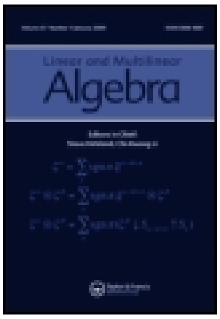

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The Stable Behavior of Some Characters of SL(n,$\mathbb{C}$)

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1. INTRODUCTION

Let $\mathfrak{g} = \mathrm{sl}(n, \mathbb{C})$ denote the Lie algebra of all $n \times n$ complex matrices of trace 0. Let $\mathrm{ad} : \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(\mathfrak{g})$ denote the adjoint representation of $\mathrm{SL}(n, \mathbb{C})$, defined by

$$(\mathrm{ad}\, X)(A) = XAX^{-1},$$

where $X \in \mathrm{SL}(n, \mathbb{C})$ and $A \in \mathfrak{g}$. Introduce two infinite sets $u = (u_1, u_2, \dots)$ and $v = (v_1, v_2, \dots)$ of variables, and the following function on $\mathrm{SL}(n, \mathbb{C})$ with values in the formal power series ring $\mathbb{Q}[[u, v]]$:

$$\det \prod_k \frac{1 - u_k ad\, X}{1 - v_k ad\, X}. \tag{1}$$

This function is a virtual character of $\mathrm{SL}(n, \mathbb{C})$, and a wide variety of problems in combinatorics and representation theory involve its decomposition into irreducibles. In particular, the q-Dyson conjecture for equal exponents and the computation of the generalized exponents of $\mathrm{SL}(n, \mathbb{C})$ are special cases of this problem, discussed in more detail in Sections 8 and 9. We will study the behavior of (1) as $n \to \infty$, and obtain what amounts to a decomposition into irreducibles in this limiting case.

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The virtual character (1) is a symmetric function of the eigenvalues of X. Our approach will be based on the theory of symmetric functions, which we now briefly review. Two basic references on symmetric functions are [14] and [20]. In general we will adhere to the notation and terminology of [14].

2. SYMMETRIC FUNCTIONS AND THE CHARACTERS OF $\mathrm{SL}(n,\mathbb{C})$

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a *partition*, i.e., a decreasing sequence $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ of nonnegative integers with only finitely many λ_i unequal to zero. If $\lambda_{n+1} = \lambda_{n+2} = \cdots = 0$, then we also write $\lambda = (\lambda_1, \ldots, \lambda_n)$. The number of nonzero λ_i is the *length* of λ, denoted $l(\lambda)$. If $m = \lambda_1 + \lambda_2 + \cdots$ then we write $\lambda \vdash m$ or $|\lambda| = m$ and say that λ is a *partition* of m. The *conjugate* partition $\lambda' = (\lambda_1', \lambda_2', \ldots)$ to λ has $\lambda_i - \lambda_{i+1}$ parts equal to i. We also let $m_k(\lambda)$ denote the number of parts of λ which are equal to k, so $|\lambda| = \sum k \cdot m_k(\lambda)$.

Let $\Lambda_n = \Lambda_n(x)$ denote the ring of all symmetric polynomials with rational coefficients in the variables $x = (x_1, \ldots, x_n)$, and let Ω_n denote Λ_n modulo the ideal generated by $x_1 x_2 \cdots x_n - 1$. A vector space basis for Ω_n consists of all *Schur functions* $s_\lambda(x) = s_\lambda(x_1, \ldots, x_n)$, where λ ranges over all partitions of length $\leqslant n-1$. For the definition and basic properties of Schur functions, see [14] or [20].

A *polynomial representation* of $\mathrm{SL}(n,\mathbb{C})$ of dimension N is a homomorphism $\phi: \mathrm{SL}(n,\mathbb{C}) \to \mathrm{GL}(N,\mathbb{C})$ such that for $X \in \mathrm{SL}(n,\mathbb{C})$, the entries of the matrix $\phi(X)$ are polynomial functions of the entries of X. For instance,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix} \tag{2}$$

is a polynomial representation of $\mathrm{SL}(2,\mathbb{C})$ of dimension 3. Every continuous representation $\phi: \mathrm{SL}(n,\mathbb{C}) \to \mathrm{GL}(N,\mathbb{C})$ is equivalent to a polynomial representation, so for all practical purposes it costs us nothing to consider only polynomial representations. If ϕ is a polynomial representation of $\mathrm{SL}(n,\mathbb{C})$, then there is a unique polynomial in Ω_n, denoted $\operatorname{char}\phi$ and called the *character* of ϕ, satisfying $(\operatorname{char}\phi)(x_1, \ldots, x_n) = \operatorname{tr}\phi(X)$ for any $X \in \mathrm{SL}(n,\mathbb{C})$ with eigenvalues

$x_1, \ldots, x_n$, where tr denotes trace. Note that since $x_1 \cdots x_n = 1$ for $X \in \mathrm{SL}(n, \mathbb{C})$ with eigenvalues $x_1, \ldots, x_n$, the value $(\mathrm{char}\,\phi)(x_1, \ldots, x_n)$ is well-defined. For instance, if ϕ is given by (2) then $\mathrm{char}\,\phi = x_1^2 + x_1x_2 + x_2^2$ (or equivalently, say, $\mathrm{char}\,\phi = x_1x_2^{-1} + 1 + x_1^{-1}x_2$, which is the same element of Ω_2).

A basic theorem (e.g., [21, Thm. 1.3]) on the representations of SL($n, \mathbb{C}$) states that the irreducible (polynomial) characters of SL($n, \mathbb{C}$) are precisely the Schur functions $s_\lambda(x) \in \Omega_n$, $l(\lambda) \leqslant n - 1$. Thus the problem of decomposing char ϕ into irreducible characters is equivalent to expanding $\mathrm{char}\,\phi$ as a linear combination of Schur functions in the ring Ω_n. More generally, any $f \in \Omega_n$ (or $f \in \Omega_n \otimes K$ for some extension field K of $\mathbb{Q}$) may be regarded as a virtual character (over $\mathbb{Q}$ or K) of SL($n, \mathbb{C}$), and expanding f as a linear combination of Schur functions (over $\mathbb{Q}$ or K) $s_\lambda(x) \in \Omega_n$ is equivalent to decomposing f into irreducibles.

Sometimes it will prove convenient to work with symmetric functions (= formal power series) in infinitely many variables $x = (x_1, x_2, \ldots)$. We let $\Lambda = \Lambda(x)$ denote the ring of all symmetric formal power series of bounded degree with rational coefficients in the variables x. Λ is the inverse limit of the rings Λ_n in the category of *graded* rings. The Schur functions $s_\lambda(x)$, for all partitions λ, form a vector space basis for $\Lambda(x)$. The completion $\hat{\Lambda}$ of Λ (with respect to the ideal of symmetric functions with zero constant term) consists of all symmetric formal power series with no restriction on the degree. $\hat{\Lambda}$ is the inverse limit of the rings Λ_n in the category of rings. For further information, see [14, Ch. I.2]. Let us remark that in [14] the elements of Λ_n, Λ, and $\hat{\Lambda}$ have *integer* coefficients, but we will find it convenient to allow rational coefficients from the start.

3. THE COEFFICIENTS $c_{\alpha\beta}(u; v)$

Introduce two sets $u = (u_1, u_2, \ldots)$ and $v = (v_1, v_2, \ldots)$ of variables. The function which sends $X \in \mathrm{SL}(n, \mathbb{C})$ to the element

$$\det \prod_k \frac{1 - u_k\, ad\, X}{1 - v_k\, ad\, X} \tag{3}$$

of $\mathbb{Q}[[u, v]]$ is a virtual character of SL($n, \mathbb{C}$) (over, say, the quotient field $\mathbb{Q}((u, v))$ of $\mathbb{Q}[[u, v]]$), i.e., it is a symmetric function of the eigenvalues of X. In fact, suppose X has eigenvalues $x_1, \ldots, x_n$.

Then (e.g., [21, eqn. (8)]) *ad X* has eigenvalues $x_i x_j^{-1}$ (once each for $i \neq j$) and 1 ($n-1$ times). Thus (3) can be rewritten

$$\prod_k \left[\prod_{i \neq j} \frac{1 - u_k x_i x_j^{-1}}{1 - v_k x_i x_j^{-1}} \right] \left(\frac{1 - u_k}{1 - v_k} \right)^{n-1} \tag{4}$$

Our main object here is to decompose (4) into irreducibles (equivalently, expand (4) in terms of $s_\lambda(x), l(\lambda) \leqslant n-1$, in the ring Ω_n) as $n \to \infty$. It will prove convenient to multiply (4) by a factor $\prod_k (1 - u_k)(1 - v_k)^{-1}$, so (4) now becomes

$$\prod_k \prod_{i,j} \frac{1 - u_k x_i x_j^{-1}}{1 - v_k x_i x_j^{-1}}.$$

Thus define formal power series $c_\lambda^n(u; v) \in \mathbb{Z}[[u, v]]$ by

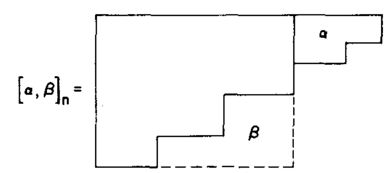
$$\left[\prod_k \frac{1 - u_k}{1 - v_k} \right] \det \prod_k \frac{1 - u_k \, ad\, X}{1 - v_k \, ad\, X} = \sum_\lambda c_\lambda^n(u; v) s_\lambda(x), \tag{5}$$

where λ ranges over all partitions of length $\leqslant n-1$.

In order to consider $c_\lambda^n(u; v)$ as $n \to \infty$, one must vary λ with n in a suitable way, or else the limit becomes zero or undefined. The correct way of passing to the limit was suggested by R. Gupta (in the somewhat less general setting of Section 8). Given any two partitions α and β of lengths k and l of the same integer m, and given $n \geqslant k + l$, define the partition

$$[\alpha, \beta]_n = \Big(\beta_1 + \alpha_1, \beta_1 + \alpha_2, \ldots, \beta_1 + \alpha_k, \underbrace{\beta_1, \ldots, \beta_1}_{n-k-l}, \beta_1 - \beta_l, \beta_1 - \beta_{l-1}, \ldots, \beta_1 - \beta_2 \Big),$$

of length $\leqslant n-1$.



It follows from Gupta's work (and will be shown below) that

$$\lim_{n\to\infty} c^n_{[\alpha,\beta]_n}(u; v)$$

exists as a formal power series, which we denote by $c_{\alpha\beta}(u; v)$. Our main goal here is formula for $c_{\alpha\beta}(u; v)$.

4. INTERNAL PRODUCTS AND SKEW SCHUR FUNCTIONS

In order to give our formula for $c_{\alpha\beta}(u; v)$, we first review some more background from the theory of symmetric functions. The irreducible characters χ^λ of the symmetric group S_m are indexed in a natural way by partitions λ of m. If $w \in S_m$, then define $\rho(w)$ to be the partition of m whose parts are the cycle lengths of w. For any partition λ of m of length l, define the power-sum symmetric function $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}$, where $p_n(x) = \sum_i x_i^n$. For brevity write $p_w = p_{\rho(w)}$. The Schur functions and power-sums are related by a famous result of Frobenius (e.g., [14, Ch. I.7]):

$$s_\lambda = \frac{1}{m!} \sum_{w \in S_m} \chi^\lambda(w) p_w .$$

Now let

$$\chi^\alpha \chi^\beta = \sum_\gamma g_{\alpha\beta\gamma} \chi^\gamma,$$

where each $g_{\alpha\beta\gamma}$ is a nonnegative integer. (It is an important open problem to obtain a nice combinatorial interpretation of $g_{\alpha\beta\gamma}$. For some recent work on this problem, see [3].) D. E. Littlewood [13], in order to incorporate the Kronecker product $\chi^\alpha \chi^\beta$ into the theory of symmetric functions, defined an associative, commutative product $f * g$ on symmetric functions by

$$s_\alpha * s_\beta = \sum_\gamma g_{\alpha\beta\gamma} s_\gamma , \tag{6}$$

and extending to all of Λ by bilinearity. We will call $f * g$ the *internal product* of f and g. (Littlewood uses the term "inner product." Since the product $f * g$ has nothing to do with the usual definition of inner product in linear algebra, we have followed a suggestion of I. G. Macdonald in calling it the internal product. Littlewood uses the notation $f \circ g$ for our $f * g$. Since we are adhering to the notation of [14], where $f \circ g$ denotes plethysm, we have introduced the new notation $f * g$). Note that $s_\alpha * s_m = s_\alpha$ and $s_\alpha s_{1^m} = s_{\alpha'}$, where α' denotes the conjugate partition to α. A table of internal products appears in [17] and [24, appendix].

In terms of the power-sums we have the expansion

$$s_\alpha * s_\beta = \frac{1}{m!} \sum_{w \in S_m} \chi^\alpha(w) \chi^\beta(w) p_w . \tag{7}$$

The following basic property of the internal product is due to Schur [19] (p. 69 of Dissertation; p. 65 of *GA*); see also [14, (7.9), p. 63].

PROPOSITION 4.1 We have

$$\prod_{i,j,k} (1 - x_i y_j v_k)^{-1} = \sum_{\alpha,\beta} s_\alpha * s_\beta(v) s_\alpha(x) s_\beta(y). \tag{8}$$

Now define a scalar product $\langle f, g \rangle$ in the ring Λ by letting the Schur functions form an orthonormal basis, i.e.,

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu} .$$

Given partitions λ, α, define a symmetric function $s_{\lambda/\alpha} \in \Lambda$, called a *skew Schur function*, by the rule

$$\langle s_{\lambda/\alpha}, s_\mu \rangle = \langle s_\lambda, s_\alpha s_\mu \rangle. \tag{9}$$

In other words, multiplication by s_α is adjoint to the linear transformation sending s_λ to $s_{\lambda/\alpha}$. It is not difficult to show that $s_{\lambda/\alpha} = 0$ unless $\alpha \leqslant \lambda$, i.e., $\alpha_i \leqslant \lambda_i$ for all i. For further information, see [14, Ch. I.5].

Let us remark (as brought to my attention by R. Gupta) that the Schur function $s_{[\alpha,\beta]_n}(x)$ was considered by D. E. Littlewood [10], who essentially showed that in the ring Ω_n we have

$$s_{[\alpha,\beta]_n}(x) = \sum_\lambda (-1)^{|\lambda|} s_{\alpha/\lambda}(x) s_{\beta/\lambda'}(1/x),$$

where $x = (x_1, \ldots, x_n)$ and $1/x = (1/x_1, \ldots, 1/x_n)$. For instance, the adjoint representation of $\mathrm{SL}(n, \mathbb{C})$ corresponds to the partition $[1,1]_n$, with character

$$\begin{aligned} s_{[1,1]_n}(x) &= s_1(x) s_1(1/x) - 1 \\ &= (x_1 + \cdots + x_n)(x_1^{-1} + \cdots + x_n^{-1}) - 1 \\ &= n - 1 + \sum_{i \neq j} x_i x_j^{-1}. \end{aligned}$$

5. A SYMMETRIC FUNCTION IDENTITY

Our evaluation of $c_{\alpha\beta}(u,v)$ will depend on a new identity involving symmetric functions.

THEOREM 5.1 *Let* $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$, $v = (v_1, v_2, \ldots)$ *be three infinite sets of variables. Then*

$$\prod_{i,j} \prod_{r \geq 1} \prod_{a_1, \ldots, a_r} (1 - x_i y_j v_{a_1} \cdots v_{a_r})^{-1}$$
$$= \left[\prod_{k \geq 1} (1 - p_k(v)) \right] \sum_{\lambda, \mu, \alpha} s_\lambda * s_\mu(v) s_{\lambda/\alpha}(x) s_{\mu/\alpha}(y). \quad (10)$$

Here $a_1, \ldots, a_r$ *range independently over the positive integers,* $p_k(v) = \sum v_i^k$, *and we set* $s_\lambda * s_\mu(v) = 0$ *if* $|\lambda| \neq |\mu|$.

Proof Let R denote the ring $\mathbb{Q}((v)) \otimes \Lambda(x) \otimes \Lambda(y)$, which should be regarded as a vector space of formal power series of bounded degree, symmetric in the x's and y's separately, over the field $\mathbb{Q}((v))$ (the quotient field of $\mathbb{Q}[[v]]$). Define a scalar product on R by letting the elements $s_\alpha(x)s_\beta(y)$ form an orthonormal basis. If $f \in R$, then following [14, Ex. 3, p. 43] let $D(f): R \to R$ denote the linear transformation which is adjoint to multiplication by f, i.e.,

$$\langle D(f)g, h \rangle = \langle g, fh \rangle.$$

Note that $D(f+g) = D(f) + D(g)$. Let

$$P(v) = \prod_{k \geq 1} (1 - p_k(v)).$$

The right-hand side of (10) is given by

$$P(v) \sum_\alpha D(s_\alpha(x)s_\alpha(y)) \sum_{\lambda,\mu} s_\lambda * s_\mu(v) s_\lambda(x) s_\mu(y)$$
$$= P(v) D\Big(\prod_{i,k} (1 - x_i y_j)^{-1} \Big) \prod_{i,j,k} (1 - x_i y_j v_k)^{-1},$$

by Proposition 4.1 and the "Cauchy identity" [14, (4.3), p. 33] [20, Cor. 7.2]

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\alpha s_\alpha(x) s_\alpha(y). \quad (11)$$

Thus writing F for the left-hand side of (10), we need to show that for all $f \in R$,

$$\langle F, f \rangle = \langle P(v) \prod (1 - x_i y_j v_k)^{-1}, f \cdot \prod (1 - x_i y_j)^{-1} \rangle.$$

It suffices to check this for all f forming a $\mathbb{Q}((v))$-basis for R, and we will choose $f = p_\alpha(x)p_\beta(y)$.

Now [14, (4.7), p. 35] [20, Prop. 4.2]

$$\langle p_\lambda, p_\mu \rangle = \begin{cases} 0, \lambda \neq \mu \\ 1^{m_1} m_1! 2^{m_2} m_2! \cdots, \lambda = \mu, \end{cases}$$

where $m_i = m_i(\lambda)$. It follows that for any power series $G = \sum_{n \geq 1} t_n / n$ we have

$$\exp G = \sum_\lambda \langle p_\lambda, p_\lambda \rangle^{-1} \prod_i t_{\lambda_i}. \tag{12}$$

Now

$$\begin{aligned} \log F &= \sum_{n \geq 1} \sum_{r \geq 1} \sum_{a_1, \ldots, a_r} \frac{1}{n} p_n(x) p_n(y) (v_{a_1} \cdots v_{a_r})^n \\ &= \sum_{n \geq 1} \frac{1}{n} p_n(x) p_n(y) p_n(v) (1 - p_n(v))^{-1}. \end{aligned}$$

Thus by (12),

$$F = \sum_\lambda \langle p_\lambda, p_\lambda \rangle^{-1} p_\lambda(x) p_\lambda(y) p_\lambda(v) \prod_i (1 - p_{\lambda_i}(v))^{-1}.$$

It follows that

$$\langle F, p_\alpha(x) p_\beta(y) \rangle = \delta_{\alpha\beta} \langle p_\alpha, p_\alpha \rangle p_\alpha(v) \prod_i (1 - p_{\alpha_i}(v))^{-1}. \tag{13}$$

Similarly

$$\begin{aligned} \prod (1 - x_i y_j)^{-1} &= \sum_\lambda \langle p_\lambda, p_\lambda \rangle^{-1} p_\lambda(x) p_\lambda(y) \\ \prod (1 - x_i y_j v_k)^{-1} &= \sum_\lambda \langle p_\lambda, p_\lambda \rangle^{-1} p_\lambda(x) p_\lambda(y) p_\lambda(v). \end{aligned}$$

Write $\mu \cup \lambda$ for the partition whose parts are those of λ and of μ, arranged in descending order [14, p. 5]. There follows

$$\begin{aligned} &P(v) \langle \prod (1 - x_i y_j v_k)^{-1}, p_\alpha(x) p_\beta(y) \prod (1 - x_i y_j)^{-1} \rangle \\ &= P(v) \Big\langle \sum_\lambda \langle p_\lambda, p_\lambda \rangle^{-1} p_\lambda(x) p_\lambda(y) p_\lambda(v), \\ &\qquad \sum_\mu \langle p_\mu, p_\mu \rangle^{-1} p_{\mu \cup \alpha}(x) p_{\mu \cup \beta}(y) \Big\rangle \\ &= \delta_{\alpha\beta} P(v) \sum_\mu \langle p_\mu, p_\mu \rangle^{-1} \langle p_{\mu \cup \alpha}, p_{\mu \cup \alpha} \rangle p_{\mu \cup \alpha}(v) \end{aligned} \tag{14}$$

Let $r_i = m_i(\mu)$ and $s_i = m_i(\alpha)$. Then (14) becomes

$$\delta_{\alpha\beta}P(v)\sum_{r_1,r_2,\ldots}\prod_{i\geq 1}\frac{p_i(v)^{r_i+s_i}(r_i+s_i)!\,i^{r_i+s_i}}{r_i!\,i^{r_i}}$$
$$=\delta_{\alpha\beta}P(v)\langle p_\alpha, p_\alpha\rangle p_\alpha(v)\sum_{r_1,r_2,\ldots}\prod_{i\geq 1}\binom{r_i+s_i}{r_i}p_i(v)^{r_i}$$
$$=\delta_{\alpha\beta}P(v)\langle p_\alpha, p_\alpha\rangle p_\alpha(v)\prod_{i\geq 1}(1-p_i(v))^{-s_i-1}$$
$$=\delta_{\alpha\beta}\langle p_\alpha, p_\alpha\rangle p_\alpha(v)\prod_{i}(1-p_{\alpha_i}(v))^{-1}. \tag{15}$$

Comparing (13) and (15) completes the proof. ■

6. A FORMULA FOR $c_{\alpha\beta}(u;v)$

We will first obtain a formula for the generating function

$$C(x,y):=\sum_{\alpha,\beta}c_{\alpha\beta}(u;v)s_\alpha(x)s_\beta(y)\in\mathbb{Q}[[u,v]]\otimes\hat{\Lambda}(x)\otimes\hat{\Lambda}(y). \tag{16}$$

It will be more convenient to work with

$$C_0(x,y)=\sum_{\alpha,\beta}c_{\alpha\beta}(0;v)s_\alpha(x)s_\beta(y),$$

and later to apply a standard trick to obtain $C(x,y)$ from $C_0(x,y)$. (Here $c_{\alpha\beta}(0;v)$ denotes the result of substituting $u_k=0$ in $c_{\alpha\beta}(u;v)$.)

LEMMA 6.1 *We have*

$$C_0(x,y)=\sum_{\lambda,\mu,\alpha}s_\lambda * s_\mu(v)s_{\lambda/\alpha}(x)s_{\mu/\alpha}(y). \tag{17}$$

Proof For the present we fix $n\geq 1$ and use only the variables $x=(x_1,\ldots,x_n)$. All our computations are done modulo the relation $x_1\cdots x_n=1$ (i.e., in the ring Ω_n) until further notice. Set $x_j=y_j=0$ for $j>n$ in (8), and set $y_j=x_j^{-1}$ for $1\leq j\leq n$. Comparing with (5) (when each $u_k=0$), we obtain

$$\sum_\nu c_\nu^n(0;v)s_\nu(x)=\sum_{\lambda,\mu}s_\lambda * s_\mu(v)s_\lambda(x)s_\mu(1/x). \tag{18}$$

Here ν,λ,μ range over all partitions of length $\leq n-1$. Let $\mu=(\mu_1,\mu_2,\ldots,\mu_{n-1})$, and define $\tilde{\mu}=(\mu_1-\mu_n,\mu_1-\mu_{n-1},\ldots,\mu_1-\mu_2)$. It is well-known (e.g., [21, (11)]) that

$$s_\mu(1/x) = s_{\tilde{\mu}}(x) \qquad (\text{in } \Omega_n).$$

The ring Ω_n has a scalar product for which $\{s_\lambda : l(\lambda) \leqslant n-1\}$ is an orthonormal basis. Take the scalar product of (18) with $s_\nu(x)$. We obtain

$$c_\nu^n(0; v) = \sum_{\lambda,\mu} s_\lambda * s_\mu(v)\langle s_\lambda s_{\tilde{\mu}}, s_\nu\rangle. \tag{19}$$

We now wish to invoke (9). Some care must be taken since we are working in Ω_n and not Λ (or Λ_n). Suppose that $l(\gamma)$, $l(\delta)$, $l(\epsilon) \leqslant n-1$, and that $|\epsilon| + rn = |\gamma| + |\delta|$ for some integer $r \geqslant 0$. Let $\bar{\epsilon} = (\epsilon_1 + r, \epsilon_2 + r, \ldots, \epsilon_{n-1} + r, r)$, so $|\bar{\epsilon}| = |\gamma| + |\delta|$. It is easily seen that

$$\langle s_\gamma s_\delta, s_\epsilon\rangle_{\Omega_n} = \langle s_\gamma s_\delta, s_{\bar{\epsilon}}\rangle_\Lambda,$$

where the subscript indicates the ring in which the scalar product is taken. Moreoever, if $|\gamma| + |\delta| - |\epsilon| \neq rn$ for some integer $r \geqslant 0$, then $\langle s_\gamma s_\delta, s_\epsilon\rangle_{\Omega_n} = 0$. For instance, $\langle s_2 s_{32}, s_1\rangle_{\Omega_3} = \langle s_2 s_{32}, s_{322}\rangle_\Lambda = 1$.

We can therefore replace $\langle s_\lambda s_{\tilde{\mu}}, s_\nu\rangle$ in (19) by $\langle s_\lambda s_{\tilde{\mu}}, s_{\bar{\nu}}\rangle_\Lambda$. Now let $\nu = [\alpha, \beta]_n$. When n is sufficiently large (namely, $n \geqslant l(\mu) + l(\alpha)$), the shape $\bar{\nu}/\tilde{\mu}$ splits up into a disjoint union of the two shapes α and $(\mu/\beta)^\vee$, where $^\vee$ denotes the operation of rotating 180°. Since $s_{\rho/\sigma} = \prod s_{\rho^i/\sigma^i}$ when the shape ρ/σ is a disjoint union of the ρ^i/σ^i (e.g., by [14, (5.7), p. 40]), we conclude $s_{[\alpha,\beta]_n/\tilde{\mu}} = s_\alpha s_{(\mu/\beta)^\vee}$. But for any μ and β we have $s_{(\mu/\beta)^\vee} = s_{\mu/\beta}$, as follows immediately from, e.g., [14, (5.11), p. 41] and the fact that skew Schur functions are symmetric. We conclude that $s_{[\alpha,\beta]_n/\tilde{\mu}} = s_\alpha s_{\mu/\beta}$ (for $n \geqslant l(\mu) + l(\alpha)$). Hence from (19),

$$\begin{aligned} c_{\alpha\beta}(0; v) &= \lim_{n\to\infty} c^n_{[\alpha,\beta]_n}(0; v) \\ &= \sum_{\lambda,\mu} s_\lambda * s_\mu(v)\langle s_\lambda, s_\alpha s_{\mu/\beta}\rangle \\ &= \sum_{\lambda,\mu} s_\lambda * s_\mu(v)\langle s_{\lambda/\alpha}, s_{\mu/\beta}\rangle, \end{aligned} \tag{20}$$

by (9), where the scalar product is now taken in Λ.

Multiply (20) by $s_\alpha(x)s_\beta(y)$ and sum on all α and β to obtain

$$\begin{aligned} C_0(x, y) &= \sum_{\lambda,\mu} \sum_{\alpha,\beta} s_\lambda * s_\mu(v)\langle s_{\lambda/\alpha}, s_{\mu/\beta}\rangle s_\alpha(x) s_\beta(y) \\ &= \sum_{\lambda,\mu} s_\lambda * s_\mu(v)\left\langle \sum_\alpha s_{\lambda/\alpha}(z)s_\alpha(x), \sum_\beta s_{\mu/\beta}(z)s_\beta(y)\right\rangle_z, \end{aligned}$$

where $\langle\ ,\ \rangle_z$ denotes that the scalar product is taken with respect to the variables z.

By, e.g. [14, p. 41], we have

$$\sum_{\alpha} s_{\lambda/\mu}(z)s_\alpha(x) = s_\lambda(z,x) = \sum_{\alpha} s_\alpha(z)s_{\lambda/\alpha}(x)$$

(since $s_\lambda(z,x)$ is symmetric in the x's and z's together), and similarly for μ/β and β. Hence

$$\begin{aligned} C_0(x,y) &= \sum_{\lambda,\mu} s_\lambda * s_\mu(v)\left\langle \sum_{\alpha} s_\alpha(z)s_{\lambda/\alpha}(x), \sum_{\beta} s_\beta(z)s_{\mu/\beta}(y)\right\rangle_z \\ &= \sum_{\lambda,\mu,\alpha} s_\lambda * s_\mu(v)s_{\lambda/\alpha}(x)s_{\mu/\alpha}(y), \end{aligned}$$

since $\langle s_\alpha(z), s_\beta(z)\rangle_z = \delta_{\alpha\beta}$. This completes the proof. ■

We can now give a formula for $c_{\alpha\beta}(0;v)$.

THEOREM 6.2 *We have*

$$c_{\alpha\beta}(0;v) = P(v)^{-1}s_\alpha * s_\beta\left(p_k \to \frac{p_k(v)}{1-p_k(v)}\right). \tag{21}$$

The notation indicates that we are to expand $s_\alpha * s_\beta$ in terms of the p_k's, as given explicitly by (7), and substitute $p_k(v)/(1-p_k(v))$ for p_k.

Equivalently, we have

$$c_{\alpha\beta}(0;v) = P(v)^{-1}s_\alpha * s_\beta(v_{a_1}\cdots v_{a_r} : r>0) \tag{22}$$

where the notation now indicates that we are to evaluate $s_\alpha * s_\beta$ at the variables $v_{a_1}\cdots v_{a_r}$ for all $r>0$, where $a_1,\ldots,a_r$ range independently over all indices of the v's. In the notation of plethysm [14, Ch. I.8], (22) *can be rewritten*

$$c_{\alpha\beta}(0;v) = P(v)^{-1}(s_\alpha * s_\beta)\circ\left(\frac{s_1(v)}{1-s_1(v)}\right). \tag{23}$$

Let us note that [14, p. 62]

$$\frac{s_1(v)}{1-s_1(v)} = \sum_{\lambda\neq\varnothing} \chi^\lambda(1)s_\lambda(v).$$

Proof of Theorem 6.2 Comparing (10), (16), (17), we see that $c_{\alpha\beta}(0;v)$ is equal to $P(v)^{-1}$ times the coefficient of $s_\alpha(x)s_\beta(y)$ when the left-hand side F of (10) is expanded as a linear combination of

$s_\lambda(x)s_\mu(y)$. If we substitute the products $v_{a_1}\cdots v_{a_r}$ for the variables v_k in (8), then we obtain

$$F = \sum_{\alpha,\,\beta} s_\alpha * s_\beta(v_{a_1}\cdots v_{a_r} : r > 0)s_\alpha(x)s_\beta(y),$$

and the proof follows. ■

In order to find a similar formula for $c_{\alpha\beta}(u;v)$, consider the algebra automorphism ω_u described in [14] acting on symmetric functions in u (regard all other variables as scalars commuting with ω_u). It follows, e.g. from [14, p. 14] that

$$\omega_u \prod_k (1+u_k t)^{-1} = \prod_k (1-u_k t).$$

Since ω_u is an algebra automorphism, we get

$$\omega_u\left[\prod_k \frac{1}{(1+u_k)(1-v_k)}\right]\det\prod_k \frac{1}{(1+u_k\operatorname{ad}X)(1-v_k\operatorname{ad}X)}$$
$$=\left[\prod_k \frac{1-u_k}{1-v_k}\right]\det\prod_k \frac{1-u_k\operatorname{ad}X}{1-v_k\operatorname{ad}X}.$$

Hence if we set $u=0$ in (5), next replace v by the two sets of variables $-u$ and v, and finally apply ω_u, then we obtain

$$\omega_u \sum_\lambda c_\lambda^n(0;-u,v)s_\lambda(x) = \sum_\lambda c_\lambda^n(u;v)s_\lambda(x).$$

Therefore $\omega_\mu c_\lambda^n(0;-u,v) = c_\lambda^n(u;v)$, so in particular

$$\omega_u c_{\alpha\beta}(0;-u,v) = c_{\alpha\beta}(u;v).$$

Now replace v by $-u,v$ in (21) and apply ω_u. Since $p_k(-u,v) = p_k(-u)+p_k(v)$ and $\omega_u p_k(-u) = -p_k(u)$ (e.g., [14, p. 16]), we obtain our main result:

THEOREM 6.3 *We have*

$$c_{\alpha\beta}(u;v) = P(u;v)^{-1}s_\alpha * s_\beta\left(p_k \to \frac{-p_k(u)+p_k(v)}{1+p_k(u)-p_k(v)}\right),$$

where

$$P(u;v) = \prod_{k\geqslant 1}(1+p_k(u)-p_k(v)).$$

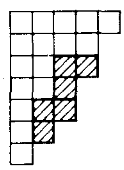
There is no direct generalization of (22) for $c_{\alpha\beta}(u;v)$, i.e., $c_{\alpha\beta}(u;v)$ cannot be written as $P(u;v)^{-1}$ times $s_\alpha * s_\beta$ in a certain set of variables. However, we do have a generalization of (23), viz.,

$$c_{\alpha\beta}(u;v) = P(u;v)^{-1}(s_\alpha * s_\beta) \circ \left(\frac{s_1(v)}{1-s_1(v)} - \frac{s_1(u)}{1-s_1(u)} \right).$$

A result essentially equivalent to Theorem 6.3 was proved independently by P. Hanlon [26]. He computed maximal weight vectors for certain virtual representations of $SL(n,\mathbb{C})$ as $n \to \infty$. When these weight vectors are used to decompose the character of the corresponding representation into irreducibles, a special case of Theorem 6.3 results. There is no difficulty, however, in extending Hanlon's work to obtain all of Theorem 6.3.

7. A LEMMA ON THE CHARACTERS OF THE SYMMETRIC GROUP

We next want to consider the form of the generating function $c_{\alpha\beta}(u;v)$ in certain special cases. We first need some information on when the character value $\chi^\lambda(\mu)$ is nonzero. We will use the formula of Littlewood and Richardson [11, p. 70] [14, ex. 5, p. 64] (equivalent to the "Murnagham–Nakayama formula") for computing the irreducible characters of the symmetric group S_m. We now review this result. Given a shape λ, a *border-strip* (or *skew hook* or *rim hook*) of λ is a nonvoid skew shape $\gamma = \lambda/\mu$ which is rookwise-connected and which contains no 2×2 block of squares. For instance, with λ as below a typical border strip is shaded. The total number of border strips of λ is $|\lambda|$.



A *border-strip tableau* (or *rim-hook tableau*) B of shape λ is a sequence $\lambda = \lambda^0 \supset \lambda^1 \supset \cdots \supset \lambda^j = \emptyset$ of shapes such that each skew-shape λ^i/λ^{i+1} $(0 \leqslant i \leqslant j-1)$ is a border strip of λ^i. We say that B has *type* $\nu = (\nu_1, \nu_2, \ldots, \nu_j)$ if $\nu_{i+1} = |\lambda^i/\lambda^{i+1}|$. Note that ν need not be a partition. We can denote a border-strip tableau by filling in the squares of λ^i/λ^{i+1} with $i+1$'s. For instance,

$$\begin{array}{cccccc} 6 & 6 & 5 & 3 & 3 & 3 \\ 6 & 4 & 2 & 2 & 2 & \\ 4 & 4 & 2 & 1 & & \\ 4 & 1 & 1 & 1 & & \\ 4 & 1 & & & & \\ 4 & 1 & & & & \\ 4 & & & & & \end{array}$$

is a border-strip tableau of shape $(6,5,4,4,2,2,1)$ and type $(6,4,3,7,1,3)$.

As in Section 4 let χ^λ denote the irreducible character of S_m corresponding to $\lambda \vdash m$. An element $w \in S_m$ has *type* $\nu = (\nu_1, \nu_2, \ldots, \nu_j)$ if its cycle lengths (in its disjoint cycle decomposition) are $\nu_1, \nu_2, \ldots, \nu_j$. Write $\nu = \nu(w)$. We do not require ν to be a partition. Each w has exactly one type which is a partition, which we denoted by $\rho(w)$ in Section 4. Since $\chi^\lambda(w)$ depends only on the type of w, we can write $\chi^\lambda(\nu)$ for $\chi^\lambda(w)$ when $\nu = \nu(w)$. We can now state the formula of Littlewood and Richardson.

THEOREM 7.1 *Let $\lambda \vdash m$ and let $\nu = (\nu_1, \ldots, \nu_l)$ be any sequence of positive integers summing to m. Then*

$$\chi^\lambda(\nu) = \sum_B (-1)^{ht\,B},$$

where B ranges over all border-strip tableaux of shape λ and type ν, and where $ht\ B$ is an integer whose value is irrelevant here.

COROLLARY 7.2 *Suppose there does not exist a border-strip tableau of shape λ and type ν. Then $\chi^\lambda(\nu) = 0$.*

LEMMA 7.3 *Suppose $\gamma = \lambda/\mu$ is a border-strip of λ with $|\gamma| = pk$, where p and k are positive integers. Then there exists a sequence $\lambda = \lambda^0 \supset \lambda^1 \supset \cdots \supset \lambda^k = \mu$ of shapes such that* (a) *λ^i/λ^{i+1} is a border strip of λ^i, $0 \leqslant i \leqslant k-1$, and* (b) $|\lambda^i/\lambda^{i+1}| = p$, $0 \leqslant i \leqslant k-1$.

Proof (in collaboration with A. Garsia) Let the successive squares of γ, reading from left-to-right and bottom-to-top, be $s_1, s_2, \ldots, s_{pk}$. By induction on k it suffices to find a border-strip λ/λ^1 of λ contained in γ, such that $\lambda/\lambda^1| = p$ and such that when we remove λ/λ^1 from γ, the connected components thus formed (either one or two of them) will have a number of squares divisible by p.

Define $\gamma_i = \{s_{ip-p+1}, s_{ip-p+2}, \ldots, s_{ip}\}$, $1 \leqslant i \leqslant k$. Let r be the least positive integer for which s_{ip+1} does not lie to the right of s_{ip}. The integer r exists since s_{kp+1} is undefined and hence doesn't lie to the right of s_{kp}. Since r is minimal, s_{ip-p+1} lies to the right of s_{ip-p}. Hence γ_r is a border-strip with the desired properties ■

Now recall [14, p. 2] that the *diagram* of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is the subset of $\mathbb{Z}^2$ given by

$$\{(i, j): 1 \leqslant i \leqslant l(\lambda), 1 \leqslant j \leqslant \lambda_i\}.$$

If $x = (i, j)$ is an element of the diagram of λ (written $x \in \lambda$), then the *hook-length* of λ at x is defined [14, p. 9] to be

$$h(x) = h(i, j) = \lambda_i + \lambda_j' - i - j + 1.$$

Note that if $\gamma = \lambda/\mu$ is a border strip of λ with its left-most square(s) in column j and topmost square(s) in row i, then $|\gamma| = h(i, j)$.

Define also the *hook polynomial* [14, p. 28]

$$H_\lambda(q) = \prod_{x \in \lambda} (1 - q^{h(x)}).$$

It is well-known (e.g., [14, Ex. 2, p. 28]) that

$$s_\lambda(q, q^2, \ldots) = q^{\hat{n}(\lambda)} H_\lambda(q)^{-1}, \tag{24}$$

where

$$\hat{n}(\lambda) = \sum i\lambda_i = \sum \binom{\lambda_i' + 1}{2}. \tag{25}$$

LEMMA 7.4 *Fix an integer $p > 0$. Given an integer j, define*

$$j* = \begin{cases} j/p, & \text{if } \; p \mid j \\ 0, & \text{if } \; p \nmid j. \end{cases}$$

Let $h_p(\lambda)$ be the number of hook lengths of λ divisible by p. If $\chi^\lambda(\nu) \neq 0$, then $\sum_i \nu_i^* \leqslant h_p(\lambda)$.

Proof Relabel the ν_i's so that $\nu_1, \ldots, \nu_j$ are divisible by p and $\nu_{j+1}, \ldots, \nu_l$ aren't, where $l = l(\nu)$. If $\chi^\lambda(\nu) \neq 0$, then by Corollary 7.2 there exists a border-strip tableau of shape λ and type ν. By the previous lemma, there exists a border strip tableau of shape λ and type $(p, p, \ldots, p, \nu_{j+1}, \ldots, \nu_l)$, where the number of p's is $\sum_i \nu_i^*$. It is well-known and easy to see that when a border strip of size p is removed from λ, the number of hook lengths divisible by p decreases by exactly one. Hence to remove $\sum_i \nu_i^*$ successive border strips of size p from λ, we must have $\sum_i \nu_i^* \leqslant h_p(\lambda)$. ■

COROLLARY 7.5 *Let $l = l(\nu)$. If $\chi^\lambda(\nu) \neq 0$, then $H_\lambda(q)$ is divisible by* $\prod_{i=1}^{l}(1 - q^{\nu_i})$.

Proof Fix $p > 0$. The multiplicity of a primitive pth root of unity as a root of $\prod (1 - q^{\nu_i})$ is card $\{i: p \mid \nu_i\}$, and of $H_\lambda(q)$ is $h_p(\lambda)$. Clearly

$$\operatorname{card}\{i: p \mid \nu_i\} \leqslant \sum_i \nu_i^*,$$

so the proof follows from the previous lemma. ∎

8. GENERALIZED EXPONENTS

This section and the next will be devoted to two special cases of the power series $c_{\alpha\beta}(u; v)$. Let $\mathfrak{g} = \mathrm{sl}(n, \mathbb{C})$ and $G = \mathrm{SL}(n, \mathbb{C})$. The adjoint action of $\mathrm{SL}(n, \mathbb{C})$ extends to an action on the symmetric algebra $S(\mathfrak{g}) = \coprod_{k \geqslant 0} S^k(\mathfrak{g})$, where S^k denotes the kth symmetric power and $\coprod$ the direct sum of vector spaces. It is well-known that the ring

$$J = S(\mathfrak{g})^G = \{f \in S(\mathfrak{g}): X \cdot f = f \text{ for all } X \in G\}$$

of invariants of this action is a polynomial ring in $n - 1$ variables $f_2, \ldots, f_n$, where $f_i \in S^i(\mathfrak{g})^G$. Namely, for $A \in \mathfrak{g}$, $f_i(A)$ is the coefficient of t^{n-i} in the characteristic polynomial $\det(A - tI)$ of A.

By a theorem of Kostant [9, Thm. 0.2], we can write

$$S(\mathfrak{g}) = J \otimes H, \tag{26}$$

were $H = \coprod H^k$ is a graded G-invariant subspace of $S(\mathfrak{g})$ (so $H^k = H \cap S^k(\mathfrak{g})$). Let H_λ denote the isotypic component of H corresponding to λ, i.e., the sum of all subspaces of H which afford the character $s_\lambda(x)$. We may then decompose H_λ into isomorphic irreducible subspaces $H_{\lambda,i}$,

$$H_\lambda = \coprod_i H_{\lambda,i},$$

where each $H_{\lambda,i}$ can be chosen to be *homogeneous*, i.e., to lie in $S^{d_i}(\mathfrak{g})$ (or H^{d_i}) for some d_i. The numbers d_i are called the *generalized exponents* of λ. Define

$$G_\lambda(q) = \sum_i q^{d_i},$$

the generating function for the generalized exponents of λ. Kostant also shows in [9, Thm. 0.11] (when applied to $\mathrm{SL}(n, \mathbb{C})$) that $G_\lambda(1)$ is equal to the dimension of the zero-weight space of the representation

λ and is therefore finite. Thus $G_\lambda(q)$ is a polynomial in q, which vanishes unless $|\lambda|$ is a multiple of n. In terms of generating functions it is easy to see from the above discussion that

$$\det(1 - q \cdot \operatorname{ad} X)^{-1} = \frac{1}{(1-q^2)\cdots(1-q^n)} \sum_\lambda G_\lambda(q) s_\lambda(x_1, \ldots, x_n)$$
$$(\text{modulo } x_1 \cdots x_n - 1). \quad (27)$$

Ranee Gupta conceived the idea of studying $G_{[\alpha,\beta]_n}(q)$ as $n \to \infty$, and showed that

$$G_{\alpha\beta}(q) := \lim_{n\to\infty} G_{[\alpha,\beta]_n}(q)$$

exists as a formal power series. She conjectured that $G_{\alpha\beta}(q)$ is a rational function $P_{\alpha\beta}(q)H_\alpha(q)^{-1}$, where $P_{\alpha\beta}(q)$ is a polynomial with nonnegative integral coefficients satisfying $P_{\alpha\beta}(1) = \chi^\beta(1)$, the number of standard Young tableaux of shape β [14, p. 5], and where $H_\alpha(q)$ is as in the previous section. Later she and I conjectured on the basis of numerical evidence that $G_{\alpha\beta}(q) = s_\alpha * s_\beta(q, q^2, \ldots)$. We will indicate how all these conjectures follow immediately from our previous discussion, except for the nonnegativity of the coefficients of $P_{\alpha\beta}(q)$, which remains open.

PROPOSITION 8.1 *We have*

$$G_{\alpha\beta}(q) = s_\alpha * s_\beta(q, q^2, \ldots).$$

Proof Comparing (5) (in the case $v_1 = q$, $v_2 = v_3 = \cdots = u_1 = u_2 = \cdots = 0$) with (27), we see that

$$G_{\alpha\beta}(q) = \left[\prod_{k\geq 1}(1-q^k)\right] c_{\alpha\beta}(0; q).$$

Thus by Theorem 6.2 (or 6.3),

$$\begin{aligned} G_{\alpha\beta}(q) &= s_\alpha * s_\beta\left(p_k \to \frac{q^k}{1-q^k}\right) \\ &= s_\alpha * s_\beta\left(p_k \to p_k(q, q^2, \ldots)\right) \\ &= s_\alpha * s_\beta(q, q^2, \ldots). \end{aligned}$$ ■

Note that by (6) and (24) there follows

$$G_{\alpha\beta}(q) = \sum_\gamma g_{\alpha\beta\gamma} q^{\hat{n}(\gamma)} H_\gamma(q)^{-1}.$$

Similarly from (7) and the fact that

$p_k(q, q^2, \ldots) = q^k/(1-q^k)$ we have

$$G_{\alpha\beta}(q) = \frac{q^m}{m!} \sum_{w \in S_m} \chi^\alpha(w)\chi^\beta(w) \prod_i (1 - q^{\rho_i(w)})^{-1}, \quad (28)$$

where $\rho(w) = (\rho_1(w), \rho_2(w), \ldots)$.

PROPOSITION 8.2 (i) *There is a polynomial $P_{\alpha\beta}(q) \in \mathbb{Z}[q]$ for which*

$$G_{\alpha\beta}(q) = P_{\alpha\beta}(q) H_\alpha(q)^{-1}.$$

(ii) $P_{\alpha\beta}(1) = \chi^\beta(1)$, *the number of standard Young tableaux of shape β (often denoted by f^β).*

(iii) $P_{\alpha'\beta'}(q) = P_{\alpha\beta}(q)$

(iv) $q^{m+h(\alpha)} P_{\alpha\beta}(1/q) = P_{\alpha\beta'}(q)$, *where* $|\alpha| = |\beta| = m$ *and* $h(\alpha) = \sum_{x \in \alpha} h(x)$.

(v) $\deg P_{\alpha\beta}(q) \leqslant h(\alpha)$, *with equality if and only if* $\beta = \alpha'$ *(in which case $P_{\alpha\beta}(q)$ is monic).*

(vi) $P_{\alpha\beta}(q)$ *is divisible by* q^m, *and the coefficient of* q^m *is the Kronecker delta* $\delta_{\alpha\beta}$.

(vii) $P_{\beta\alpha}(q) = P_{\alpha\beta}(q) H_\beta(q) H_\alpha(q)^{-1}$.

(viii) *Let β consist of the single part m, and write $P_{\alpha m}(q)$ for $P_{\alpha\beta}(q)$. Then $P_{\alpha m}(q) = q^{\hat{n}(\alpha)}$, where $\hat{n}(\alpha)$ is defined by* (25).

Proof (i) By Corollary 7.5, every term of (28) for which $\chi^\alpha(w) \neq 0$ has the property that its denominator $\prod(1 - q^{\rho_i(w)})$ divides $H_\alpha(q)$. Hence $G_{\alpha\beta}(q)$ itself has the common denominator $H_\alpha(q)$, and it is easily seen that the numerator has integer coefficients (e.g. because $G_{\alpha\beta}(q)$ has integer coefficients).

(ii) One of several ways to prove this result is to multiply (28) by $H_\alpha(q)$ and set $q = 1$. Since $H_\alpha(q)$ is divisible by $(1-q)^m$, every term of the right-hand side of (28) will vanish except for $w = 1$, where we obtain $m!^{-1} f^\alpha f^\beta \prod_{x \in \alpha} h(x)$. The well-known hook-length formula of Frame–Robinson–Thrall asserts that $f^\alpha = m!/\prod_{x \in \alpha} h(x)$, so the proof follows.

(iii) Since $H_\alpha(q) = H_{\alpha'}(q)$, we need to show $G_{\alpha\beta}(q) = G_{\alpha'\beta'}(q)$. This is immediate from (28), since $\chi^{\alpha'}(w)\chi^{\beta'}(w) = (\operatorname{sgn} w)^2 \chi^\alpha(w)\chi^\beta(w) = \chi^\alpha(w)\chi^\beta(w)$.

(iv) From (28) we have

$$G_{\alpha\beta}(1/q) = \frac{1}{m!} \sum_w \chi^\alpha(w)\chi^\beta(w)(-1)^{c(w)} \prod_i (1 - q^{\rho_i(w)})^{-1},$$

where $c(w)$ denotes the number of cycles of w. But $(-1)^{c(w)} =$

$(-1)^m(\operatorname{sgn} w)$ and $(\operatorname{sgn} w)\chi^{\beta}(w) = \chi^{\beta'}(w)$ (e.g., [14, Ex. 2, p. 63]), so

$$G_{\alpha\beta}(1/q) = \frac{(-1)^m}{m!} \sum_w \chi^{\alpha}(w)\chi^{\beta'}(w) \prod_i \left(1 - q^{\rho_i(w)}\right)^{-1}$$
$$= (-1)^m q^{-m} G_{\alpha\beta'}(q).$$

The proof follows from (i).

(v) Since $H_\alpha(q)\prod_i(1 - q^{\rho_i(w)})^{-1}$ is a polynomial of degree $h(\alpha) - m$, it is evident from (28) and the definition of $P_{\alpha\beta}(q)$ that $\deg P_{\alpha\beta}(q) \leqslant h(\alpha)$. Now the coefficient of $q^{h(\alpha)}$ in (28) is

$$\frac{1}{m!} \sum_w \chi^{\alpha}(w)\chi^{\beta}(w)(-1)^{m+c(w)} = \frac{1}{m!} \sum_w \chi^{\alpha}(w)\chi^{\beta'}(w) = \delta_{\alpha\beta'},$$

and the proof follows.

(vi) Follows from (iv) and (v), or by dividing (28) by q^m and setting $q = 0$.

(vii) Follows from the definition $G_{\alpha\beta}(q) = P_{\alpha\beta}(q)H_\alpha(q)^{-1}$ and the fact (evident e.g. from (28)) that $G_{\alpha\beta}(q) = G_{\beta\alpha}(q)$.

(viii) We have $s_\alpha * s_m = s_\alpha$ (since χ^m is the trivial character of S_m), so $G_{\alpha m}(q) = s_\alpha(q, q^2, \ldots)$, and the proof follows from (24). ■

To conclude this section we state explicitly the conjecture mentioned above.

CONJECTURE 8.3 (Gupta–Stanley) *The coefficients of $P_{\alpha\beta}(q)$ are nonnegative.*

Alain Lascoux has proved the above conjecture when β is a "hook", i.e., a partition of the form $(m - k, 1^k)$ for some $0 \leqslant k \leqslant m - 1$. He has shown that in this case $P_{\alpha\beta}(q)$ is the coefficient of t^k in the product

$$q \prod_{\substack{(i,j)\in\alpha \\ (i,j)\neq(1,1)}} (q^i + tq^j).$$

9. APPLICATION TO THE Q-DYSON CONJECTURE

Let $a_1, \ldots, a_n$ be nonnegative integers. In 1962 Dyson [2] conjectured that when the product

$$\prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - x_i x_j^{-1}\right)^{a_i}$$

is expanded as a Laurent polynomial in the variables $x_1, \ldots, x_n$, then the constant term is equal to the multinomial coefficient $(a_1 + \cdots + a_n)!/a_1! \cdots a_n!$. This conjecture was proved in 1962 by Gunson [5] and Wilson [23], and in 1970 an exceptionally elegant proof was given by Good [4].

In 1975 G. Andrews [1, p. 216] formulated a "q-analogue" of the Dyson conjecture, which reduces to the original conjecture when $q = 1$. Write $(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$, so $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$.

Q-DYSON CONJECTURE 9.1 *When the product*

$$\prod_{1 \leqslant i < j \leqslant n} \left(q \cdot x_i x_j^{-1}\right)_{a_i} \left(x_j x_i^{-1}\right)_{a_j} \tag{29}$$

is expanded as a Laurent polynomial in the variables $x_1, \ldots, x_n$, then the constant term is equal to the q-multinomial coefficient

$$(q)_{a_1 + \cdots + a_n}/(q)_{a_1} \cdots (q)_{a_n}.$$

This conjecture was proved for $n \leqslant 3$ by Andrews [1, pp. 216–217] and for $n = 4$ by Kadell [7]. It was also proved for $a_1 = \cdots = a_n = 1$, 2, or ∞ by Macdonald [15], who formulated a far-reaching generalization. See also [6] and [16]. Recently D. Bressoud and D. Zeilberger [25] have found a proof in general, while P. Hanlon has made considerable progress on many of the conjectures in [15]. We will obtain what may be regarded as the entire expansion of (29) in the case $a_1 = a_2 = \cdots = l$ in the limit $n \to \infty$. In the form stated above, the expansion of (29) as $n \to \infty$ becomes meaningless. However, (29) can be rewritten to make sense in this limit. Here we will rewrite (29) in terms of representation theory in a form due to Macdonald [15, Conj. 3.1′].

Q-DYSON CONJECTURE 9.2 FOR $a_1 = \cdots = a_n = l$ (reformulated). *The multiplicity of the trivial character of $SL(n, \mathbb{C})$ in the virtual character*

$$\det(1 - q \cdot \operatorname{ad} X)(1 - q^2 \cdot \operatorname{ad} X) \cdots (1 - q^{l-1} \cdot \operatorname{ad} X) \tag{30}$$

is equal to

$$\prod_{i=1}^{n-1} \prod_{j=1}^{l-1} (1 - q^{li+j}). \tag{31}$$

PROPOSITION 9.3 *Let α, $\beta \vdash m$. The coefficient of the character $s_{[\alpha,\beta]_n}$ in the expansion of the virtual character* (30) *of $SL(n, \mathbb{C})$ approaches, as*

$n\to\infty$, the value

$$\left[\prod_{k=1}^{l-1}(1-q^k)^{-1}\right]c_{\alpha\beta}(q,q^2,\ldots,q^{l-1};0)$$

$$=\left[\prod_{i\geqslant 1}\prod_{j=1}^{l-1}(1-q^{li+j})\right]s_\alpha * s_\beta\left(p_k\to\frac{-q^k(1-q^{(l-1)k})}{1-q^{lk}}\right) \tag{32}$$

Proof Compare (30) with (5), and apply Theorem 6.3 in the case

$$u_i=\begin{cases} q^i, & 1\leqslant i\leqslant l-1\\ 0, & i\geqslant l\end{cases}$$

$$v_i=0. \quad ■$$

If we let $\alpha=\beta=\emptyset$ (the void partition) above, then $s_\emptyset * s_\emptyset = s_\emptyset = 1$, so we obtain:

COROLLARY 9.4 (q-Dyson conjecture for $a_i=l-1$ and $n=\infty$) *The coefficient of the trivial character $s_\emptyset$ in (30) approaches, as $n\to\infty$, the value*

$$\prod_{i\geqslant 1}\prod_{j=1}^{l-1}(1-q^{li+j}). \quad ■$$

We now wish to obtain information on the form of the generating function $c_{\alpha\beta}(q,q^2,\ldots,q^{l-1};0)$ analogous to Proposition 8.2. Regarding l as fixed, define

$$D_{\alpha\beta}(q)=s_\alpha * s_\beta\left(p_k\to\frac{-q^k(1-q^{(l-1)k})}{1-q^{lk}}\right), \tag{33}$$

so by (32) we have

$$c_{\alpha\beta}(q,q^2,\ldots,q^{l-1};0)=\left[\prod_{k\geqslant 1}\frac{1-q^k}{1-q^{kl}}\right]D_{\alpha\beta}(q).$$

PROPOSITION 9.5 (i) *There is a polynomial $L_{\alpha\beta}(q)\in\mathbb{Z}[q]$ (which depends on l) for which*

$$D_{\alpha\beta}(q)=L_{\alpha\beta}(q)H_\alpha(q^l)^{-1}.$$

(ii) *Let ζ be an l-th root of unity. Then $L_{\alpha\beta}(\zeta)=(1-\zeta)^m f^\beta$.*

(iii) $L_{\alpha'\beta'}(q)=L_{\alpha\beta}(q)$

(iv) $q^{m+l\cdot h(\alpha)}L_{\alpha\beta}(1/q)=L_{\alpha\beta}(q)$

(v) $\deg L_{\alpha\beta}(q) \leqslant l \cdot h(\alpha)$, with equality if and only if $\alpha = \beta'$ (in which case $L_{\alpha\beta}(q)$ is monic)

(vi) $L_{\alpha\beta}(q)$ is divisible by $q^m(1-q)^{m-1}(1-q^{l-1})$, and the coefficient of q^m is $(-1)^m\delta_{\alpha\beta'}$.

(vii) $L_{\beta\alpha}(q) = L_{\alpha\beta}(q)H_\beta(q^l)H_\alpha(q^l)^{-1}$.

(viii) Let β consist of the single part m, and write $L_{\alpha m}(q)$ for $L_{\alpha\beta}(q)$. Then

$$L_{\alpha m}(q) = \prod_{(i,j)\in\alpha}\left(q^{il} - q^{(j-1)l+1}\right). \tag{34}$$

Example of (viii): Let $\alpha = (3,2,2)$. The factors $q^{il} - q^{(j-1)l+1}$ are given by

$$\begin{array}{lll} q^l - q & q^l - q^{l+1} & q^l - q^{2l+1} \\ q^{2l} - q & q^{2l} - q^{l+1} & \\ q^{3l} - q & q^{3l} - q^{l+1} & \end{array}$$

Thus

$$L_{\alpha 7}(q) = -q^{5+4l}(1-q^{l+1})(1-q)(1-q^{l-1})^2(1-q^{2l-1})^2(1-q^{3l-1}).$$

Proof (i) By (7), (32), and the fact that $(-1)^{c(w)} = (-1)^m(\operatorname{sgn} w)$, we have

$$D_{\alpha\beta}(q) = \frac{(-1)^m q^m}{m!}\sum_{w\in S_m}\chi^\alpha(w)\chi^{\beta'}(w)\prod_i\left(\frac{1-q^{(l-1)\rho_i(w)}}{1-q^{l\rho_i(w)}}\right), \tag{35}$$

where $\rho(w) = (\rho_1(w), \rho_2(w), \ldots)$. The proof now follows from Corollary 7.5 just as did Proposition 8.2(i).

(ii) Multiply (35) by $H_\alpha(q^l)$ and set $q = \zeta$. Now $H_\alpha(q^l)$ is divisible by $(1-q^l)^m$, while the multiplicity of ζ as a root of $\prod_i(1-q^{l\rho_i(w)})$ is equal to the number of cycles of w. Hence every term vanishes except for $w = 1$; giving

$$\begin{aligned} L_{\alpha\beta}(\zeta) &= \frac{(-1)^m\zeta^m}{m!} f^\alpha f^\beta\left[(1-\zeta^{l-1})^m\right]\frac{H_\alpha(q^l)}{(1-q^l)^m}\bigg|_{q=\zeta} \\ &= m!^{-1}(1-\zeta)^m f^\alpha f^\beta\prod_{x\in\alpha}h(x) \\ &= (1-\zeta)^m f^\beta. \end{aligned}$$

(iii) Analogous to Proposition 8.2 (iii).
(iv) Analogous to Proposition 8.2 (iv).
(v) Analogous to Proposition 8.2 (v).
(vi) For each $w \in S_m$, the product $\prod_i(1 - q^{(l-1)\rho_i(w)})$ is divisible by $1 - q^{l-1}$. Since $H_\alpha(q^l)\prod_i(1 - q^{l\rho_i(w)})^{-1}$ is a polynomial, it follows that $L_{\alpha\beta}(q)$ is divisible by $1 - q^{l-1}$. Since 1 is not a pole of the product in (35) but is a pole of $H_\alpha(q^l)$ of multiplicity m, it follows that $L_{\alpha\beta}(q)$ is divisible by $(1 - q)^m$. Finally it is obvious from (35) that $L_{\alpha\beta}(q)$ is divisible by q^m, and the coefficient of q^m is computed as in Proposition 8.2(vi).
(vii) Analogous to Proposition 8.2 (vii).
(viii) Let $f(x) = \sum_{n \geqslant 0} b_n x^n$ be a power series. Littlewood [11, pp. 99ff] discusses the *Schur function* (or "S-function") *of the series* f, corresponding to the partition λ, which we denote by s_λ^f. It follows easily from Littlewood's discussion that if $f(x)$ has the form

$$f(x) = \prod \frac{1 - c_i x}{1 - d_i x},$$

then

$$s_\lambda^f = s_\lambda(p_k \to p_k(d) - p_k(c)).$$

(In the λ-ring notation of [14, pp. 26–27], s_λ^f corresponds to the operation $S^\lambda(\sum d_i - \sum c_i)$.)

Now $s_\alpha * s_m = s_\alpha$, so

$$L_{\alpha m}(q) = H_\alpha(q) s_\alpha\left(p_k \to \frac{-q^k(1 - q^{(l-1)k})}{1 - q^{lk}}\right)$$
$$= H_\alpha(q) s_\alpha^f,$$

where

$$f(x) = \frac{(1 - qx)(1 - q^{l+1}x)(1 - q^{2l+1}x)\cdots}{(1 - q^l x)(1 - q^{2l}x)(1 - q^{3l}x)\cdots}$$

The proof follows from [11, Thm. II, p. 125] by substituting $z \to q$, $w \to q^l$, $q \to q^l$, and using the fact that Littlewood's product

$$\frac{\prod(1 - q^{\lambda_r - \lambda_s - r + s})}{\prod[\lambda_r + p - r]!}$$

is equal to $H_\lambda(q)^{-1}$. ∎

Remark Equation (35) shows that we may rewrite (33) as

$$D_{\alpha\beta}(q) = (-1)^m s_\alpha * s_{\beta'}\left(p_k \to \frac{q^k(1-q^{(l-1)k})}{1-q^{lk}}\right).$$

Remark on $\Lambda(\mathfrak{g})$ Consider now the special case of Proposition 9.3 obtained by setting $l = 2$ and $q \to -q$, so (30) becomes $\det(1 + q \cdot \operatorname{ad} X)$. Letting $\mathfrak{g} = \mathrm{sl}(n, \mathbb{C})$ as in Section 8, it follows that the coefficient of q^k in $\det(1 + q \cdot \operatorname{ad} X)$ is the character of $G = \mathrm{SL}(n, \mathbb{C})$ acting on $\Lambda^k(\mathfrak{G})$, the kth exterior power of the adjoint representation of G. Then (31) becomes $(1 + q^3)(1 + q^5) \cdots (1 + q^{2n-1})$, the well-known Hilbert series for the invariant subalgebra $\Lambda(\mathfrak{g})^G$. (See, e.g., [8, Cor. 8.7] [12] [18] [21, (17)], [22, p. 233].) There is no analogue of (26) known which would describe the decomposition of $\Lambda^k(\mathfrak{g})$ into irreducibles. But Proposition 9.3 essentially gives such a decomposition as $n \to \infty$, viz., the multiplicity of $s_{[\alpha, \beta]_n}$ in the character of the representation $\Lambda^k(\mathfrak{g})$ of $\mathrm{SL}(n, \mathbb{C})$ approaches, as $n \to \infty$, the coefficient of q^k in the power series

$$\begin{aligned}
&(1+q)^{-1} c_{\alpha\beta}(-q; 0) \\
&= \left[\prod_{i \geqslant 1} (1 + q^{2i+1})\right] s_\alpha * s_\beta \left(p_j \to \frac{(-1)^{j+1} q^j}{1 + (-1)^j q^j}\right) \\
&= \left[\prod_{i \geqslant 1} (1 + q^{2i+1})\right] s_\alpha * s_{\beta'} \left(p_j \to \frac{q^j}{1 + (-1)^j q^j}\right). \qquad (36)
\end{aligned}$$

Of course Proposition 9.5, in the case $l = 2$ and $q \to -q$, gives some properties of the power series (36).

Remark There is considerable evidence to suggest that the polynomial $L_{\alpha\beta}(q)$ of Proposition 9.5 has in many cases in addition to (34) a simple explicit expression. It may even be true that for any α and β there is a fairly simple description of $L_{\alpha\beta}(q)$ which would involve a product analogous to (34), together with an additional factor with direct combinatorial or algebraic significance.

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