tilde{\mathfrak{S}}(\mathbf{y}) = \mathfrak{S}(\mathbf{y}) = \prod_{d=2-n}^{n-2} \prod_{\substack{i-j=d \\ i+j \leqslant n}} h_{i+j-1}(y)$$

(by rearranging factors). Hence, Lemma 4.3 gives

$$\mathfrak{S}(y, ..., y)\, \mathfrak{S}(x, ..., x) = \prod_{d=2-n}^{n-2} \prod_{\substack{i-j=d \\ i+j \leqslant n}} h_{i+j-1}(y + x) = \mathfrak{S}(y + x, ..., y + x). \quad ▮$$

Proof of Lemma 2.3. Lemma 5.1 implies that there is an element $f = f(u_1, u_2, ...)$ of $\mathcal{N}_n$ such that

$$\mathfrak{S}(x, ..., x) = \exp(xf). \tag{5.1}$$

Differentiation of (5.1) with respect to x shows that

$$f = \frac{d}{dx}\, \mathfrak{S}(x, ... x)_{x=0}.$$

From definition (2.4)/(2.2) and the Leibniz rule we obtain $f = u_1 + 2u_2 + 3u_3 + \cdots$, as desired. ▮

5.2. LEMMA. $\mathfrak{S}(q^i, q^{i+1}, ..., q^{i+n-2}) = \mathfrak{S}(q^{i+1}, q^{i+2}, ..., q^{i+n-1}) \prod_{j=n-1}^{1} h_j(q^i(1-q^j))$, *where in the (non-commutative) product the factors are multiplied in decreasing order (with respect to j).*

Proof. First note that the relations (2.1) are homogeneous and therefore all the arguments of any formula involving h_j's may be simultaneously multiplied by one and the same term (in our case, by q^i). So it suffices to prove the lemma for $i=0$.

Induction on n. The cases $n=2, 3$ are easly checked. Now

$$\begin{aligned}
&\mathfrak{S}(q, q^2, ..., q^{n-1})\, h_{n-1}(1-q^{n-1}) \cdots h_1(1-q) \\
&= A_1(q)\, A_2(q^2) \cdots A_{n-1}(q^{n-1})\, h_{n-1}(1-q^{n-1}) \cdots h_1(1-q) \\
&= A_2(q)\, h_1(q) \cdots A_{n-1}(q^{n-2})\, h_{n-2}(q^{n-2}) \\
&\quad \times h_{n-1}(q^{n-1})\, h_{n-1}(1-q^{n-1}) \cdots h_1(1-q) \\
&= A_2(q) \cdots A_{n-1}(q^{n-2})\, h_1(q)\, h_2(q^2) \cdots \\
&\quad h_{n-2}(q^{n-2})\, h_{n-1}(1)\, h_{n-2}(1-q^{n-2}) \cdots h_1(1-q) \\
&= A_2(q) \cdots A_{n-1}(q^{n-2})\, h_{n-1}(1-q^{n-2}) \cdots \\
&\quad h_2(1-q)\, h_1(1)\, h_2(q)\, h_3(q^2) \cdots h_{n-1}(q^{n-2}) \\
&\quad \text{(by repeated use of Lemma 3.1(iii))} \\
&= A_2(1) \cdots A_{n-1}(q^{n-3})\, h_1(1)\, h_2(q) \cdots \\
&\quad h_{n-1}(q^{n-2}) \qquad \text{(by the induction hypothesis)} \\
&= A_1(1) \cdots A_{n-2}(q^{n-3})\, h_{n-1}(q^{n-2}) = \mathfrak{S}(1, q, ..., q^{n-2}). \qquad \blacksquare
\end{aligned}$$

The next lemma is a q-analogue of Lemma 2.3. A standard q-analogue of the exponential function is

$$\prod_{k \geqslant 0} (1 + q^k(1-q)\, t) = \sum_{p \geqslant 0} q^{\binom{p}{2}} \frac{t^p}{[p]!},$$

where $[p]!$ is defined in Theorem 2.4. One can thus recognize the right-hand side of the next lemma as a (non-commutative) q-analogue of the right-hand side of Lemma 2.3.

5.3. Lemma. $\mathfrak{S}(1, q, q^2, ..., q^{n-2}) = \prod_{k=\infty}^{0} \prod_{j=n-1}^{1} h_j(q^k(1-q^j))$, *where in the (non-commutative) products the factors are multiplied in decreasing order (with respect to k and j).*

Proof. Repeatedly use Lemma 5.2. $\blacksquare$

5.4. LEMMA. *Let $t_1, ..., t_{n-1}$ be some elements of an associative algebra over K with identity I. (So we disregard the relations between t_i's, if any.) Let $q, z_1, ..., z_{n-1}$ be formal variables. Then*

$$\prod_{k=\infty}^{0} \prod_{j=n-1}^{1} (I+q^k z_j t_j) = \sum_{p\geq 0} \sum_{a_1,...,a_p} \frac{z_{a_1}\cdots z_{a_p}}{[p]!\,(1-q)^p} q^{\sum_{a_i\leq a_{i+1}} i} t_{a_1}\cdots t_{a_p}.$$

(Recall $[p]! = [1][2]\cdots[p]$, $[m]=(1-q^m)/(1-q)$.)

Proof. Actually the lemma claims that the coefficient $c_{a_1\cdots a_p}$ of $t_{a_1\cdots a_p}$ in

$$\prod_{k=\infty}^{0} \prod_{j=n-1}^{1} (I+q^k t_j)$$

is given by

$$c_{a_1\cdots a_p} = \frac{1}{[p]!\,(1-q)^p} q^{\sum_{a_i\leq a_{i+1}} i}.$$

Denote

$$\varepsilon_i = \begin{cases} 1 & \text{if } a_i \leq a_{i+1} \\ 0 & \text{if } a_i > a_{i+1}. \end{cases}$$

Then

$$\begin{aligned}
c_{a_1\cdots a_p} &= \sum_{\substack{k_1\geq k_2\geq \cdots \geq k_p \geq 0 \\ a_i\leq a_{i+1} \Rightarrow k_i > k_{i+1}}} \prod_{i=1}^{p} q^{k_i} \\
&= \sum_{k_p=0}^{\infty} q^{k_p} \sum_{k_{p-1}=k_p+\varepsilon_{p-1}}^{\infty} q^{k_{p-1}} \cdots \sum_{k_1=k_2+\varepsilon_1}^{\infty} q^{k_1} \\
&= \sum_{k_p=0}^{\infty} q^{k_p} \cdots \sum_{k_2=k_3+\varepsilon_2}^{\infty} q^{2k_2+\varepsilon_1} \cdot \frac{1}{1-q} \\
&= \sum_{k_p=0}^{\infty} q^{k_p} \cdots \sum_{k_3=k_4+\varepsilon_3}^{\infty} q^{3k_3+2\varepsilon_2+\varepsilon_1} \cdot \frac{1}{1-q^2} \cdot \frac{1}{1-q} \\
&= \cdots = \sum_{k_p=0}^{\infty} q^{pk_p+(p-1)\varepsilon_{p-1}+\cdots+2\varepsilon_2+\varepsilon_1} \cdot \frac{1}{1-q^{p-1}} \cdots \frac{1}{1-q} \\
&= q^{p\varepsilon_p+\cdots+2\varepsilon_2+\varepsilon_1} \cdot \frac{1}{1-q^p} \cdots \frac{1}{1-q} \\
&= q^{\sum_{a_i\leq a_{i+1}} i} \cdot \frac{1}{[p]!\,(1-q)^p}. \quad \blacksquare
\end{aligned}$$

Proof of Theorem 2.4. Combine Lemma 5.3, Lemma 5.4 (with $t_j = u_j$, $z_j = 1 - q^j$), and our definition of Schubert polynomials. ∎

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