Log-Concave and Unimodal Sequences in Algebra, Combinatorics, and Geometry[a]

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INTRODUCTION

A sequence $a_0, a_1, \ldots, a_n$ of real numbers is said to be *unimodal* if for some $0 \le j \le n$ we have $a_0 \le a_1 \le \cdots \le a_j \ge a_{j+1} \ge \cdots \ge a_n$, and is said to be *logarithmically concave* (or *log-concave* for short) if $a_i^2 \ge a_{i-1}a_{i+1}$ for all $1 \le i \le n-1$. Clearly a log-concave sequence of *positive* terms is unimodal. Let us say that the sequence $a_0, a_1, \ldots, a_n$ has *no internal zeros* if there do not exist integers $0 \le i < j < k \le n$ satisfying $a_i \ne 0, a_j = 0, a_k \ne 0$. Then in fact a *nonnegative* log-concave sequence with no internal zeros is unimodal. The sequence $a_0, a_1, \ldots, a_n$ is called *symmetric* if $a_i = a_{n-i}$ for $0 \le i \le n$. Thus a symmetric unimodal sequence $a_0, a_1, \ldots, a_n$ has its maximum at the middle term (n even) or middle two terms (n odd). We also say that a polynomial $a_0 + a_1q + \cdots + a_nq^n$ has a certain property (such as unimodal, log-concave, or symmetric) if its sequence $a_0, a_1, \ldots, a_n$ of coefficients has that property.

Our object here is to survey the surprisingly rich variety of methods for showing that a sequence is log-concave or unimodal. For each method we will give examples of its applicability to combinatorially defined sequences that arise naturally from problems in algebra, combinatorics, and geometry. We make no attempt, however, to give a comprehensive account of *all* work done in this area.

DIRECT COMBINATORIAL METHODS

In this section we collect a hodgepodge of "direct" methods for showing log-concavity or unimodality. Consider first what is probably the best-known unimodal (or log-concave) sequence—the nth row of Pascal's triangle:

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}. \tag{1}$$

Here log-concavity is easy to show because of the explicit formula $\binom{n}{k} = n!/k!(n-k)!$. Indeed,

$$\binom{n}{k}^2 \Big/ \binom{n}{k-1}\binom{n}{k+1} = \frac{(k+1)(n-k+1)}{k(n-k)} > 1.$$

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A similar direct proof of the log-concavity of some related sequences appears in [119, 120] (with further results in [121]).

We could also ask for a *combinatorial proof* that the sequence (1) is log-concave. In general, if $a_0, a_1, \ldots, a_n$ is any sequence of nonnegative integers for which a combinatorial meaning is known (i.e., we have sets $S_0, S_1, \ldots, S_n$ such that $\#S_i = a_i$), then a construction of an explicit injection $\phi_k: S_{k-1} \times S_{k+1} \to S_k \times S_k$ yields a combinatorial proof that $a_k^2 \geq a_{k-1}a_{k+1}$. Similarly, a collection of injections $\rho_k: S_k \to S_{k+1}$ $(0 \leq k < j)$ and surjections $\rho_k: S_k \twoheadrightarrow S_{k+1}$ $(j \leq k \leq n-1)$ shows combinatorially that the sequence $a_0, a_1, \ldots, a_n$ is unimodal. For the case $a_k = \binom{n}{k}$, we choose S_k to be the set $\binom{\mathbf{n}}{k}$of k-element subsets of $\mathbf{n} = \{1, 2, \ldots, n\}$. For any set $X \subseteq \mathbf{n}$ define $X_j = X \cap \mathbf{j}$. Given $(A, B) \in S_{k-1} \times S_{k+1}$, let j be the largest integer (easily seen to exist) for which $\#A_j = \#B_j - 1$. Define $C = A_j \cup (B - B_j)$, $D = B_j \cup (A - A_j)$, and set $\phi_k(A, B) = (C, D)$. It is easy to verify that ϕ_k is an injection, as desired. See [89] for more general results that can be proved by this technique, such as the log-concavity of the Stirling numbers of the first kind (which are further discussed in Example 1).

A well-known generalization of the binomial coefficients $\binom{n}{k}$ are the q-*binomial coefficients* $\left[{n \atop k}\right]$, defined for $0 \leq k \leq n$ by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!}, \tag{2}$$

where

$$\begin{aligned} [j]! &= [1][2] \cdots [j], \\ [i] &= 1 - q^i. \end{aligned} \tag{3}$$

Here q may be regarded as an indeterminate (in which case $\left[{n \atop k}\right]$ is a polynomial in q of degree $k(n-k)$) or as a number. If $q = 1$, then $\left[{n \atop k}\right]$ reduces to the binomial coefficient $\binom{n}{k}$. If q is a prime power, then $\left[{n \atop k}\right]$ is equal to the number of k-dimensional subspaces of an n-dimensional vector space V_n over the finite field $\mathcal{F}_q$ with q elements. Using the explicit formula (2), it is easily seen that the sequence $\left[{n \atop 0}\right], \left[{n \atop 0}\right], \ldots, \left[{n \atop n}\right]$ is log-concave for $q \geq 0$ (and hence unimodal, since clearly $\left[{n \atop k}\right] > 0$ for $q \geq 0$). A combinatorial proof that $\left[{n \atop 0}\right], \left[{n \atop 0}\right], \ldots, \left[{n \atop n}\right]$ is log-concave was given by L. Butler (private communication). The unimodality of the sequence $\left[{n \atop 0}\right], \left[{n \atop 1}\right], \ldots, \left[{n \atop n}\right]$ for $q \geq 0$ should not be confused with the problem of showing that for fixed k and n, the coefficients $a_0, a_1, \ldots, a_{k(n-k)}$ of the polynomial

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{i=0}^{k(n-k)} a_i q^i$$

are unimodal. This much more difficult problem is discussed in later sections. (The example $\left[{4 \atop 2}\right] = 1 + q + 2q^2 + q^3 + q^4$ shows that the coefficients of $\left[{n \atop k}\right]$ need not be log-concave.)

An extraordinary example of a combinatorial proof of unimodality was given recently by L. Butler [19, chap. 2], [20], and is a generalization of the unimodality of $\left[{n \atop 0}\right], \left[{n \atop 1}\right], \ldots, \left[{n \atop n}\right]$. To state Butler's result, it is convenient to use terminology from the theory of posets (partially ordered sets). A finite poset P is *graded of rank* n if every maximal chain of P has length n (or cardinality $n + 1$). In this case, we define the *rank*

$\rho(x)$ of $x \in P$ to be the length i of the longest chain $x_0 < x_1 < \cdots < x_i = x$ of P with top element x. Let a_i be the number of elements of P of rank i, and call P *rank-unimodal* if the sequence $a_0, a_1, \ldots, a_n$ is unimodal. Now consider the case where P is the lattice of subgroups of a finite Abelian group G. If the prime factorization of $|G|$ is given by $p_1^{b_1} p_2^{b_2} \cdots$, then P is graded of rank $b_1 + b_2 + \cdots$. If $H \in P$ and $|H| = p_1^{c_1} p_2^{c_2} \cdots$, then $\rho(H) = c_1 + c_2 + \cdots$ in P.

THEOREM 1: The lattice of subgroups of a finite Abelian group is rank-unimodal.

To prove the theorem, one easily reduces to the case $|G| = p^b$ where p is prime. If G is of type $\lambda = (\lambda_1, \lambda_2, \ldots)$ (i.e., a product of cyclic groups of orders $p^{\lambda_1}, p^{\lambda_2}, \ldots$), then the number of subgroups of G of order p^i is readily seen to be a polynomial function $f_{\lambda,i}(p)$ of p. Butler in fact proves that for $0 \le i < b/2$, the polynomial $f_{\lambda,i+1}(p) - f_{\lambda,i}(p)$ has nonnegative coefficients, by giving a combinatorial interpretation of the coefficients. The validity of this combinatorial interpretation rests on deep results from the theory of symmetric functions and will not be given here. For further examples of posets proved to be rank-unimodal or rank-log-concave by combinatorial means, see [43], [50].

Let us now consider a further direct method for proving log-concavity or unimodality, namely, mathematical induction. We give a simple example of the use of this method, answering a question once raised by P. Diaconis. For integers $n \ge 0$ and $k \ge 1$, define $a_{-k}, a_{-k+1}, \ldots, a_k$ by the condition

$$(x + x^{-1})^{2n} \equiv a_{-k}x^{-2k} + a_{-k+1}x^{-2k+2} + \cdots + a_k x^{2k} \pmod{x^{4k+2} - 1}. \quad (4)$$

Clearly the sequence $a_{-k}, a_{-k+1}, \ldots, a_k$ is symmetric (i.e., $a_i = a_{-i}$); we claim it is also unimodal. This assertion is clear for $n = 0$; assume it now for some $n \ge 0$. Let a_i be as in (4), and define

$$(x + x^{-1})^{2(n+1)} \equiv b_{-k}x^{-2k} + b_{-k+1}x^{-2k+2} + \cdots + b_k x^{2k} \pmod{x^{4k+2} - 1}.$$

Then $b_i = a_{i-1} + 2a_i + a_{i+1}$ (subscripts taken modulo $2k + 1$). It is straightforward to verify, by checking a small number of cases, that the unimodality and symmetry of the a_i implies the same of the b_i. Hence by induction the claim is proved.

A more elaborate example of an inductive proof is given in [23, theorem 3.1]. There it is proved that if w is a finite word of length n (whose terms come from some alphabet A), and if a_k denotes the number of *distinct* subwords (= subsequences) of w of length k, then the sequence $a_0, a_1, \ldots, a_n$ is log-concave.

A further interesting example of the use of induction appears in [28]. (See also [29].) Let P be a finite poset and n a positive integer. A map $\sigma: P \to \mathbf{n}$ is *order-preserving* (respectively, *strict order-preserving*) if $x < y$ in P implies $\sigma(x) \le \sigma(y)$ [respectively, $\sigma(x) < \sigma(y)$]. Fix $v \in P$, and define f_k (respectively, $\bar{f}_k$) to be the number of order-preserving (respectively, strict order-preserving) maps $\sigma: P \to \mathbf{n}$ satisfying $\sigma(v) = k$. Then the sequences $f_1, \ldots, f_n$ and $\bar{f}_1, \ldots, \bar{f}_n$ are log-concave (with no internal zeros). If we choose $n = \#P$ and insist that σ be an order-preventing *bijection*, then the corresponding result remains true but the proof uses much deeper techniques. See Theorem 5.

We conclude this section with a discussion of certain operations on sequences that preserve log-concavity or unimodality.

PROPOSITION 1: If $A(q)$ and $B(q)$ are symmetric unimodal polynomials with nonnegative coefficients, then so is $A(q)B(q)$.

Proof: Let

$$A(q) = \sum_{i=0}^{m} a_i q^i, \qquad B(q) = \sum_{j=0}^{n} b_j q^j.$$

Set $r = \lfloor m/2 \rfloor$, $s = \lfloor n/2 \rfloor$. Then

$$A(q) = \sum_{i=0}^{r} (a_i - a_{i-1})(q^i + q^{i+1} + \cdots + q^{m-i}),$$

$$B(q) = \sum_{j=0}^{s} (b_j - b_{j-1})(q^j + q^{j+1} + \cdots + q^{n-j}),$$

whence

$$A(q)B(q)$$

$$= \sum_{i=0}^{r} \sum_{j=0}^{s} (a_i - a_{i-1})(b_j - b_{j-1}) \cdot (q^i + \cdots + q^{m-i})(q^j + \cdots + q^{n-j}). \quad (5)$$

Now the polynomial $(q^i + \cdots + q^{m-i})(q^j + \cdots + q^{n-j})$ is immediately seen to be symmetric and unimodal with center of symmetry at $(m + n)/2$. Since $(a_i - a_{i-1})(b_j - b_{j-1}) \geq 0$ for $0 \leq i \leq r$ and $0 \leq j \leq s$, it follows from (5) that $A(q)B(q)$ is unimodal (and symmetric with center $(m + n)/2$). $\square$

Note that the assumption of symmetry in Proposition 1 cannot be dropped. For instance,

$$(q^2 + q + 3)^2 = q^4 + q^3 + 7q^2 + 6q + 9.$$

EXERCISE: Does Proposition 1 remain true if we assume that only $A(q)$ is symmetric?

PROPOSITION 2: If $A(q)$ and $B(q)$ are log-concave polynomials with nonnegative coefficients and no internal zero coefficients, then so is $A(q)B(q)$.

Proof: Write $A(q) = \Sigma_0^{m+n} a_i q^i$ and $B(q) = \Sigma_0^{m+n} b_i q^i$, where $\deg A = m$ and $\deg B = n$. If X and Y are $r \times r$ real matrices all of whose $k \times k$ minors are nonnegative, then the Cauchy–Binet theorem shows that the same is true for the matrix XY. Moreover, it is easily seen that if $c_0, c_1, \ldots, c_n$ is nonnegative and log-concave with no internal zeros, then $c_i c_j \geq c_{i-r} c_{j+r}$ whenever $i \leq j$ and $r \geq 0$. Now take $k = 2$,

$$X = \begin{bmatrix} a_0 & a_1 & \cdots & a_{m+n} \\ & a_0 & \cdots & a_{m+n-1} \\ & & \ddots & \vdots \\ \bigcirc & & & a_0 \end{bmatrix} \qquad Y = \begin{bmatrix} b_0 & b_1 & \cdots & b_{m+n} \\ & b_0 & \cdots & b_{m+n-1} \\ & & \ddots & \vdots \\ \bigcirc & & & b_0 \end{bmatrix}$$

and the proof follows. $\square$

Propositions 1 and 2 (which have many other proofs) go far back (I am unaware of their precise origin) and are frequently rediscovered (e.g., [6]). For some results similar to Proposition 2, but only assuming unimodality of the coefficients, see [79]. For a direct combinatorial application of Proposition 2, see [40]. A tremendous number of variations and extensions of Propositions 1 and 2 may be found in [60] and more recently [16]; in particular, our Proposition 1 is part of [60, theorem 1.2, p. 394]. Moreover, the following result can be deduced from the case $r = 1$ of [60, theorem 3.1, p. 21].

PROPOSITION 3: Let $\mathcal{A}$ be the class of all polynomials with nonnegative coefficients with the following property: If $A(q) \in \mathcal{A}$ and $B(q)$ is any polynomial with nonnegative unimodal coefficients, then $A(q)B(q)$ has unimodal coefficients. Then $A(q) \in \mathcal{A}$ if and only if $A(q)$ has log-concave nonnegative coefficients with no internal zeros.

Note that Proposition 2 is an immediate consequence of Proposition 3, since $\mathcal{A}$ is clearly closed under multiplication.

POLYNOMIALS WITH REAL ZEROS

The following basic result goes back to Newton; see [25, p. 270] or [53, p. 52].

THEOREM 2: Let

$$P(x) = \sum_{j=0}^{n} \binom{n}{j} a_j x^j$$

be a (real) polynomial with real zeros. Then $a_j^2 \geq a_{j-1}a_{j+1}$.

Note: If we write $P(x) = \Sigma\, b_j x^j$ (so $b_j = \binom{n}{j}$), then the condition $a_j^2 \geq a_{j-1}a_{j+1}$ becomes

$$b_j^2 \geq b_{j-1}b_{j+1}\left(1 + \frac{1}{j}\right)\left(1 + \frac{1}{n-j}\right),$$

which is stronger than $b_j^2 \geq b_{j-1}b_{j+1}$.

Proof of Theorem 2: Let $D = (d/dx)$. By Rolle's theorem, $Q(x) = D^{j-1}P(x)$ has real zeros, and thus also $R(x) = x^{n-j+1}Q(1/x)$. Again by Rolle's theorem, $D^{n-j-1}R(x)$ has real zeros. But one computes easily that

$$D^{n-j-1}R(x) = \frac{n!}{2}(a_{j-1}x^2 + 2a_jx + a_{j+1}).$$

In order for this quadratic polynomial to have real zeros, we must have $a_j^2 \geq a_{j-1}a_{j+1}$. □

There are various methods for showing that polynomials have real zeros, leading to several results of combinatorial interest. One basic method for showing that a sequence $P_0(x)$, $P_1(x)$, . . . of polynomials has real zeros is to show by induction that the polynomials form a *Sturm sequence*, that is, they have interlaced simple (real) zeros.

For instance, the Hermite polynomials

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!}$$

satisfy

$$H_n(x) = -e^{x^2} \frac{d}{dx} (e^{-x^2} H_{n-1}(x)).$$

By induction $H_{n-1}(x)$ has $n - 1$ real zeros. Since $e^{-x^2}H_{n-1}(x) \to 0$ as $x \to \pm\infty$, it follows that $H_n(x)$ has n real zeros interlaced by the zeros of $H_{n-1}(x)$. Let us consider a more substantial example.

EXAMPLE 1: Let G be a finite graph (multiple edges allowed), and let t_j be the number of matchings of size j (i.e., the number of j-element sets M of edges of G, no two edges in M having a common vertex). Heilmann and Lieb [55, theorem 4.2] give essentially three different proofs, based on Sturm sequences, that the polynomial $\Sigma\, t_j x^j$ has real zeros. Hence the sequence

$$\frac{t_0}{\binom{m}{0}}, \frac{t_1}{\binom{m}{1}}, \ldots, \frac{t_m}{\binom{m}{m}}$$

is log-concave, where m denotes the largest size of a matching in G. This result was proved independently by Gruber and Kunz [51] and by Nijenhuis [76] for bipartite graphs, and was given an entirely different proof in [46, corollary 5.2]. A direct proof of the unimodality of the sequence $t_0, t_1, \ldots, t_m$ appears in [92]. Some more general results are discussed in [128]. If we take G to be the bipartite graph on vertices $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ with u_i connected to $v_1, v_2, \ldots, v_i$, then t_j is the Stirling number $S(n+1, n+1-j)$ of the second kind [25, p. 204] [113, chap. 1.4]. Thus the polynomial $\Sigma_i\, S(n, i)x^i$ has real zeros, a result originally due to Harper [54] (see also [22, 33, 62, 65, 67]). Incidentally, if $s(n, i)$ denotes a Stirling number of the first kind [so $(-1)^{n-i}s(n, i)$ is the number of permutations of n objects with i cycles], then $\Sigma_i\, (-1)^{n-i}s(n, i)x^i = x(x+1) \cdots (x+n-1)$, which trivially has real zeros.

Let us consider another combinatorial situation in which polynomials with real zeros arise. Let P be a finite partially ordered set, with elements $x_1, \ldots, x_n$ labeled so that if $x_i < x_j$ in P, then $i < j$ in $\mathbb{Z}$. Let $e_j = e_j(P)$ denote the number of surjective order-preserving maps $\sigma: P \to \{1, 2, \ldots, j\}$. Let $\bar{e}_j = \bar{e}_j(P)$ denote the number of *strict* surjective order-preserving maps $\tau: P \to \{1, 2, \ldots, j\}$. Let $w_j = w_j(P)$ denote the number of permutations $\pi = a_1 a_2 \cdots a_n$ of $1, 2, \ldots, n$ satisfying:

(a) if $x_{a_r} < x_{a_s}$ in P, then $r < s$ (i.e., π is a *linear extension* of P),
(b) $j = \#\{r : a_r > a_{r+1}\}$, the number of *descents* of π, denoted $d(\pi)$.

Define $E(q) = \Sigma\, e_j q^j$, $\bar{E}(q) = \Sigma\, \bar{e}_j q^j$, $W(q) = \Sigma\, w_j q^j$. Using the theory of P-partitions developed in [97], it follows that

$$qW(q) = (1-q)^n E(q/(1-q)),$$

$$q^n W(1/q) = (1-q)^n \bar{E}(q/(1-q)).$$

Hence for fixed P, either all or none of $E(q)$, $\bar{E}(q)$, $W(q)$ have all their zeros real (essentially a result of R. Simion [96, p. 19]). The following conjecture appears in [75, p. 114].

CONJECTURE 1: For any finite poset P, the polynomial $E(q)$ [and hence $\bar{E}(q)$ and $W(q)$] has real zeros.

This conjecture has been proved by R. Simion [96] in the case where P is a disjoint union of chains, using the concept of multiindexed Sturm sequences. Further special cases are proved in [16], where a more general conjecture is discussed. In the case where P is an antichain (disjoint union of points), we have

$$W(q) = \sum_{\pi \in \Sigma_n} q^{d(\pi)} = q^{-1}A_n(q), \tag{6}$$

where $A_n(q)$ is an *Eulerian polynomial* [25, pp. 244–246], [113, chap. 1.3] and where Σ_n denotes the symmetric group $\mathrm{Sym}(\mathbf{n})$ of all permutations of $\mathbf{n}$. It is well known (e.g., [25, p. 292]) that Eulerian polynomials have real zeros. A refinement of the unimodality of $A_n(q)$ will be given in Proposition 12 in the eighth section. Let us also mention [1] as a combinatorial application of Theorem 2.

Another well-known class of polynomials with real zeros consists of characteristic polynomials of real symmetric matrices. More generally, we have the following somewhat less well-known result. An even more general result is given by Schneider [91] and is also discussed in [45, theorem 3.2].

PROPOSITION 4: If A and B are $n \times n$ real matrices with A positive semidefinite (written $A \geq 0$) and B symmetric, then the product AB has real eigenvalues. Equivalently, if B is symmetric and $P \geq 0$, then the polynomial $\det(B + qP)$ has real zeros.

Proof: Since $A \geq 0$, it has a square root $C \geq 0$. Then

$$\begin{aligned}\det(AB - qI) &= \det(C^2B - qI)\\ &= \det(CBC - qI),\end{aligned}$$

since $C(CB)$ and $(CB)C$ have the same characteristic polynomial. But CBC is symmetric, and the proof follows. □

Note that Proposition 4 is false if we assume only that A and B are symmetric. For instance, take

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad AB = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let us now mention some combinatorial applications of Proposition 4. The first involves only the characteristic polynomial of a *symmetric* matrix. Let G be a finite loopless graph, with multiple edges allowed, on the vertex set $V = \{v_1, \ldots, v_n\}$. Let C be the matrix with rows and columns indexed by V given by

$$C_{uv} = \begin{cases} -(\text{number of edges between } u \text{ and } v), & \text{if } u \neq v \\ \text{degree } (u), & \text{if } u = v. \end{cases} \tag{7}$$

Then $C \geq 0$, and

$$\det(C + qI) = \sum_i a_i(G)q^i,$$

where $a_i(G)$ is the number of spanning forests F of G with each connected component of F rooted at a vertex, such that F has exactly i components. This is a result of Kelmans (see [26, theorem 1.4, p. 38]). Hence we obtain (since $a_0(G) = 0$):

PROPOSITION 5: Let G be a finite graph (allowing multiple edges) with p vertices, and set

$$a_i(G) = \sum_F \gamma(F),$$

where F ranges over all i-component spanning forests of G and where

$$\gamma(F) = t_1 t_2 \cdots t_i,$$

with $t_1, \ldots, t_i$ being the number of vertices of the components of F. Then the sequence

$$a_1(G)\bigg/\binom{p-1}{0}, a_2(G)\bigg/\binom{p-1}{1}, \ldots, a_p(G)\bigg/\binom{p-1}{p-1}$$

is log-concave (with no internal zeros).

The preceding result suggests naturally that we look at spanning forests without the condition that the components are rooted.

CONJECTURE 2: Let b_i be the number of spanning forests with i edges (or $p - i$ components) of a graph G with p vertices and c components. Then the sequence

$$b_0\bigg/\binom{p-c}{0}, b_1\bigg/\binom{p-c}{1}, \ldots, b_{p-c}\bigg/\binom{p-c}{p-c}$$

is log-concave. (It is in fact not known whether the sequence $b_0, b_1, \ldots, b_{p-c}$ is log-concave or even unimodal.)

Conjecture 2 has a natural matroid-theoretic generalization, due to J. H. Mason [72], where spanning forests with i edges are replaced by independent sets of i points. See [95] and [129, p. 298]. Some progress on this conjecture has been made by Dowling [34] and Mahoney [71]. See also [27].

For our second example of the use of Proposition 4, let G be a finite connected graph (say without loops), and let B be the adjacency matrix of some orientation of G, that is, the rows of B are indexed by the vertices of G, and the columns by the (directed) edges, with

$$B_{ve} = \begin{cases} 1, & \text{if } e \text{ points out of } v \\ -1, & \text{if } e \text{ points into } v \\ 0, & \text{otherwise.} \end{cases}$$

Thus $C = BB^t$, where C is as in (7). Let S be a subset of the edge set E of G, and let $\bar{S} = E - S$. Define diagonal matrices D, $\bar{D}$ indexed by $E \times E$, as follows:

$$D_{ee} = \begin{cases} 1, & \text{if } e \in S \\ 0, & \text{if } e \in \bar{S} \end{cases}$$

$$\bar{D}_{ee} = \begin{cases} 0, & \text{if } e \in S \\ 1, & \text{if } e \in \bar{S}. \end{cases}$$

Then $BDB^t \geq 0$ and $B\bar{D}B^t \geq 0$. Let M_0 denote the matrix M with its first row and column removed, and set $C = (BDB^t)_0$ and $\bar{C} = (B\bar{D}B^t)_0$. Thus $C \geq 0$ and $\bar{C} \geq 0$. A special case of a result of C. Godsil [45, sec. 3] asserts that if

$$\det(qC + \bar{C}) = \Sigma\, a_i q^i,$$

then a_i is the number of spanning trees of G that intersect S in exactly i edges. Hence by Proposition 4, we conclude the following.

PROPOSITION 6: Let a_i be as previously, and let $s = \#S$. Then the polynomial $\Sigma_0^s\, a_i q^i$ has real zeros, so the sequence

$$a_0 \Big/ \binom{s}{0}, a_1 \Big/ \binom{s}{1}, \ldots, a_s \Big/ \binom{s}{s} \tag{8}$$

is log-concave.

The log-concavity of (8) was first proved in [104, corollary 2.4] using the techniques of the fifth section. The stronger result that $\Sigma\, a_i q^i$ has real zeros is due to Godsil [45, sec. 3]. More general results of this nature also appear in [104] and [45]. There are a host of conjectures related to Propositions 4 and 6, in addition to Conjecture 2. We mention only the most striking here.

CONJECTURE 3: (a) (R. C. Read [86, p. 68] for unimodality, D. J. A. Welsh [129, exercise 5, p. 266] for log-concavity.) Let $P_G(q) = a_p q^p - a_{p-1} q^{p-1} + \cdots + (-1)^p a_0$ be the chromatic polynomial of a finite graph (or more generally the characteristic polynomial of a finite matroid [129, p. 262]). Then the sequence $a_0, a_1, \ldots, a_p$ is log-concave (or even just unimodal).

(b) (G.-C. Rota for unimodality, D. J. A. Welsh [129, p. 289] for log-concavity.) Let W_k be the number of flats of rank k of a finite matroid of rank n. Then the sequence $W_0, W_1, \ldots, W_n$ is log-concave (or even just unimodal). Equivalently, finite geometric lattices are rank-log-concave (or even just rank-unimodal).

For further work related to (a), see [56], while for (b) see [35], [94], [113, exercise 3.37], [116], [117]. A further possible unimodal sequence arising from graph theory is discussed in [13, prob. 2], but a counterexample (the truncated dodecahedron) was recently found by M. Watkins and J. Shearer.

Theorem 2 deals with the log-concavity of the sequence $a_0, a_1, \ldots, a_n$ defined by the polynomial $P(x) = \Sigma_{i=0}^n \binom{n}{i} a_i x^i$. We could also ask under what (weaker) circumstances the coefficients themselves are log-concave. For the sake of completeness we give such a result, although we know of no combinatorial applications. This result appears, for example, as the case $r = 2$ of [60, theorem 7.1, p. 415].

PROPOSITION 7: Let $Q(x) = \Sigma_{i=0}^{n} b_i x^i$, where b_i is real, $b_0 \neq 0$, and $b_n > 0$. Suppose that the zeros ζ of $Q(x)$ lie in the sector $\{\zeta:(2\pi/3) \leq \arg \zeta \leq (4\pi/3)\}$. Then each $b_i > 0$ and the sequence $b_0, b_1, \ldots, b_n$ is log-concave.

Proof: We may assume $b_n = 1$. Factor $Q(x)$ into irreducible (over $\mathbb{R}$) linear and quadratic factors with real coefficients. The linear factors must be of the form $x + a$, $a > 0$. One easily checks that if $x^2 + cx + d$ is a quadratic factor, then the hypothesis on its zeros yields $c > 0$, $d > 0$, and $c^2 \geq d$. The proof follows from Proposition 2. $\square$

ANALYTIC TECHNIQUES

The basic idea here is to obtain an analytic expression (such as a contour integral) for each term a_k of a sequence $a_0, a_1, \ldots, a_n$, and then use analytic techniques to estimate a_k accurately enough to prove unimodality. It is rather surprising that sufficiently accurate estimates can be made for a wide variety of problems. The prototype for this method is a beautiful result of Szekeres [118] concerning the number $p(n, k)$ of partitions of n into k parts. It is well known and easy to see that

$$\sum_{n\geq 0} p(n, k)x^n = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

Hence

$$p(n, k) = \frac{1}{2\pi i}\int \frac{s^{k-n-1}\,ds}{(1-s)(1-s^2)\cdots(1-s^k)},$$

the integral being around a circle $|s| = \rho < 1$ in the complex plane. By an intricate use of the method of steepest descent, Szekeres showed the following.

THEOREM 3: For n sufficiently large (conjecturally for all n), the sequence $p(n, 1)$, $p(n, 2), \ldots, p(n, n)$ is unimodal. In fact, there exists a real number $k_1 = k_1(n)$ such that $p(n, k) < p(n, k+1)$ for $k < k_1$, and $p(n, k) > p(n, k+1)$ for $k > k_1$. Moreover,

$$k_1(n) = c\sqrt{n}\,L + c^2\left(\frac{3}{2} + \frac{3}{2}L - \frac{1}{4}L^2\right) - \frac{1}{2} + 0\left(\frac{\log^4 n}{\sqrt{n}}\right),$$

where $L = \log(c\sqrt{n})$, $c = \sqrt{6}/\pi$.

Actually, an examination of Szekeres's proof shows that he overlooked the fact that $p(n, n) = p(n, n-1)$, but otherwise his result appears valid. It follows that we can have $p(n, k) = p(n, k+1)$ for at most one value of $k \neq n-1$, and it is still open whether equality can ever occur. Szekeres's paper also contains analogous results for partitions of n into k *distinct* parts. A variation of Theorem 3 due to Odlyzko and Richmond [77] treats the case of *ordered* partitions (compositions) of n into k parts.

Entringer [41] used analytic techniques to show that the polynomial $(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2$ is unimodal. This result was greatly extended by Odlyzko and

Richmond [78] to show that for "nicely behaved" sequences $a_1, a_2, \ldots$ of positive integers, the polynomial $(1 + q^{a_1})(1 + q^{a_2}) \cdots (1 + q^{a_n})$ is "almost" unimodal for all n, that is, there is an integer $m \geq 0$ (independent of n) such that if $(1 + q^{a_1}) \cdots (1 + q^{a_n}) = \Sigma\, c_{k,n}q^k$, then the sequence

$$c_{m,n}, c_{m+1,n}, \ldots, c_{s-m,n}$$

is unimodal, where $s = \Sigma\, a_i$. For any given "nice" sequence $a_1, a_2, \ldots$, a finite amount of computation (albeit sometimes rather lengthy and requiring a computer) will decide whether the polynomials $(1 + q^{a_1}) \cdots (1 + q^{a_n})$ are indeed unimodal, not just "almost" unimodal. For instance, Odlyzko and Richmond show by these means that the polynomial $(1 + q)(1 + q^2) \cdots (1 + q^n)$ is unimodal, a result previously obtained by algebraic techniques (see Example 3 in the seventh section). Along the same lines, Almkvist [4] shows that the polynomial $(1 + q)(1 + q^3) \cdots (1 + q^{2n-1})$ is unimodal, except at the coefficient of q^2 and q^{n^2-2}. Almkvist also conjectures that for even $r \geq 2$ or for odd $r \geq 3$ and n large enough (probably $n \geq 11$), the polynomial $\Pi_{k=1}^{n} \cdot (1 - q^{rk})/(1 - q^k)$ is unimodal. For $r = 2$ we get the polynomial $(1 + q)(1 + q^2) \cdots (1 + q^n)$ just discussed, while the case $r = 4$ is settled in [3]. In [5] the conjecture is proved for $3 \leq r \leq 20$ and for $r = 100$ and 101 by refining the methods of Odlyzko–Richmond.

Many of the polynomials that appear later in this paper, when there is a reasonable formula for them, can be proved to be unimodal using the Odlyzko–Richmond techniques. We will mention several such examples at the appropriate points.

MIXED VOLUMES

Let K and L be convex bodies (= nonempty, compact, convex sets) in $\mathbb{R}^n$. For $x, y \geq 0$, define the *Minkowski sum*

$$xK + yL = \{x\alpha + y\beta \colon \alpha \in K, \beta \in L\}.$$

Let V denote n-dimensional volume (or Lebesgue measure). It was shown essentially by Minkowski (though he treated only the case $n \leq 3$) that there are real numbers $V_i(K, L) \geq 0$ satisfying

$$V(xK + yL) = \sum_{i=0}^{n} \binom{n}{i} V_i(K, L)x^{n-i}y^i,$$

for all $x, y \geq 0$. The number $V_i(K, L)$ is called the *ith mixed volume* of K and L. Setting $x = 1, y = 0$ or $x = 0, y = 1$ shows that $V_0(K, L) = V(K)$ and $V_n(K, L) = V(L)$, so the mixed volumes $V_i(K, L)$ may be regarded as "interpolating" between $V(K)$ and $V(L)$.

The basic result that we need concerning mixed volumes was proved independently by A. D. Aleksandrov [2] and W. Fenchel [42], and is known as the *Aleksandrov–Fenchel* inequalities. The proof is difficult and will not be given here. For further information on mixed volumes and the Aleksandrov–Fenchel inequalities, see [14, 18, 38, 66, 81, 123, 124].

THEOREM 4 (The Aleksandrov–Fenchel inequalities): For any convex bodies K, L in $\mathbb{R}^n$, the sequence

$$V_0(K, L), V_1(K, L), \ldots, V_n(K, L) \tag{9}$$

is log-concave (with no internal zeros).

It remains to find convex bodies K, L for which the sequence (9) is of combinatorial interest. In [104, corollary 2.4] a proof was given of Proposition 6 (in fact, of a slight generalization) based on this idea. For a different application [104, theorem 3.1] (compare with the result of [28] mentioned in the second section), let P be a finite poset with n elements, and fix $v \in P$. Let $N_i = N_i(v)$ by the number of order-preserving bijections $\sigma: P \to \{1, 2, \ldots, n\}$ satisfying $\sigma(v) = i$.

THEOREM 5: The sequence $N_1, N_2, \ldots, N_n$ is log-concave (with no internal zeros).

Sketch of proof: Let $P = \{v_1, \ldots, v_{n-1}, v\}$. Let K be the set of all points $(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$ satisfying:

(a) $0 \leq t_i \leq 1$,
(b) if $v_i \leq v_j$ in P, then $t_i \leq t_j$,
(c) if $v_i < v$, then $t_i = 0$.

Similarly define $L \subset \mathbb{R}^{n-1}$ by (a), (b), and:

(c′) if $v_i > v$, then $t_i = 1$.

Then K and L are convex polytopes. By an explicit decomposition of $xK + yL$ into products of simplices, it can be computed that $V_i(K, L) = N_{i+1}/(n-1)!$. The proof follows from Theorem 4. □

Theorem 5 settles a conjecture of Chung, Fishburn, and Graham [24], strengthening an earlier unpublished conjecture of R. Rivest that the sequence $N_1, \ldots, N_n$ is unimodal.

As pointed out in [104, corollary 3.3], a suitable choice of P and v yields the following result.

COROLLARY 1: Let S be a subset of $\{1, 2, \ldots, n-1)$ and fix $1 \leq j \leq n$. Define $\omega_i = \omega_i(S, j)$ to be the number of permutations $\pi = a_1 a_2 \cdots a_n$ of $\mathbf{n}$ satisfying $\{k: a_k > a_{k+1}\} = S$ and $a_j = i$. Then the sequence $\omega_1, \omega_2, \ldots, \omega_n$ is log-concave.

The following variant of Theorem 5 was proved by J. Kahn and M. Saks [59] using the same technique.

THEOREM 6: Let P be a finite n-element poset, and fix u and v in P. Let $N_i' = N_i'(u, v)$ be the number of order-preserving bijections $\sigma: P \to \{1, 2, \ldots, n\}$ satisfying $\sigma(v) - \sigma(u) = i$. Then the two sequences $N_1', N_2', \ldots, N_{n-1}'$ and $N_{-1}', N_{-2}', \ldots, N_{-(n-1)}'$ are log-concave.

Kahn and Saks use this result to prove the following remarkable theorem, which settles a conjecture of M. Fredman.

THEOREM 7: Let P be a finite n-element poset that is not a chain. Then there exists elements $u, v \in P$ such that if β denotes the fraction of order-preserving bijections $\sigma: P \to \{1, 2, \ldots, n\}$ satisfying $\sigma(u) < \sigma(v)$, then $3/11 < \beta < 8/11$.

A further variant of Theorem 5 appears in [111, theorem 6.2], again proved by choosing a suitable K and L.

THEOREM 8: Let P be a finite n-element poset, and fix $v \in P$. Let $M_i = M_i(v)$ be the number of order-preserving bijections $\sigma: P \to \{1, \ldots, n\}$ such that if $\sigma(v) = k$, then i is the largest integer $< k$ for which $\sigma^{-1}(k-1), \sigma^{-1}(k-2), \ldots, \sigma^{-1}(k-i)$ are all incomparable with v. Then the sequence $M_0, M_1, \ldots, M_{n-1}$ is log-concave.

While no proof is known of Theorem 5–Theorem 8 avoiding the Aleksandrov–Fenchel inequalities, in the case of Theorem 8 it is easy to give a direct proof [111, theorem 6.5] that $M_0 \geq M_1 \geq \cdots \geq M_{n-1}$.

Let us also mention a curious result of Rees and Sharp [87], which in a special case considered by Teissier [112] is equivalent to a special case of a "complementary" version of the Aleksandrov–Fenchel inequalities.

THEOREM 9: Let R be a (commutative, noetherian) local ring (with identity), with maximal ideal m and Krull dimension d. Let I and J be m-primary ideals of R. It is known that for sufficiently large positive integers r and s, the length $l(R/I^rJ^s)$ of the quotient ring R/I^rJ^s is a polynomial function of r and s of total degree d. Write the terms of total degree d of this polynomial as

$$\frac{1}{d!}\sum_{k=0}^{d}\binom{d}{k}E_k(I,J)r^{d-k}s^k.$$

Then the sequence $E_0(I,J), E_1(I,J), \ldots, E_d(I,J)$ is log-*convex*, that is, $E_k^2 \leq E_{k-1}E_{k+1}$. (Moreover, each E_k is a nonnegative integer.)

For the benefit of readers knowledgeable about commutative algebra, we mention some further sequences arising from this area.

CONJECTURE 4: (a) Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded (noetherian) Cohen–Macaulay (or perhaps Gorenstein) *domain* over a field $K = R_0$, which is generated by R_1 and has Krull dimension d. Let $H(R, m) = \dim_K R_m$ be the Hilbert function of R, and write

$$\sum_{m\geq 0} H(R,m)x^m = (1-x)^{-d}\sum_{i=0}^{s} h_i x_i.$$

Then the sequence $h_0, h_1, \ldots, h_s$ is log-concave.

(b) Let A be a regular local ring with residue field K, and let I be an ideal of A for which $R = A/I$ is Cohen–Macaulay (or perhaps Gorenstein). (Or perhaps we should take $A = K[x_1, \ldots, x_n]$ and I generated by homogeneous polynomials.)

Let $\beta_i = \dim_K \mathrm{Tor}_i^A(R, K)$, the ith Betti number of R (as an A-module). Then the sequence $\beta_0, \beta_1, \ldots, \beta_{n-d}$ is log-concave, where $n = \dim A$ and $d = \dim R$.

Conjecture 4 would have many combinatorial applications. For instance, the validity of Conjecture 4(a) for normal rings generated by monomials of the same degree would imply that the numerators of the rational functions $F_G(Y)$ (for G bipartite) of [99] are log-concave, as well as the polynomials $W(q)$ of Conjecture 1. Let us mention, however, that we are not too confident about Conjecture 4 and would not be at all surprised if it turned out to be false. If in Conjecture 4(a) we assume that

R is Gorenstein but not a domain, then [100, example 4.3] gives a counterexample to the unimodality (and hence log-concavity) of the sequence $h_0, h_1, \ldots, h_s$. Related results appear in [63].

LINEAR ALGEBRA AND FINITE GROUPS

Recall that in the second section we discussed the possibility of proving that two sets S and T satisfy $\#S \leq \#T$ by constructing an explicit injection $\rho: S \to T$. Rather than dealing directly with S and T, we can greatly extend the possibilities for ρ by considering formal linear combinations of elements of S and T. More generally, let $a_0, a_1, \ldots, a_n$ be a sequence that we want to prove is symmetric and unimodal. (The methods we are about to discuss only apply to symmetric sequences. Usually symmetry will be quite easy to prove directly.) Suppose we can find vector spaces $V_0, V_1, \ldots, V_n$ (over $\mathbb{C}$, say) and linear transformations $\phi_k: V_k \to V_{k+1}$, $0 \leq k \leq \lfloor (n-1)/2 \rfloor$, satisfying:

(a) $\dim V_i = a_i, \qquad 0 \leq i \leq n,$
(b) ϕ_k is injective for $0 \leq k \leq \lfloor (n-1)/2 \rfloor$
(c) $V_i \cong V_{n-i}$.

Then clearly the sequence $a_0, a_1, \ldots, a_n$ is symmetric and unimodal. Let us call this method for showing unimodality the *linear algebra paradigm* (LAP), and denote it by LAP($V_0, \ldots, V_n$).

There are some minor variants of LAP that sometimes arise (and that we will also regard as instances of LAP).

VARIATION 1: We now have linear transformations ϕ_k for $0 \leq k \leq n-1$, and (b) and (c) are replaced by:

(b′) For $0 \leq k \leq \lfloor (n-1)/2) \rfloor$, the composition

$$\phi_{n-k-1} \cdots \phi_{k+1}\phi_k: V_k \to V_{n-k}$$

is a bijection.

Clearly (b′) implies (b) and (c). We also get that ϕ_k is surjective for $\lfloor n/2 \rfloor \leq k \leq n-1$. In all known applications of LAP, there will be "natural" isomorphisms $V_j \cong V_{n-j}$ and pairings $V_i \times V_{i+1} \to \mathbb{C}$ that make ϕ_k and ϕ_{n-k-1} adjoints of one another. Hence injectivity of ϕ_k for $0 \leq k \leq \lfloor (n-1)/2 \rfloor$ automatically yields surjectivity for $\lfloor n/2 \rfloor \leq k \leq n-1$. Since symmetry will be easy to prove directly, however, we have no reason here to consider ϕ_k for $\lfloor n/2 \rfloor \leq k \leq n-1$.

VARIATION 2: Sometimes it is convenient to consider a single graded vector space $V = V_0 \oplus V_1 \oplus \cdots \oplus V_n$, and a single linear transformation ϕ with $\phi(v) = \phi_k(v)$ if $v \in V_k$, $0 \leq k < n$, and $\phi(v) = 0$ if $v \in V_n$. This amounts only to a change in notation.

The most direct way of constructing the vector spaces V_k (as alluded to at the beginning of this section) is the following. If we have sets $S_0, S_1, \ldots, S_n$ with $a_k = \#S_k$ and $a_k = a_{n-k}$, then let $V_k = \mathbb{C}S_k$, the complex vector space with basis S_k. Thus, $\mathbb{C}S_k$

consists of all formal linear combinations

$$\sum_{x \in S_k} \alpha_x x, \qquad \alpha_x \in \mathbb{C}.$$

(Much of what we say remains valid for any field K replacing $\mathbb{C}$, but for our purposes it suffices to consider only $K = \mathbb{C}$. Conceivably, however, there might be an application of modular representations of finite groups to unimodality.) In order to apply LAP, it suffices to construct linear transformations

$$\phi_k : \mathbb{C}S_k \to \mathbb{C}S_{k+1}, \qquad 0 \leq k \leq \lfloor (n-1)/2 \rfloor, \tag{10}$$

and then to prove their injectivity. The most important example for our purposes is the following. Let M be a finite multiset (set with repeated elements) of cardinality n, and let $S_k = \binom{M}{k}$ denote the set of k-element submultisets of M. For example, if $M = \{1, 1, 2, 3\}$, then $n = 4$ and $S_3 = \{\{1, 1, 2\}, \{1, 1, 3\}, \{1, 2, 3\}\}$.)

PROPOSITION 8: Let S_k be as before, and define $\phi_k : \mathbb{C}S_k \to \mathbb{C}S_{k+1}$ by

$$\phi_k(T) = \sum_{\substack{U \in S_{k+1} \\ T \subset U}} U, \qquad T \in S_k. \tag{11}$$

Then ϕ_k is injective for $0 \leq k \leq \lfloor (n-1)/2 \rfloor$ (and surjective for $\lfloor n/2 \rfloor \leq k \leq n - 1$).

Proof: This is a fairly easy exercise in linear algebra, essentially the same argument as that used to prove [85, theorem 3.1] (also proved in [21]). For the case where M is a *set*, a particularly elegant proof and additional references appear in [49]. □

We conclude from Proposition 8 that if a_k denotes the number of k-element submultisets of M, then $a_0 \leq a_1 \leq \cdots \leq a_{\lfloor n/2 \rfloor}$. Since $a_k = a_{n-k}$ is clear, we have that $a_0, a_1, \ldots, a_n$ is unimodal. This result is easy to prove by other means. For instance, if the multiplicities of the elements of M are $m_1, m_2, \ldots, m_r$, then

$$\sum_{k=0}^{n} a_k q^k = \prod_{i=1}^{r} (1 + q + q^2 + \cdots + q^{m_i}),$$

so unimodality follows from Proposition 1. We shall soon see, however, some deeper applications of Proposition 8.

In order to get more interesting examples, we introduce some simple concepts from group theory. Let G be a finite group acting on a complex vector space V. [In other words, we are given a representation $G \to GL(V)$.] Let V^G denote the fixed subspace, that is,

$$V^G = \{v \in V : \pi \cdot v = v \text{ for all } \pi \in G\}. \tag{12}$$

Suppose also that G acts on the vector space W and that

$$\phi : V \to W \tag{13}$$

is an injective linear transformation commuting with the action of G, that is,

$$\pi \cdot \phi(v) = \phi(\pi \cdot v), \qquad \pi \in G, \quad v \in V. \tag{14}$$

If $v \in V^G$, then by (14) $\phi(v) \in W^G$. Hence ϕ maps V^G to W^G, and of course still is injective. Thus we obtain the following.

PROPOSITION 9: Suppose $V_0, V_1, \ldots, V_n$ are complex vector spaces and that we have linear transformations ϕ_k: $V_k \to V_{k+1}$ that are injective for $0 \le k \le \lfloor (n-1)/2 \rfloor$. Suppose the finite group G acts on each V_k and commutes with each ϕ_k, and that $V_k \overset{G}{\cong} V_{n-k}$ (i.e., V_k and V_{n-k} are isomorphic as G-modules). Let $a_k = \dim V_k^G$. Then the sequence $a_0, a_1, \ldots, a_n$ is symmetric and unimodal.

Proposition 9 says in effect that if LAP$(V_0, \ldots, V_n)$ "commutes" with the action of G, then there follows LAP$(V_0^G, \ldots, V_n^G)$.

Now suppose that $S_0, S_1, \ldots, S_n$ are finite sets and that $V_k = \mathbb{C}S_k$ in the preceding proposition. Suppose that G acts on each S_k. (We may regard G as a subgroup of the symmetric group Sym(S_k) of all permutations of S_k, since nothing is gained by allowing G to act nonfaithfully.) Thus G acts on V_k by

$$\pi \cdot \sum_{x \in S_k} \alpha_x x = \sum_{x \in S_k} \alpha_x (\pi \cdot x).$$

(This is just the usual way of identifying a permutation representation with a linear representation.) Let X/G denote the set of G-orbits of X. By a standard result in group theory (equivalent to Burnside's lemma), $\#(X/G) = \dim(\mathbb{C}X)^G$. Hence from Proposition 9 we obtain the following.

COROLLARY 2: Suppose G acts on finite sets $S_0, S_1, \ldots, S_n$ and that there are injective linear transformations ϕ_k:$\mathbb{C}S_k \to \mathbb{C}S_{k+1}$, $0 \le k \le \lfloor (n-1)/2 \rfloor$ that commute with the action of G. Suppose also that the actions of G on S_k and S_{n-k} are isomorphic, $0 \le k \le n$. Let $a_k = \#(S_k/G)$, the number of G-orbits of S_k. Then the sequence $a_0, a_1, \ldots, a_n$ is symmetric and unimodal.

Let us apply Corollary 2 to the situation of Proposition 8. Let X be a finite set on which G acts. Let M be an n-element multiset with elements from X such that if $x, y \in X$ belong to the same G-orbit, then x and y have the same multiplicity in M. We call M a *G-compatible multiset*. Let $S_k = \binom{M}{k}$ as in Proposition 8, and let $N \in S_k$, say $N = \{x_1^{n_1}, \ldots, x_r^{n_r}\}$ (the notation meaning that x_i has multiplicity n_i in N, so $\Sigma\, n_i = k$). Then G acts on S_k by the rule

$$\pi \cdot N = \{(\pi \cdot x_1)^{n_1}, \ldots, (\pi \cdot x_r)^{n_r}\}.$$

(Since M is G-compatible, we have $\pi \cdot N \in S_k$, so G does indeed act on S_k.)

THEOREM 10: Let $a_k = \#(S_k/G)$, the number of G-orbits of $S_k = \binom{M}{k}$. Then the sequence $a_0, a_1, \ldots, a_n$ is symmetric and unimodal.

Proof: The linear transformation ϕ_k:$\mathbb{C}S_k \to \mathbb{C}S_{k+1}$ given by (11) clearly commutes with the action of G, and by Proposition 8 ϕ_k is injective for $0 \le k \le \lfloor (n-1)/2 \rfloor$. Moreover, the permutation representations of G on S_k and S_{n-k} are isomorphic, via the map sending $N \in S_k$ to its complement $M - N$. The proof follows from Corollary 2. $\square$

Theorem 10 in the case when M is a set is originally due to Livingstone and Wagner [68]. See also [82, 109, 131] and the references therein for further information. Let us consider some applications, the first relatively straightforward. Let X be an n-element set, and let $S = \binom{X}{2}$ be the set of 2-element subsets of S. An element of $\binom{S}{k}$ may be regarded as a *graph* with k edges. Let G be the group of permutations of S induced by Sym(X). Then two graphs $\Gamma, \Gamma' \in \binom{S}{k}$ are in the same G-orbit if and only if they are isomorphic. Hence we have proved the following proposition.

PROPOSITION 10: If a_k is the number of nonisomorphic graphs with n vertices and k edges, then the sequence $a_0, a_1, \ldots, a_{\lfloor n/2 \rfloor}$ is symmetric and unimodal.

Of course, the same reasoning applies to many other structures besides graphs. J. Sheehan and E. M. Wright have raised the question as to whether $a_i < a_{i+1}$ for $1 \leq i \leq \lfloor \frac{1}{2}\binom{n}{2} \rfloor - 1$ in Proposition 10.

For a less obvious application of Theorem 10, consider an n-element poset P. An *order ideal* of P is a subset I of P such that if $x \in I$ and $y \leq x$, then $y \in I$. Let $j_k = j_k(P)$ denote the number of k-element order ideals of P. It is often an interesting problem to decide whether the sequence $j_0, j_1, \ldots, j_n$ is unimodal. Here we will be concerned with the case $P = \mathbf{l} \times \mathbf{m}$, the product of the l-element chain $1 < 2 < \cdots < l$ and the m-element chain $1 < 2 < \cdots < m$. A k-element order ideal of $\mathbf{l} \times \mathbf{m}$ may be identified with a partition $m \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0$ of the integer $k = \Sigma \lambda_i$ into at most l parts $\lambda_i > 0$, and with largest part λ_1 at most m. Namely, let

$$\lambda_i = \max\{j: (i, j) \in I\}.$$

It is well known (e.g., [7, theorem 3.1] and [113, proposition 1.3.19]) that

$$\sum_{k=0}^{lm} j_k(\mathbf{l} \times \mathbf{m}) q^k = \begin{bmatrix} l+m \\ l \end{bmatrix}, \tag{15}$$

where $\left[\begin{smallmatrix} l+m \\ l \end{smallmatrix}\right]$ denotes the q-binomial coefficient, as given by (2).

THEOREM 11: The q-binomial coefficient $\left[\begin{smallmatrix} l+m \\ l \end{smallmatrix}\right]$ is symmetric and unimodal.

Proof: Symmetry is clear, either from the combinatorial definition of $j_k(\mathbf{l} \times \mathbf{m})$ [by associating an order ideal I with its complement in the dual poset $(\mathbf{l} \times \mathbf{m})^*$] or the formula (2). (Symmetry will also follow from Theorem 10.) To show unimodality, consider all permutations π of $\mathbf{l} \times \mathbf{m}$ of the form

$$\pi \cdot (i, j) = (\sigma \cdot i, \rho_i \cdot j),$$

where $\sigma \in \Sigma_l$ and $\rho_1, \ldots, \rho_l \in \Sigma_m$. (Here Σ_l and Σ_m denote the symmetric groups on $\mathbf{l}$ and $\mathbf{m}$, respectively.) The set of all $l!m!^l$ such permutations forms a subgroup of Sym($\mathbf{l} \times \mathbf{m}$) that by definition is the *wreath product* $\Sigma_m \S \Sigma_l$. A little thought shows that each orbit of $\binom{\mathbf{l} \times \mathbf{m}}{k}$ contains a unique order ideal (of cardinality k). Hence the number $\#\binom{\mathbf{l} \times \mathbf{m}}{k} / \Sigma_l \S \Sigma_m$ of orbits of k-element subsets of $\mathbf{l} \times \mathbf{m}$ is $j_k(\mathbf{l} \times \mathbf{m})$, and the proof follows from Theorem 10. ☐

A result immediately implying Theorem 11, known as the *Cayley–Sylvester theorem*, was conjectured by Cayley in the 1850s and proved by Sylvester in 1878. The first explicit statement that the Cayley–Sylvester theorem implies the unimodality of $\left[\begin{smallmatrix} l+m \\ l \end{smallmatrix}\right]$ is due to Elliott in 1913. The work of Cayley and Sylvester takes place in the context of the invariant theory of binary forms and is essentially equivalent to the technique given in the next section. The proof we have just given is the simplest proof to date and first appeared in [105, corollary 9.6]; an intricate combinatorial proof was recently given by O'Hara [80] and further expounded in [132]. (A combinatorial proof for the case $l = 4$ is included in stronger results of [88] and [130]. The paper [127] is possibly relevant, but I have not seen a copy.) For historical references, see [84]. It is possible (as mentioned to me by A. Odlyzko) to give an analytic proof along the lines of [78]. Other proofs of Theorem 11 have been given using sophisticated techniques from representation theory, algebraic geometry, etc., but they all are essentially equivalent to the original proof of Sylvester. By analyzing more closely the linear transformations $\phi_k:\mathbb{C}\binom{\mathbf{l}\times\mathbf{m}}{k} \to \mathbb{C}\binom{\mathbf{l}\times\mathbf{m}}{k+1}$, one can obtain much additional information about the poset $\mathbf{l} \times \mathbf{m}$ and related posets. See, for instance, [102, 103, 109] for further details. An interesting conjectured variation of Theorem 11 involving rook configurations appears in [44, p. 250].

EXERCISE: Let the symmetric group Σ_n act on the multiset $M = \{1^l, \ldots, n^l\}$ by permuting $1, \ldots, n$. Show that

$$\sum_{k=0}^{ln}\left(\#\binom{M}{k}\Big/\Sigma_n\right)q^k = \begin{bmatrix} n+l \\ l \end{bmatrix},$$

and deduce from this Theorem 11.

Theorem 10 can be generalized by appealing to the representation theory of the finite group G. Let $\hat{G}$ denote the set of inequivalent irreducible complex linear representations of G. The cardinality of $\hat{G}$ is thus equal to the number of conjugacy classes of G. (A good reference for the basic concepts of representation theory that we will use is [93].) If G acts on a complex vector space V (equivalently, we are given a representation $\sigma:G \to GL(V)$), then V breaks up into a direct sum of isotypic components,

$$V = \coprod_{\rho\in\hat{G}} V_\rho,$$

where each V_ρ is a direct sum of some number $m_\sigma(\rho) = m_V(\rho)$ of copies of irreducible subspaces affording ρ. The number $m_\sigma(\rho)$ is the *multiplicity* of ρ in σ, and

$$\dim V = \deg\sigma = \sum_\rho \dim V_\rho = \sum_\rho m_\sigma(\rho)(\deg\rho).$$

If ι denotes the trivial representation of G, then the space V^G of (**12**) is just V_ι.

Now let $\phi: V \to W$ as in (**13**). If ϕ commutes with the action of G, then Schur's lemma [93, p. 13] ensures that $\phi(V_\rho) \subseteq W_\rho$ for all $\rho \in \hat{G}$. Hence if ϕ is injective, then $\dim V_\rho \leq \dim W_\rho$, so

$$m_V(\rho) \leq m_W(\rho). \tag{16}$$

THEOREM 12 (see [105, proposition 9.4]): Let G act on the finite set X, and let M be a G-compatible n-element multiset on X. If $0 \leq k \leq n$ and $\rho \in \hat{G}$, then let $m_k(\rho)$ denote the multiplicity of ρ in the action of G on the set $\binom{M}{k}$ of k-element submultisets of M.

Then the sequence

$$m_0(\rho), m_1(\rho), \ldots, m_n(\rho)$$

is symmetric and unimodal.

Proof: We have $m_0(\rho) \leq m_1(\rho) \leq \cdots \leq m_{\lfloor n/2 \rfloor}(\rho)$ by the preceding discussion and the fact that the map ϕ_k of (**11**) commutes with the action of G and is injective for $0 \leq k \leq \lfloor (n-1)/2 \rfloor$. Symmetry follows as in Theorem 10; the actions of G on $\binom{M}{k}$ and $\binom{M}{n-k}$ are isomorphic, so $m_\rho(k) = m_\rho(n-k)$. $\square$

An application of Theorem 12 of great combinatorial significance arises from letting $M = \{1^l, 2^l, \ldots, n^l\}$ and $G = \Sigma_n$, as in the previous exercise. The irreducible representations of Σ_n are indexed by partitions λ of n. Let $s_\lambda(x) = s_\lambda(x_1, x_2, \ldots)$ be the *Schur function* indexed by λ, as defined, for example, in [70, 98]. We merely state without proof the following result, whose proof involves knowledge of the representation theory of Σ_n. See [61, Satz 7.2.2]. A proof can also be given using the representation theory of the Lie algebra $sl(2, \mathbb{C})$ and goes back to Dynkin, as discussed in the next section (Example 2).

THEOREM 13: Let $M = \{1^l, 2^l, \ldots, n^l\}$ and $G = \Sigma_n$ as before. Let $m_k(\lambda)$ denote the multiplicity of the irreducible representation of Σ_n corresponding to λ in the action of Σ_n on $\binom{M}{k}$. Then

$$\sum_{k \geq 0} m_k(\lambda) q^k = s_\lambda(1, q, q^2, \ldots, q^{l-1}).$$

Hence the polynomial $s_\lambda(1, q, q^2, \ldots, q^{l-1})$ is symmetric and unimodal.

The polynomial $s_\lambda(1, q, q^2, \ldots, q^{l-1})$ can be explicitly evaluated, and also has a combinatorial interpretation. Namely, the coefficient $m_k(\lambda)$ of q^k in $s_\lambda(1, q, q^2, \ldots, q^{l-1})$ is the number of column-strict plane partitions (as defined, e.g., in [70, 98]) of k of shape λ and parts chosen from $0, 1, \ldots, l-1$. Moreover

$$s_\lambda(1, q, q^2, \ldots, q^{l-1}) = q^{d(\lambda)} \prod_{x \in \lambda} \frac{1 - q^{c(x)}}{1 - q^{h(x)}},$$

where

$$d(\lambda) = \Sigma\,(i-1)\lambda_i = \Sigma \binom{\lambda_i'}{2}, \tag{17}$$

and where if $x = (i, j)$ with $1 \leq i \leq \lambda_1'$ and $1 \leq j \leq \lambda_i$, then $c(x) = j - i$ (the *content* of x) and $h(x) = \lambda_i + \lambda_j' - i - j + 1$ (the *hook-length* of x). Here $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\lambda' = (\lambda_1', \lambda_2', \ldots)$ (the conjugate partition to λ). See [70, p. 48] and [98, theorem 15.3] for further details.

A special case of Theorem 13 is of particular interest. Let $\lambda = (c^r)$, the partition with r parts equal to c. (The diagram of λ is a rectangle with r rows and c columns.) It is easily seen from the preceding combinatorial description of $m_k(\lambda)$ in terms of plane partitions that

$$q^{-d(\lambda)} s_\lambda(1, q, q^2, \ldots, q^{t+r-1}) = \sum_{k=0}^{crt} j_k(\mathbf{c} \times \mathbf{r} \times \mathbf{t}) q^k,$$

where $j_k(\mathbf{c} \times \mathbf{r} \times \mathbf{t})$ is the number of k-element order ideals of $\mathbf{c} \times \mathbf{r} \times \mathbf{t}$. Hence we obtain the following corollary.

COROLLARY 3: For any positive integers c, r, t, the sequence

$$j_0(\mathbf{c} \times \mathbf{r} \times \mathbf{t}), j_1(\mathbf{c} \times \mathbf{r} \times \mathbf{t}), \ldots, j_{crt}(\mathbf{c} \times \mathbf{r} \times \mathbf{t})$$

is symmetric and unimodal.

Corollary 3 (and the special case Theorem 11) naturally suggest the following conjecture, which is open even for $m = 4$ or for $r_1 = r_2 = \cdots = r_m = 2$.

CONJECTURE 5: For any positive integers $r_1, r_2, \ldots, r_m$ write $j_k = j_k(\mathbf{r}_1 \times \mathbf{r}_2 \times \cdots \times \mathbf{r}_m)$ and $r = r_1 r_2 \cdots r_m$. Then the sequence $j_0, j_1, \ldots, j_r$ is unimodal. (The sequence is clearly symmetric.)

Now let G be any subgroup of Σ_n and let $\sigma: G \to \mathbb{C}$ be any complex-valued function of G. Suppose $\pi \in G$ has $c_i(\pi)$ cycles of length i (so $\Sigma_i\, ic_i(\pi) = n$). Define the *generalized cycle index* polynomial of G with respect to σ by

$$\mathrm{Cyc}(G, \sigma) = \frac{1}{|G|} \sum_{\pi \in G} \sigma(\pi^{-1}) \prod_i x_i^{c_i(\pi)}.$$

If $f(q)$ is any polynomial in q, then define the *Pólya composition* $\mathrm{Cyc}(G, \sigma)[f(q)]$ of $\mathrm{Cyc}(G, \sigma)$ with $f(q)$ to be the polynomial obtained by substituting $f(q^i)$ for x_i in $\mathrm{Cyc}(G, \sigma)$, that is,

$$\mathrm{Cyc}(G, \sigma)[f(q)] = \mathrm{Cyc}(G, \sigma)(x_i \to f(q^i)).$$

In particular, it follows from the theory of Schur functions that

$$\mathrm{Cyc}(\Sigma_n, \chi^\lambda)[1 + q + \cdots + q^t] = s_\lambda(1, q, \ldots, q^t),$$

where χ^λ denotes the irreducible character of Σ_n corresponding to the partition λ. The following result is not difficult to deduce from Theorem 13. It was first proved by A. Kerber [61, sec. 7.2.3] and also appears in [110, corollary 3.3].

THEOREM 14: Let G be a subgroup of Σ_n, χ an ordinary character of G (i.e., the character of a complex linear representation of G), and $f(q)$ a polynomial with nonnegative integral unimodal coefficients, satisfying $q^t f(1/q) = f(q)$. Let

$$F(q) = \mathrm{Cyc}(G, \chi)[f(q)].$$

Then $F(q)$ has nonnegative integral unimodal coefficients, and $q^{nt} F(1/q) = F(q)$.

As noted in [110, p. 268], the assumption that the coefficients of $f(q)$ are *integers* is essential for $F(q)$ to be unimodal. For instance [writing $\mathrm{Cyc}(G)$ for $\mathrm{Cyc}(G, \iota)$ when ι is the trivial character],

$$\mathrm{Cyc}(\Sigma_2)[\tfrac{1}{2} + \tfrac{1}{2}q] = \tfrac{1}{8}(3 + 2q + 3q^2).$$

We can ask whether there are other examples of LAP of combinatorial interest besides those arising from Proposition 8. We state one very special result of this nature [105, proposition 9.11].

PROPOSITION 11: Let n be a positive integer. Then the two polynomials

$$\left[{2n \atop n}\right] + (1 + q)(1 + q^3)(1 + q^5) \cdots (1 + q^{2n-1}),$$

$$\left[{2n \atop n}\right] - (1 + q)(1 + q^3)(1 + q^5) \cdots (1 + q^{2n-1})$$

are symmetric and unimodal.

Idea of proof: Choose S_k to be the set $J_k(\mathbf{n} \times \mathbf{n})$ of k-element order ideals of $\mathbf{n} \times \mathbf{n}$. Define $\phi_k : \mathbb{C}S_k \to \mathbb{C}S_{k+1}$ by

$$\phi_k(I) = \sum_{\substack{I' \in S_{k+1} \\ I' \supset I}} I', \qquad I \in S_k.$$

Let G be the cyclic group of order 2, acting on S_k in the obvious way. The difficult part of the proof is to show that ϕ_k is injective for $0 \le k \le \lfloor (n^2 - 1)/2 \rfloor$. This being done, the proof follows from (**16**) by a straightforward computation. $\square$

A further connection between linear algebra and unimodality is provided by the theory of association schemes. We will not enter into this theory here, but refer the reader to [8, pp. 205 and 374–376] for the details.

REPRESENTATIONS OF $sl(2, \mathbb{C})$

The Lie algebra $sl(2) = sl(2, \mathbb{C})$ is defined to be the set of all complex 2×2 matrices of trace 0, with the binary operation $[A, B] = AB - BA$. Thus $sl(2)$ is a vector space over $\mathbb{C}$ of dimension three. The representation theory of $sl(2, \mathbb{C})$ is a powerful tool for proving unimodality of certain sequences (which will always be symmetric). We first review the basic facts about $sl(2)$ (see, e.g., [58, sec. 7]).

Let $gl(n) = gl(n, \mathbb{C})$ denote the Lie algebra of all complex $n \times n$ matrices (with the operation $[A, B] = AB - BA$). A (finite-dimensional, complex, linear) *representation* of $sl(2)$ of *degree n* is a Lie algebra homomorphism $\psi : sl(2) \to gl(n)$ (so $\psi[A, B] = [\psi A, \psi B]$). Let h denote the matrix

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in sl(2).$$

The eigenvalues of the $n \times n$ matrix ψh are integers, say $e_1, e_2, \ldots, e_n$. Define the *character* of ψ to be the Laurent polynomial

$$\operatorname{char} \psi = \sum_{i=1}^{n} q^{e_i}. \tag{18}$$

We may identify $gl(n)$ with the set End V of linear endomorphisms of V (i.e., linear transformations $V \to V$), for some n-dimensional complex vector space V. We may then think of the representation ψ as defining an *action* of $sl(2)$ on V, that is, if $A \in sl(2)$ and $v \in V$, then $A \cdot v = \psi(A)(v)$. The representation ψ is *irreducible* if V contains no proper $sl(2)$-invariant subspace. Every representation ψ is *completely reducible*, that is, we can write $V = V \oplus V_2 \oplus \cdots \oplus V_k$, where each V_i is an irreducible invariant subspace. Thus $\psi = \psi_1 \oplus \cdots \oplus \psi_k$ and

$$\text{char}\,\psi = \text{char}\,\psi_1 + \cdots + \text{char}\,\psi_k$$

where ψ_i denotes the action of $sl(2)$ on V restricted to V_i. Two representations ψ_1, $\psi_2: sl(2) \to gl(n)$ are *equivalent* if there exists a nonsingular $n \times n$ matrix X for which

$$\psi_2(A) = X\psi_1(A)X^{-1}, \qquad \text{for all } A \in sl(2).$$

We then have that for each $j \geq 0$, $sl(2)$ has exactly one irreducible representation (up to equivalence)

$$\rho_j: sl(2) \to gl(j+1)$$

of degree $j + 1$. Moreover,

$$\text{char}\,\rho_j = q^{-j} + q^{-j+2} + q^{-j+4} + \cdots + q^j. \tag{19}$$

From these basic facts there follows the next theorem.

THEOREM 15: Let $\psi: sl(2) \to gl(n)$ be a representation of $sl(2)$ with

$$\text{char}\,\psi = \Sigma\, b_i q^i. \tag{20}$$

Then the two sequences

$$\begin{aligned} &\ldots, b_{-4}, b_{-2}, b_0, b_2, b_4, \ldots \\ &\ldots, b_{-3}, b_{-1}, b_1, b_3, \ldots \end{aligned} \tag{21}$$

are symmetric (i.e., $b_i = b_{-i}$) and unimodal.

Proof: Let m_j be the multiplicity of ϕ_j in ψ, so

$$\begin{aligned} \text{char}\,\psi &= \Sigma\; m_j(\text{char}\,\rho_j) \\ &= \Sigma\, m_j(q^{-j} + q^{-j+2} + \cdots + q^j). \end{aligned} \tag{22}$$

Clearly from (19) char ψ is symmetric. Moreover, comparing (20) and (22) yields $m_i = b_i - b_{i+2}$, $i \geq 0$. Since $m_i \geq 0$, the proof follows. □

REMARK: Let $x = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in sl(2)$. Given $\psi: sl(2) \to \text{End}\,V$, let V^i be the eigenspace of $\psi(h)$ corresponding to the eigenvalue i, so if char $\psi = \Sigma\, b_i q^i$, then $b_i = \dim V^i$. (V^i is called a *weight* space.) Then $x \cdot V^i \subseteq V^{i+2}$, and x is injective for $i < 0$. Hence, the graded vector spaces $V = \amalg\, V^{2i}$ and $V = \amalg\, V^{2i+1}$ and linear transformation x (acting on V) provide an example of LAP (Variation 2). Many of the examples of LAP in the previous section turn out to coincide precisely with an $sl(2)$ representation after a suitable choice of basis. See, for instance, [83].

We know of two main applications of Theorem 15, the second of which is the subject of the following section. For the first, let $\mathcal{G}$ be any finite-dimensional complex semisimple Lie algebra. Then $\mathcal{G}$ contains a certain subalgebra (unique up to conjugation) isomorphic to $sl(2)$, known as a *principal three-dimensional subalgebra* (see [64]). Now let $\psi: \mathcal{G} \to gl(n)$ be an irreducible representation. We can compose the inclusion $sl(2) \hookrightarrow \mathcal{G}$ with ψ to get a representation $\tilde{\psi}: sl(2) \to gl(n)$. The character of $\tilde{\psi}$ was essentially first computed by Dynkin [36], [37, p. 332]. In particular, char $\tilde{\psi}$ is either an odd or an even polynomial in q, so one of the sequences (21) is trivial. A simple description of char $\tilde{\psi}$ in terms of root systems was given in [101]. (An alternative description of char $\tilde{\psi}$ for the root systems B_n, C_n, D_n is implicit in [39], though only the case $q = 1$ is treated explicitly.) We will simply state the result here. The reader unfamiliar with root systems may think of a root system as a finite set R of vectors in a real vector space V satisfying certain axioms. See, for example, [15] or [58] for further information on root systems.

Let $B = \{\alpha_1, \ldots, \alpha_n\}$ be a base for the root system R of rank n. Then R decomposes as a disjoint union $R = R_+ \cup R_-$ of positive and negative roots. Every vector $\beta \in R_+$ can be written uniquely in the form

$$\beta = \sum_{i=1}^{n} c_i \alpha_i,$$

where c_i is a *nonnegative* integer. Moreover, $\beta \in R_+$ if and only if $-\beta \in R_-$. Let us write $x^\beta = x_1^{c_1} \cdots x_n^{c_n}$, and define the polynomial

$$P_R(x_1, \ldots, x_n) = \sum_{\beta \in R_+} (1 - x^\beta).$$

THEOREM 16: Let R be a root system of rank n, and let $m_1, \ldots, m_n$ be any positive integers. Define

$$Q_R(m_1, m_2, \ldots, m_n) = \frac{P_R(q^{m_1}, q^{m_2}, \ldots, q^{m_n})}{P_R(q, q, \ldots, q)}.$$

Then $Q_R(m_1, \ldots, m_n)$ is a symmetric unimodal polynomial in the variable q with nonnegative integer coefficients and constant term 1.

EXAMPLE 2: Choose R to be the root system A_n. Then, with a suitable ordering of the base,

$$Q_R(m_1, \ldots, m_n) = q^{-d(\lambda)} s_\lambda(1, q, \ldots, q^{n-1}),$$

where λ is the partition with $\lambda_i - \lambda_{i+1} = m_i$, $d(\lambda)$ is given by (**17**), and s_λ is as in Theorem 13. Hence we have a second proof of Theorem 13, essentially the proof in [110, theorem 3.1] and [70, p. 67].

EXAMPLE 3: Choose $R = C_n$. Then, with a suitable ordering of the base,

$$Q(C_n; 1, 1, \ldots, 1, 2) = (1 + q)(1 + q^2) \cdots (1 + q^n). \qquad \textbf{(23)}$$

It is remarkable that there is no simple proof (e.g., similar to our proof of Theorem 11) of the unimodality of this polynomial. Hughes [57] was the first to realize the relevance of Dynkin's unimodality result to combinatorics and to observe that (**23**) is a special case. Proctor [84] gives, among other things, a version of Dynkin's proof of the unimodality of (**23**) that avoids explicit mention of Lie algebras. Recall also that in the fourth section we mentioned the analytic proof of Odlyzko–Richmond of the unimodality of (**23**). The polynomial $Q(C_n; 1, 1, \ldots, 1, m)$ for any $m \geq 1$ also has special combinatorial significance; see [101, example 3].

REMARK (for readers familiar with Lie algebras): The representation $\psi: \mathcal{G} \to gl\,(2^n)$ corresponding to (**23**) is the spin representation of $so(2n + 1, \mathbb{C})$. Note that $so(2n + 1,$

$\mathbb{C}$) is of type B_n, but we chose $R = C_n$. This is because the root system R appearing in Theorem 16 turns out to be the *dual* of the root system corresponding to $\mathcal{G}$.

EXAMPLE 4: Choose $R = B_n$. Then, with a suitable ordering of the base,

$$Q(B_n; 2, 1, \ldots, 1, 1) = \begin{bmatrix} 2n+2 \\ n+1 \end{bmatrix} \frac{[2][n+1]}{[n+2][2n+2]}$$
$$= K_{n+1}(q), \text{ say,}$$

where we use the notation of (**2**) and (3). Hence $K_n(1)$ is equal to the *Catalan number*

$$K_n = K_n(1) = \frac{1}{n+1}\binom{2n}{n},$$

and $K_n(q)$ provides a symmetric, unimodal "q-analog" of degree n^2. The most natural q-analog of K_n is

$$\begin{bmatrix} 2n \\ n \end{bmatrix} \frac{[1]}{[n+1]},$$

but this polynomial is not unimodal. The coefficient of q^k in $K_{n+1}(q)$ has the following combinatorial interpretation: it is the number of increasing integer sequences $s_1 < s_2 < \cdots < s_n$ satisfying: (a) $2i - 1 \le s_i \le 2n$, and (b) $k = \frac{1}{2}(n^2 + f(s_1) + \cdots + f(s_n))$, where

$$f(s) = \begin{cases} s - 1, & s \text{ even} \\ -s, & s \text{ odd.} \end{cases}$$

Is there a simpler interpretation of this coefficient?

An interesting variation of Theorem 16 arises when we replace Lie algebras with Lie superalgebras. We will say only a brief word about it here, referring the reader to [110] for more details. Lie superalgebras are generalizations of Lie algebras, and much of the preceding theory has a "superanalog." The analog of $sl(2)$ turns out to be a certain five-dimensional superalgebra denoted osp(1, 2). The superalgebra osp(1, 2) has one irreducible representation ρ_j, $j \ge 0$, of every odd degree $2j + 1$, and

$$\text{char}\ \rho_j = q^{-j} + q^{-j+1} + q^{-j+2} + \cdots + q^j.$$

Any (finite-dimensional) representation ψ of osp(1, 2) has a character char $\psi = \Sigma\, b_i q^i$ satisfying (a) $b_i = b_{-i}$, and (b) char ψ is unimodal. (We no longer need to consider even and odd exponents separately as in (**21**). The theory of superalgebras "unifies" the two sequences (**21**) into one, analogous to the way the theory of supersymmetry in physics unifies bosons (particles of integral spin) and fermions (half-integral spin). Our aims are not quite as lofty as those of physicists, however; physicists want to unify the laws of nature, while we wish to show that certain sequences are unimodal.)

We no longer have as rich a class of examples as in Theorem 16 (see [125] and [126] for more details), but there does turn out to be a superanalog of Example 2. The Schur function $s_\lambda(x)$ is replaced by the "super-Schur function" $s_\lambda(x/y)$ (also called a "hook Schur function" and denoted $HS_\lambda(x, y)$), as defined in [11] or [110]. Theorem 13 becomes the result that the polynomials

$$s_\lambda(1, q^2, \ldots, q^{2n}/q, q^3, \ldots, q^{2n-1})$$

are symmetric and unimodal. In particular, we get the following superanalog of the q-binomial coefficients by taking λ to consist of the single part j or of j parts all equal to 1. A. Odlyzko has informed me that the techniques of [78] can in principle also be used to prove Theorem 17, but the details have not been worked out.

THEOREM 17: Define polynomials $P_{jn}(q)$ and $P'_{jn}(q)$ by

$$\sum_{j\geq 0} P_{jn}(q)t^j = \frac{(1+qt)(1+q^3t)\cdots(1+q^{2n-1}t)}{(1-t)(1-q^2t)\cdots(1-q^{2n}t)},$$

$$\sum_{j\geq 0} P'_{jn}(q)t^j = \frac{(1+t)(1+q^2t)\cdots(1+q^{2n}t)}{(1-qt)(1-q^3t)\cdots(1-q^{2n-1}t)}.$$

Then $P_{jn}(q)$ and $P'_{jn}(q)$ are symmetric and unimodal.

The coefficient of q^k in $P_{jn}(q)$ is the number of partitions of k into at most j parts all $\leq 2n$, and with no repeated odd part. If we think of even parts as "bosons" and odd parts as "fermions" (the "spin" of a part p being $p/2$, or more accurately, $(p/2) - n$), then the condition that no odd part is repeated is just the Pauli exclusion principle, which applies to fermions but not bosons.

THE HARD LEFSCHETZ THEOREM

Our second main application of Theorem 15 arises from algebraic geometry. The fundamental underlying algebraic result is that $sl(2)$ acts in a nice way on the cohomology ring $H^*(X, \mathbb{C})$ of certain algebraic varieties X. Classically, X is an irreducible smooth complex projective variety. A much richer class of examples is obtained by replacing the condition that X is smooth by the condition that the singularities of X are not too badly behaved. Namely, we assume X is a (complex) V-*variety*, that is, locally X looks like affine space $\mathbb{C}^n$ modulo the action of a finite group G of linear transformations. (The group G depends on the point $p \in X$ under consideration; if G is trivial, then X is smooth at p.)

Thus let X be an irreducible complex projective V-variety of (complex) dimension n. Then X has associated with it its (singular) cohomology ring (over $\mathbb{C}$, say), denoted $H^*(X) = H^*(X, \mathbb{C})$. $H^*(X)$ has the structure

$$H^*(X) = H^0(X) \oplus H^1(X) \oplus \cdots \oplus H^{2n}(X)$$

of a finite-dimensional graded $\mathbb{C}$-algebra, so that each $H^i(X)$ is a finite-dimensional vector space, and $H^i(X)H^j(X) \subseteq H^{i+j}(X)$. Set $\beta_i(X) = \dim H^i(X)$, the ith *Betti number* of X. Since X is projective, we can embed it in a complex projective space $\mathbb{P}^N$. Let K be a generic hyperplane in $\mathbb{P}^N$. By a standard construction in algebraic geometry, the closed subvariety $K \cap X$ of X defines an element $\omega \in H^2(X)$. Multiplication by ω is a linear transformation on $H^*(X)$ sending $H^i(X)$ to $H^{i+2}(X)$. Define another linear transformation η on $H^*(X)$ by $\eta(x) = (i - n)x$ if $x \in H^i(X)$. One can then define a scalar product on $H^*(X)$ with the following property: if ω^* denotes the adjoint to multiplication by ω, then the map $\psi: sl(2) \to \operatorname{End} H^*(X)$ given by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \omega, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \eta, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \omega^*$$

is a Lie algebra homomorphism [i.e., a representation of $sl(2)$]. Comparing with our definition (**18**) of char ψ, we get

$$\operatorname{char} \psi = q^{-n} \sum_{i=0}^{2n} \beta_i(X) q^i.$$

Hence from Theorem 15 we obtain the following theorem.

THEOREM 18: Let X be an n-dimensional irreducible complex projective V-variety with Betti numbers $\beta_i = \beta_i(X)$. Then the two sequences

$$\beta_0, \beta_2, \ldots, \beta_{2n}$$

$$\beta_1, \beta_3, \ldots, \beta_{2n-1}$$

are symmetric and unimodal.

A consequence of the representation ψ is that for $0 \leq i \leq n$, the map $\omega^{n-i}: H^i(X) \to H^{2n-i}(X)$ (defined as multiplication by ω^{n-i}) is a bijection. This result is known as the *hard Lefschetz theorem* and is sufficient to deduce Theorem 18. The hard Lefschetz theorem and its consequence Theorem 18 were first given for *smooth* varieties by Lefschetz, but his proof was not complete. The first rigorous proof was given by Hodge using his theory of harmonic integrals, while the connection with $sl(2)$ is due to Chern. The extension of the hard Lefschetz theorem to V-varieties was published by Steenbrink, but his proof was incorrect for $n > 2$. A correct proof was given by Saito [90] using deep results from the theory of perverse sheafs (see also [74]). For further historical information and references, see [108]. Note too that the hard Lefschetz theorem applied to $H^*(X)$ may be regarded as an example of LAP($H^0(X)$, $H^2 \cdot (X), \ldots, H^{2n}(X)$), and LAP($H^1(X), H^3(X), \ldots, H^{2n-1}(X)$) (Variation 1).

If we take X to be the Grassmann variety $G_{l,l+m}(\mathbb{C})$ of all l-dimensional linear subspaces of $\mathbb{C}^{l+m}$, then Theorem 18 reduces to Theorem 11. Similarly, if we take $X = SO(2n + 1, \mathbb{C})/P$ for a certain maximal parabolic subgroup P, then we obtain the unimodality of (**23**). More generally, if G is any complex connected semisimple Lie group and P any parabolic subgroup, then we can apply Theorem 18 to the quotient space ("generalized flag manifold") $X = G/P$. We then obtain the following result (see [102] for further details, precise definitions, etc.).

THEOREM 19: Let W be a (finite) Weyl group with simple reflections S, and let $J \subseteq S$. Let W_J be the subgroup of W generated by J, and let W^J be the set of minimal length (with respect to the usual length function l on W) coset representatives of W_J. Then the polynomial

$$F(W^J, q) = \sum_{w \in W^J} q^{l(w)} \qquad (\mathbf{24})$$

is symmetric and unimodal. Explicitly, we have

$$F(W^J, q) = \frac{\prod_1^m (1 + q + q^2 + \cdots + q^{e_i})}{\prod_1^k (1 + q + q^2 + \cdots + q^{f_i})},$$

where $e_1, \ldots, e_m$ are the exponents of W and $f_1, \ldots, f_k$ the exponents of W_J. (Hence by Proposition 1, the unimodality of **(24)** in general follows from the special case $\#(S - J) = 1$.)

The varieties G/P underlying Theorem 19 are smooth. For an application of Theorem 18 when X is a nonsmooth V-variety, take Y to be a product

$$Y = \mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \times \cdots \times \mathbb{P}^{m_r}$$

of complex projective spaces $\mathbb{P}^{m_i}$, and let G be a group of permutations of $\mathbf{r}$ that acts on Y by permuting coordinates. If we let X be the quotient variety Y/G, then X is a V-variety and Theorem 18 reduces to Theorem 10.

We now give a more substantial application of Theorem 18 for which the varieties X are nonsmooth V-varieties. Let $\mathcal{P}$ be a *simplicial* d*-polytope*, that is, a d-dimensional convex polytope for which every proper face is a simplex. (See [52] or [73] for basic information on convex polytopes.) Let $f_i = f_i(\mathcal{P})$ denote the number of i-dimensional faces of $\mathcal{P}$, $0 \le i \le d - 1$. Set $f_{-1} = 1$ (corresponding to regarding the empty set as a face of dimension -1). Call the vector $f(\mathcal{P}) = (f_0, f_1, \ldots, f_{d-1})$ the f*-vector* of $\mathcal{P}$. Define the numbers $h_0, h_1, \ldots, h_d$ by

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x - 1)^{d-i}. \tag{25}$$

Clearly knowing the f_i is equivalent to knowing the h_i. The vector $h(\mathcal{P}) = (h_0, h_1, \ldots, h_d)$ is called the h*-vector* of $\mathcal{P}$. The *Dehn–Sommerville equations* (e.g., [52, secs. 9.2 and 9.8] and [73, secs. 2.4 and 5.1] assert that $h(\mathcal{P})$ is symmetric, that is, $h_i = h_{d-i}$. In 1970 McMullen and Walkup conjectured that $h(\mathcal{P})$ is unimodal. This conjecture is known as the *generalized lower bound conjecture* (GLBC), because it generalizes an earlier conjecture (subsequently proved by D. Barnette) that gives the least possible value of f_i when f_0 (the number of vertices) and d (the dimension) are specified. (The GLBC also included a condition as to when $h_i = h_{i+1}$. This part of the GLBC remains open.)

THEOREM 20: The GLBC for simplicial d-polytopes $\mathcal{P}$ is valid, that is, the h-vector $h(\mathcal{P})$ is unimodal (and symmetric).

Sketch of proof: Without loss of generality we may assume (a) $\mathcal{P} \subset \mathbb{R}^d$, (b) the origin of $\mathbb{R}^d$ lies in the interior of $\mathcal{P}$, and (because $\mathcal{P}$ is simplicial) (c) the vertices of $\mathcal{P}$ have rational coordinates. Under these circumstances we can associate an irreducible complex variety $X(\mathcal{P})$ of dimension d, called a *toric variety*, first defined by Demazure. (The varieties $X(\mathcal{P})$ are special cases of the general notion of toric varieties.) The convexity of $\mathcal{P}$ implies that $X(\mathcal{P})$ is projective (a result of Demazure), while the simplicial property implies the (much easier) result that $X(\mathcal{P})$ is a V-variety. Finally, Danilov and Jurciewicz computed the Betti numbers of $X(\mathcal{P})$, namely,

$$\beta_{2i}(X(\mathcal{P})) = h_i(\mathcal{P}),$$
$$\beta_{2i+1}(X(\mathcal{P}) = 0. \tag{26}$$

The GLBC then follows immediately from Theorem 18. □

For further details and references concerning Theorem 20, see [107]. In particular, a more detailed analysis of the cohomology ring $H^*(X(\mathcal{P}))$ yields a complete characterization (known as McMullen's g-conjecture) of the h-vectors of simplicial polytopes. There are a number of open problems related to Theorem 20. From the viewpoint of unimodality, the most significant one is as follows. Let Δ be an abstract simplicial complex that triangulates the $(d-1)$-sphere S^{d-1}. (More generally, Δ could be a $(d-1)$-dimensional nonacyclic Gorenstein complex, as discussed, e.g., in [106, 112].) We can define f_i and h_i exactly as we did for simplicial polytopes, and one still has $h_i = h_{d-i}$.

CONJECTURE 6: The sequence $h_0, h_1, \ldots, h_d$ just defined is unimodal.

A recent development in topology, namely, the development [47, 48] of *intersection (co)homology* by Goresky and MacPherson, allows Theorem 18 to be further extended. With *any* irreducible complex projective variety of dimension d, we can associate a graded vector space (over $\mathbb{C}$, say),

$$IH^*(X) = IH^0(X) \oplus IH^1(X) \oplus \cdots \oplus IH^{2d}(X),$$

called the middle intersection cohomology of X, which in general is more nicely behaved than singular cohomology $H^*(X)$. In particular [10, theorem 5.4.10]:

THEOREM 21: Let X be an irreducible complex projective variety of dimension d, and set

$$\tilde{\beta}_i = \tilde{\beta}_i(X) = \dim IH^i(X).$$

Then the two sequences

$$\tilde{\beta}_0, \tilde{\beta}_2, \ldots, \tilde{\beta}_{2d},$$
$$\tilde{\beta}_1, \tilde{\beta}_3, \ldots, \tilde{\beta}_{2d-1},$$

are symmetric and unimodal.

For instance, Lusztig [69] defines varieties for which Theorem 21 reduces to Theorem 16. For an application of Theorem 21 that wasn't previously known, let $\mathcal{P}$ and $\mathcal{Q}$ be *arbitrary* (i.e., not necessarily simplicial) convex polytopes. We say that $\mathcal{P}$ and $\mathcal{Q}$ are *combinatorially equivalent* if they have isomorphic face-lattices, that is, if there is a one-to-one correspondence between their faces that preserves inclusion. A polytope $\mathcal{P}$ is *rational* if it is combinatorially equivalent to a polytope whose vertices have rational coordinates. Although simplicial polytopes are rational, it was first shown by M. Perles that nonrational polytopes do exist (see [52, sec. 5.5.6]). Now suppose $\mathcal{P}$ is a d-dimensional polytope, embedded in $\mathbb{R}^d$ so its vertices have rational coordinates and the origin is in its interior. Then we can still define the toric variety $X(\mathcal{P})$, which is a

d-dimensional irreducible complex projective variety. However, unless $\mathcal{P}$ is simplicial, $X(\mathcal{P})$ will not be a V-variety, so Theorem 20 is invalid. In fact, if we insist upon defining the h-vector by **(25)**, then even the Dehn–Sommerville equations $h_i = h_{d-i}$ will fail. We should instead define

$$h_i = \tilde{\beta}_{2i}(X(\mathcal{P})) = \dim IH^{2i}(X(\mathcal{P})).$$

[It can be shown that $\tilde{\beta}_{2i+1}(X(\mathcal{P})) = 0$.] Theorem 21 then guarantees that the sequence $h(\mathcal{P}) = (h_0, h_1, \ldots, h_d)$ is symmetric and unimodal. An obvious question remains: How do we compute $h(\mathcal{P})$ in terms of the structure of $\mathcal{P}$? This computation was carried out independently by Bernstein, Khovanskii, MacPherson, and perhaps others. The description of $h(\mathcal{P})$ no longer depends only on the f-vector $f(\mathcal{P})$, but rather is given inductively in terms of $h(\mathcal{Q})$ for proper faces $\mathcal{Q}$ of $\mathcal{P}$. At least it is still true that $h(\mathcal{P})$ depends only on the combinatorial type of $\mathcal{P}$, and not on the embedding into $\mathbb{R}^d$. The combinatorial definition of $h(\mathcal{P})$ makes sense even if $\mathcal{P}$ is not rational, but in this case it is not even known whether $h_i \geq 0$. (It is still true, nonetheless, that $h_i = h_{d-i}$.) We refer the reader to [112] for the definition and basic properties of $h(\mathcal{P})$, as well as a host of open problems and conjectures.

Algebraic geometers have computed the Betti numbers and middle intersection Betti numbers for many other classes of algebraic varieties. We have not systematically investigated these results, but certainly we expect some to be of combinatorial interest. In particular, an intriguing generalization of Eulerian numbers arises in this way in [31] and [32]. For a further example, see [30].

Let us now consider an interesting variation of the preceding theory that arises when we have a group G (which we may assume is finite) acting on a complex projective variety X. In that case, as pointed out to me by S. Kleiman, there is a projective embedding $X \subseteq \mathbb{P}^N$ such that multiplication by a hyperplane section ω commutes with the action of G. It follows from **(16)** and Theorem 18 that if X satisfies the hard Lefschetz theorem and ρ is an irreducible character of G, then the two sequences

$$\begin{gathered}\beta_0(\rho), \beta_2(\rho), \ldots, \beta_{2n}(\rho) \\ \beta_1(\rho), \beta_3(\rho), \ldots, \beta_{2n-1}(\rho)\end{gathered} \tag{27}$$

are symmetric and unimodal, where $\beta_i(\rho)$ denotes the multiplicity of the irreducible representation ρ in the G-module $H^i(X)$. Let us mention two combinatorial applications.

THEOREM 22: Let $\mathcal{P}$ be a *centrally symmetric* (i.e., if $x \in \mathcal{P}$, then $-x \in \mathcal{P}$) simplicial d-polytope, with h-vector $(h_0, h_1, \ldots, h_d)$. Then the sequence

$$h_0 - \binom{d}{0}, h_1 - \binom{d}{1}, \ldots, h_d - \binom{d}{d}$$

is unimodal (and symmetric). In particular (since $h_0 = 1$), we have $h_i \geq \binom{d}{i}$.

Sketch of Proof: We can assume $\mathcal{P}$ satisfies conditions (a)–(c) in the proof of Theorem 20, so that the toric variety $X(\mathcal{P})$ is defined. In that case, the group $G = \mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{P}$ and therefore on $X(\mathcal{P})$ and $H^*(X(\mathcal{P}))$. Let ρ be the nontrivial irreducible representation of G. One can compute that $\beta_{2i}(\rho) = \frac{1}{2}(h_i - \binom{d}{i})$, and the proof follows from the unimodality of (**27**). □

Theorem 22 settles a conjecture of Björner (unpublished) that generalizes a conjecture of Bárány–Lovász [9, pp. 325–326]. Further details appear in [114].

For our second application, we assume familiarity with the theory of symmetric functions [70].

PROPOSITION 12: For each partition λ of a nonnegative integer, define a polynomial $P_\lambda(q)$ by

$$\sum_\lambda P_\lambda(q)s_\lambda = \frac{\sum_{k\geq 0} s_k}{1 - q\sum_{k\geq 2}(1 + q + \cdots + q^{k-2})s_k},$$

where s_λ and s_k denote Schur functions. If $\lambda \vdash n$ (i.e., $\Sigma\,\lambda_i = n$), then $P_\lambda(q)$ is symmetric about $\frac{1}{2}(n-1)$ (i.e., $q^{n-1}P_\lambda(1/q) = P_\lambda(q)$) and is unimodal.

Sketch of Proof: Here we choose P to be essentially the Coxeter complex of the symmetric group and G to be Σ_n. Take ρ to be the irreducible representation of Σ_n corresponding to the partition λ of n. The proof follows from the unimodality of (**27**) and a computation of deConcini–Procesi (unpublished). □

An elementary proof of Proposition 12 has subsequently been given by F. Brenti [17].

If f^λ denotes the number of standard Young tableaux of shape λ [70, p. 5], then it is not hard to show that

$$\sum_{\lambda\vdash n} f^\lambda P_\lambda(q) = q^{-1}A_n(q), \tag{28}$$

where $A_n(q)$ denotes an Eulerian polynomial [see Eq. (**6**)]. Hence Proposition 12, together with (**28**), provides a refinement of the unimodality of $A_n(q)$. It would be interesting to give a combinatorial interpretation of the coefficients of $P_\lambda(q)$. Along these lines, let us mention that if λ is the hook shape $(n + 1 - l, 1^l)$, then it can be shown that

$$P_{(n+1-l,1^l)}(q) = \binom{n-l}{n-2l} q^l(1 + q)^{n-2l}.$$

The following variation of Proposition 12 was conjectured by this writer and proved by F. Brenti [17], as part of a more general result that also includes Proposition 12.

PROPOSITION 13: For each partition λ, define a polynomial $R_\lambda(q)$ by

$$\sum_\lambda R_\lambda(q)s_\lambda = \frac{1}{1 - q\sum_{k\geq 2}(1 + q + \cdots + q^{k-2})s_k}.$$

Then $R_\lambda(q)$ is unimodal. [It is easy to show that if $\lambda \vdash n$, then $q^nR_\lambda(1/q) = R_\lambda(q)$.]

It can be shown that if

$$\sum_{\lambda\vdash n} f^{\lambda} R_n(q) = \sum_j d_n(j) q^j,$$

then $d_n(j)$ is equal to the number of permutations $\pi = a_1 a_2 \cdots a_n \in \Sigma_n$ satisfying:

(a) $a_i \neq i$ for all i (i.e., π is a *derangement*),
(b) $j = \#\{i : a_i > i\}$.

There is a further "refinement by partitions" of $A_n(q)$ (whose significance will not be explained here) that this writer conjectured had unimodal parts and that was proved by Brenti [16].

PROPOSITION 14: For each partition λ define polynomials $T_\lambda(q)$ by

$$\sum_\lambda T_\lambda(q) s_\lambda = \frac{1}{1 - q \sum_{n\geq 1} (1-q)^{n-1} s_n}.$$

Then the polynomials $T_\lambda(q)$ have real zeros. (They are easily seen to satisfy

$$q^{n+1} T_\lambda(1/q) = T_{\lambda'}(q)$$

$$\sum_{\lambda\vdash n} f^\lambda T_\lambda(q) = A_n(q).)$$

A FINAL WORD OF WARNING

Lest the reader think that every "reasonable" sequence turns out to be unimodal, let us mention an exception. In the late 1950s Motzkin (and later independently Welsh) conjectured that the f-vector $f(\mathcal{P}) = (f_0, \ldots, f_{d-1})$ (as defined in the last section) of any convex polytope $\mathcal{P}$ is unimodal. This conjecture was disproved by Björner [12], who showed that there exists a 24-dimensional simplicial convex polytope with 2.6×10^{11} vertices, such that $f_{14} > f_{15} < f_{16}$! (This was subsequently improved independently by Björner and Lee to a 20-dimensional simplicial polytope with around 4.2×10^{12} vertices with $f_{11} > f_{12} < f_{13}$.)

For a further example of a nonunimodal sequence, see [115].

NOTE ADDED IN PROOF: (1) A nice combinatorial interpretation of the q-Catalan numbers $K_n(q)$ of Example 4 appears in J. Fürlinger and J. Hofbauer. 1985. q-Catalan numbers. J. Comb. Theory A **40:** 248–264 (the case $\lambda = 0$).

(2) A combinatorial interpretation of the coefficients of the polynomial $P_\lambda(q)$ of Eq. (**28**) has been given by J. R. Stembridge, Eulerian numbers, tableaux, and the Betti numbers of a toric variety. Preprint dated May 1989.

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