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# Combinatorial Applications of the Hard Lefschetz Theorem

## 1. The hard Lefschetz theorem

Let $X$ be a smooth irreducible complex projective variety of (complex) dimension $n$ (or more generally a Kähler manifold), endowed with the "classical" Hausdorff topology. Let $H^*(X) = H^0(X) \oplus H^1(X) \oplus \ldots \oplus H^{2n}(X)$ denote the singular cohomology ring of $X$ with complex coefficients. (Any field of characteristic zero would do just as well for the coefficient group. In fact, for the most part we could work over $\boldsymbol{Z}$, but this is unnecessary for our purposes.) Since $X$ is projective we may imbed it in some complex projective space $P^N$. Let $H$ denote a (generic) hyperplane in $P^N$. Then $X \cap H$ is a closed subvariety of $X$ of real codimension two, and thus by a standard construction in algebraic geometry represents a cohomology class $\omega \in H^2(X)$, called the *class of a hyperplane section.*

THE HARD LEFSCHETZ THEOREM. *Let* $0 \leqslant i \leqslant n$. *The map* $H^i(X) \xrightarrow{\omega^{n-i}} H^{2n-i}(X)$ *given by multiplication by* $\omega^{n-i}$ *is an isomorphism of vector spaces.*

This result was first stated by Lefschetz in [18], but his proof was not entirely rigorous. The first complete proof was given by Hodge in [15], using his theory of harmonic integrals. The "standard" proof today uses the representation theory of the Lie algebra $\mathrm{sl}_2(\boldsymbol{C})$ and is due to Chern [4]. Lefschetz' original proof was only recently made rigorous by Deligne (see [22]), who extended it to characteristic $p$. Other references include [5], [13], [17], [32].

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## 2. Unimodality

Since $\omega^{n-i}$: $H^i(X) \to H^{2n-i}(X)$ is bijective for all $0 \leqslant i \leqslant n$, it follows that $\omega$: $H^i(X) \to H^{i+1}(X)$ is *injective* for $0 \leqslant i \leqslant n-1$ and *surjective* for $n \leqslant i \leqslant 2n-1$. Thus if $\beta_i = \beta_i(X) = \dim H^i(X)$ denotes the $i$-th Betti number of $X$, then the sequences $\beta_0, \beta_2, \ldots, \beta_{2n}$ and $\beta_1, \beta_3, \ldots, \beta_{2n-1}$ are *symmetric* and *unimodal*, i.e., $\beta_0 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_{2[n/2]}$ and $\beta_i = \beta_{2n-i}$, and similarly for $\beta_1, \beta_3, \ldots, \beta_{2n-1}$. This consequence of the hard Lefschetz theorm was well known from the beginning.

There are several examples for which $\beta_i$ has a combinatorial interpretation. The archetype is the Grassmann variety $X = G_{dn}$ of $d$-planes in $C^n$. It was rigorously known since Ehresmann [8] that $\beta_{2i+1} = 0$ and that $\beta_{2i}$ is the number $p(i, d, n-d)$ of partitions of the integer $i$ into $\leqslant d$ parts, with largest part $\leqslant n-d$. The unimodality of the sequence $p(0, d, n-d), p(1, d, n-d), \ldots, p\big(d(n-d), d, n-d\big)$ was first proved by Sylvester [31], and several subsequent proofs have been given. Perhaps the simplest is [28, Cor. 9.6], but no purely combinatorial proof is known. Such a proof would involve an explicit injection from the partitions counted by $p(i, d, n-d)$ to those counted by $p(i+1, d, n-d)$, for $0 \leqslant i \leqslant [\frac{1}{2}d(n-d)]$.

More generally, take $X$ to be a generalized flag manifold $G/P$, where $G$ is a complex semisimple algebraic group and $P$ a parabolic subgroup. The hard Lefschetz theorem then yields the unimodality of the number of elements of length $i$ in the quotient Bruhat order $W^J = W/W_J$, where $W$ is the Weyl group of $G$ and $W_J$ of $P$ ([26, § 3]).

Now let $\mathfrak{g}$ be a complex semisimple Lie algebra and $L_\lambda$ an irreducible finite-dimensional $\mathfrak{g}$-module with highest weight $\lambda$. Let $m_\lambda(\mu)$ denote the multiplicity of the weight $\mu$ in $L_\lambda$. The *height* $\mathrm{ht}\,\mu$ of $\mu$ is the number of simple roots which need to be added to $-\lambda$ to obtain $\mu$. The polynomial $P_\lambda(q) = \sum_\mu m_\lambda(\mu) q^{\mathrm{ht}\,\mu}$ was first shown by Dynkin (see [7, p. 332]) to have unimodal coefficients; see also [25]. Recently Lusztig [19] has computed the (complex) intersection cohomology $\boldsymbol{H}^*(\bar{\mathcal{O}}_\lambda)$, as defined by Goresky and Macpherson [10], [11], of certain *Schubert varieties* $\bar{\mathcal{O}}_\lambda$ (in general singular). Namely,

$$\sum_i \dim \boldsymbol{H}^i(\bar{\mathcal{O}}_\lambda) q^i = P_\lambda(q^2).$$

(See [19, Cor. 8.9]). Now the intersection cohomology $\boldsymbol{H}^*(X)$ of any complex projective variety, considered as a module over the singular cohomology

$H^*(X)$, satisfies the hard Lefschetz theorem. Hence Dynkin's result may be regarded as a consequence of the hard Lefschetz theorem for intersection cohomology. It would be interesting to investigate what other sequences of combinatorial interest arise as Betti numbers in intersection cohomology.

## 3. McMullen's $g$-conjecture

Let $\mathscr{P}$ be a $d$-dimensional simplical convex polytope ([14], [21]) with $f_i$ $i$-dimensional faces, $0 \leqslant i \leqslant d-1$. We call the vector $f(\mathscr{P}) = (f_0, \dots, f_{d-1})$ the *f-vector* of $\mathscr{P}$. The problem of obtaining information about such vectors goes back to Descartes and Euler. In 1971 McMullen [20], [21, p. 179], drawing on all the available evidence, gave a remarkable condition on a vector $(f_0, \dots, f_{d-1})$ which he conjectured was equivalent to being an $f$-vector as above.

To describe this condition, define a new vector $h(\mathscr{P}) = (h_0, \dots, h_d)$, called the *h-vector* of $\mathscr{P}$, by

$$h_i = \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_{j-1},$$

where we set $f_{-1} = 1$. The Dehn–Sommerville equations ([14, §§ 9.2, 9.8], [21, §§ 2.4, 5.1]) assert that $h_i = h_{d-i}$. McMullen's conditions, though he did not realize it, are equivalent to the Dehn–Sommerville equations together with the existence of a graded commutative $C$-algebra $R = R_0 \oplus \oplus R_1 \oplus \dots$ where $R_0 = C$, $R$ is generated as a $C$-algebra by $R_1$, and $\dim_C R_i = h_i - h_{i-1}$ for $1 \leqslant i \leqslant [d/2]$. In particular, $h_0 \leqslant h_1 \leqslant \dots \leqslant h_{[d/2]}$, and we are led to suspect the existence of a smooth $d$-dimensional complex projective variety $X(\mathscr{P})$ for which $\beta_{2i}(X(\mathscr{P})) = h_i$. If moreover $H^*(X(\mathscr{P}))$ is generated by $H^2(X(\mathscr{P}))$, then we can take $R = H^*(X(\mathscr{P}))/(\omega)$, where $(\omega)$ denotes the ideal generated by the class of a hyperplane section, to deduce the necessity of McMullen's conditions. In [6] varieties $X(\mathscr{P})$ are constructed (after some assumptions on $\mathscr{P}$ irrelevant for proving McMullen's conjecture) with all the desired properties except smoothness (despite the misleading statement in [6, Rmk. 3.8]). Although $X(\mathscr{P})$ need not be smooth, its singularities are sufficiently nice that the hard Lefschetz theorem continues to hold [30]. Namely, $X(\mathscr{P})$ is a *V-variety*, i.e., locally it looks like $C^n/G$ where $G$ is a finite group of linear transformations. Thus the necessity of McMullen's condition follows [27]. Sufficiency was proved about the same time by Billera and Lee [1], [2]. For further information see [29]. Some recent work of Kalai [16] suggests that a more elementary proof may be possible, and perhaps an extension to more general objects (such as shellable triangulations of spheres or even triangulations of homology spheres).

## 4. The Sperner property

Let $P$ be a finite poset (= partially ordered set). We say $P$ is *graded of rank* $n$ if every maximal chain of $P$ has length $n$ (or cardinality $n+1$). We then define the *rank* $\varrho(x)$ of $x \in P$ to be the length $l$ of the longest chain $x_0 < x_1 < \ldots < x_l = x$. Let $P_i = \{x \in P\colon \varrho(x) = i\}$. An *antichain* is a subset $A$ of $P$ of pairwise incomparable elements. Thus each $P_i$ is an antichain. We say that $P$ has the *Sperner property* if no antichain is larger than the largest $P_i$. This terminology stems from the theorem of E. Sperner [24] that the poset of subsets of an $n$-element set, ordered by inclusion, has the Sperner property (see also [12]).

If now $X$ is a complex projective variety, we say $X$ has a *cellular decomposition* if there exists a (finite) set $\mathscr{C} = \{C_1, \ldots, C_t\}$ of pairwise disjoint subsets $C_i$ of $X$, each isomorphic as algebraic varieties to complex affine space $\boldsymbol{C}^{n_i}$, such that $\cup C_i = X$ and the closure $\bar{C}_i$ (in the classical or Zariski topology) of each $C_i$ is a union of $C_j$'s. We then define a poset $Q^X = Q^X(\mathscr{C})$ to be the set $\mathscr{C}$ ordered by reverse inclusion of the closure of the $C_i$'s. If $X$ is irreducible of dimension $n$, then $Q^X$ is graded of rank $n$.

THEOREM 1 ([26, Thm. 2.4]). *If $X$ (as above) is smooth and irreducible, then $Q^X$ has the Sperner property.*

The crucial step in the proof is the use of the hard Lefschetz theorem, and indeed the theorem remains true for any irreducible $X$ (with a cellular decomposition) satisfying the conclusions of the hard Lefschetz theorem. The conclusion to the above theorem is not the strongest possible; see [26] for further details.

The main class of varieties to which the above theorem applies (indeed, the only known class for which the Sperner property of $Q^X$ is non-trivial) are the generalized flag manifolds $X = G/P$ mentioned above. Here the $Q^X$ are the quotient Bruhat orders $W^J$. For certain choices of $G$ and $P$ the posets $Q^X$ have a special combinatorial significance. In particular, taking $G = \mathrm{SO}(2n+1, \boldsymbol{C})$ and a certain maximal $P$, the Spernicity of the poset $Q^X$ can be used to prove the following number-theoretic conjecture of Erdös and Moser [9, eqn. (12)].

THEOREM 2. *Let $S$ be a finite subset of $\mathbf{R}$, and for $k \in \mathbf{R}$ let $f(S,k)$ denote the number of subsets of $S$ whose elements sum to $k$. Then for $|S| = 2l+1$, we have*

$$f(S, k) \leqslant f(\{-l, -l+1, \ldots, l\}, 0).$$

For further information in addition to [26], see [23] and the references therein.

It would be interesting to discover other properties of the posets $Q^X$ of Theorem 1 (in addition to the Sperner property). In particular, if $\Delta$ denotes the simplicial complex of chains of $\bar{Q}^X$, where $\bar{Q}^X$ denotes $Q^X$ with the bottom and top element removed, then is the geometric realization $|\Delta|$ always a sphere or cell? (It may not even be necessary to assume $X$ is smooth.) This is true when $\dim X \leqslant 2$ or when $X = G/P$. For the latter case see [3].

In [25, Problem 2] it is asked whether some posets arising from irreducible representations of the Lie algebra $\mathrm{sl}(n, \mathbf{C})$ have a "symmetric chain decomposition", which is stronger than the Sperner property. In fact, even the Sperner property is open. Perhaps there is a "cellular decomposition for intersection cohomology" of the Schubert varieties $\bar{\mathcal{O}}_\lambda$ discussed in Section 2 which would yield a proof of the Sperner property.

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