GL(n,ℂ) FOR COMBINATORIALISTS

R. P. Stanley
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139 U.S.A.

1. INTRODUCTION

Let $G_n = GL(n,\mathbb{C}) = GL(V_n)$ denote the group of all invertible linear transformations $A:V_n \to V_n$, where V_n is an n-dimensional complex vector space. Once we choose a basis for V_n we can regard G_n as the group of nonsingular $n\times n$ complex matrices. A (linear) representation of G_n of dimension m is a homomorphism $\phi:G_n \to G_m$. We call ϕ a polynomial (respectively, rational) representation if (after choosing bases) the matrix entries of $\phi(A)$ are fixed polynomials (respectively, rational functions) in the matrix entries of A. For instance, the map

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\phi} \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix} \qquad (1)$$

is a polynomial representation of G_2 of dimension three, while

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\rho} (ad-bc)^{-1} \qquad (2)$$

is a rational representation of dimension one. Henceforth, all representations in this paper are assumed to be rational.

The theory of such representations has close connections with combinatorics, and our object here is to present an overview of this subject from the combinatorial viewpoint. We first will state without proof the basic results (which may be gleaned from such sources as Hamermesh (1962), pp. 377-391; James & Kerber (1981), Ch. 8; Littlewood (1950), Ch. X; Macdonald (1979), pp. 74-84; and Weyl (1946), Ch. IV), and then proceed to the combinatorial ramifications.

The first result we need is that the (rational) representations of G_n are completely reducible (i.e., G_n is a reductive group). This means in effect that every representation $\phi:G_n \to G_m$ can be decomposed into irreducibles, i.e., if $G_m = GL(V)$, so that we may regard G_n as acting

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The theory of such representations has close connections with combinatorics, and our object here is to present an overview of this subject from the combinatorial viewpoint. We first will state without proof the basic results (which may be gleaned from such sources as Hamermesh (1962), pp. 377-391; James & Kerber (1981), Ch. 8; Littlewood (1950), Ch. X; Macdonald (1979), pp. 74-84; and Weyl (1946), Ch. IV), and then proceed to the combinatorial ramifications.

The first result we need is that the (rational) representations of G_n are <u>completely reducible</u> (i.e., G_n is a <u>reductive</u> group). This means in effect that every representation $\phi:G_n\to G_m$ can be decomposed into irreducibles, i.e., if $G_m = GL(V)$, so that we may regard G_n as acting on V, then $V = V_1 \oplus \cdots \oplus V_r$ where each V_i is nonzero and invariant under G_n, and no V_i has a proper G_n-invariant subspace. Although the V_i's need not be unique, the multiset $\{\phi_1,\ldots,\phi_n\}$ of irreducible representations $\phi_i : G_n \to GL(V_i)$ is unique up to equivalence. Thus to determine ϕ up to equivalence, it suffices to describe the multiplicity of each irreducible representation of G_n in ϕ.

Suppose $A \in G_n$ has eigenvalues $\theta_1,\ldots,\theta_n$. Then there exists a multiset M_ϕ of m Laurent monomials $u(x) = x_1^{a_1}\cdots x_n^{a_n}$, $a_i \in \mathbb{Z}$, independent of A, such that the eigenvalues of $\phi(A)$ are given by the multiset $\{u(\theta_1,\ldots,\theta_n) \mid u \in M_\phi\}$. For the representations ϕ and ρ of (1) and (2), the reader can check that $M_\phi = \{x_1^2, x_1x_2, x_2^2\}$ and $M_\rho = \{x_1^{-1}x_2^{-1}\}$. The Laurent polynomial $f_\phi = \sum_{u \in M_\phi} u$ (which is a symmetric function of $x_1,\ldots,x_n$) is called the <u>character</u> of ϕ; clearly

$$f_\phi(\theta_1,\ldots,\theta_n) = \operatorname{tr} \phi(A),$$

where tr denotes trace. The character f_ϕ uniquely determines ϕ up to equivalence. In other words, f_ϕ can be written uniquely as a nonnegative integral combination of irreducible characters. We now wish to describe the irreducible characters of G_n. First we reduce this problem to polynomial representations.

1.1 <u>Theorem</u>. Any rational representation $\phi : GL_n \to GL_m$ has the form $\phi(A) = (\det A)^{-r}\phi'(A)$ for some $r \in \mathbb{Z}$ and some polynomial representation ϕ'. ϕ is irreducible if and only if ϕ' is irreducible, and $f_\phi(x_1,\ldots,x_n) = (x_1\cdots x_n)^{-r} f_{\phi'}(x_1,\ldots,x_n)$.

Now let $\lambda = (\lambda_1,\ldots,\lambda_n)$ be any partition into $\leq$ n parts, i.e., $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, $\lambda_i \in \mathbb{Z}$. The number of (positive) parts $\lambda_i > 0$ of λ is called the <u>length</u> of λ, denoted $\ell(\lambda)$. We also write $|\lambda| = \lambda_1 + \cdots + \lambda_n$. Following Macdonald (1979), let $s_\lambda(x_1,\ldots,x_n)$ denote the <u>Schur function</u> corresponding to λ in the variables $x_1,\ldots,x_n$. It has the following combinatorial definition. Write down a left-justified array whose entries are the numbers 1,2,...,n (with any multiplicities), with λ_i entries in row i, such that the columns are strictly decreasing and rows weakly decreasing. With such an array T (called a <u>tableau</u> of <u>shape</u> λ and largest part $\leq$ n) associate the monomial $m(T) = x_1^{a_1}\cdots x_n^{a_n}$, where a_i i's appear in T. Then $s_\lambda(x) = s_\lambda(x_1,\ldots,x_n)$ is defined to be $\sum_T m(T)$, summed over all tableaux T of shape λ and largest part $\leq n$. Though not obvious from the definition, $s_\lambda(x)$ is a symmetric function of $x_1,\ldots,x_n$.

Example. Take $\lambda = (2,1)$, $n = 3$. The appropriate tableaux are

21	22	31	33	32	33	32	31
1	1	1	1	2	2	1	2

Hence $s_{21}(x_1,x_2,x_3) = x_1^2x_2+x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3$.

The main result on the polynomial characters of G_n is the following.

1.2 Theorem. Let $\lambda = (\lambda_1,\ldots,\lambda_n)$ be a partition, $\ell(\lambda)\leq n$. Then the Schur function $s_\lambda(x_1,\ldots,x_n)$ is an irreducible polynomial character of GL_n, different λ's yield different characters, and every irreducible polynomial character has this form.

We will denote by ρ_λ the representation of G_n whose character is s_λ. In other words, $s_\lambda = f_{\rho_\lambda}$.

The Schur functions $s_\lambda(x_1,\ldots,x_n)$, $\ell(\lambda)\leq n$, form a $\mathbb{Z}$-basis for the additive group of all symmetric polynomials in $x_1,\ldots,x_n$ with integer coefficients (Macdonald 1979, p.24). Thus every such polynomial f is a virtual character (= difference of two characters) of G_n, and expanding f in terms of Schur functions is equivalent to finding the multiplicity of each irreducible character of G_n in f.

Remarks on some other groups. The representations of the groups U(n), SL(n,$\mathbb{C}$), and SU(n) can be obtained easily from those of G_n= GL(n,$\mathbb{C}$). Since G_n is a reductive algebraic group with maximal compact subgroup U(n) it follows from general principles that the rational representations of G_n and U(n) coincide. More precisely, distinct irreducible representations of G_n restrict to distinct irreducibles of U(n), and every irreducible representation of U(n) arises in this way.

Regarding SL(n,$\mathbb{C}$), suppose $\phi:G_n\to G_m$ has character $s_\lambda(x)$. By our definition of s_λ,

$$s_\lambda(x) = (x_1\ldots x_n)^{\lambda_n} s_{\lambda^*}(x), \tag{3}$$

where $\lambda^* = (\lambda_1-\lambda_n, \lambda_2-\lambda_n,\ldots,\lambda_{n-1}-\lambda_n,0)$. If $\phi^*:G_n\to G_m$ has character ϕ^*, then the right-hand side of (3) is just the character of $(\det)^{\lambda_n}\phi^*$. Hence $\phi = (\det)^{\lambda_n}\phi^*$, so that ϕ and ϕ^* restrict to the same representation of SL(n,$\mathbb{C}$). But except for this, the irreducible representations of GL(n,$\mathbb{C}$) and SL(n,$\mathbb{C}$) coincide. More precisely:

1.3 Theorem. Let $\lambda = (\lambda_1,\ldots,\lambda_{n-1})$ be a partition into $\leq n-1$ parts. Then the Schur function $s_\lambda(x_1,\ldots,x_n)$ is an irreducible polynomial character of $SL(n,\mathbb{C})$, different λ's yield different characters, and every irreducible polynomial character has this form.

We will call any Laurent polynomial $f(x_1,\ldots,x_n)$ a character of the representation $\rho:SL(n,\mathbb{C})\to G_m$ if $\operatorname{tr}\rho(A) = f(\theta_1,\ldots,\theta_n)$ for all $A\in SL(n,\mathbb{C})$ with eigenvalues $\theta_1,\ldots,\theta_n$. Since $\theta_1\cdots\theta_n = 1$, the character is not unique (as it was for G_n). The character f_ρ of ρ is, however, a uniquely defined element of the quotient ring $\Lambda(x_1,\ldots,x_n)/(x_1\cdots x_n-1)$, where $\Lambda(x_1,\ldots,x_n)$ denotes the ring of symmetric polynomials with integer coefficients in the variables $x_1,\ldots,x_n$. Thus frequently we will carry out our computations with characters of $SL(n,\mathbb{C})$ in this quotient ring.

Finally, $SU(n)$ bears the same relation to $SL(n,\mathbb{C})$ as $U(n)$ does to $GL(n,\mathbb{C})$.

2. SOME EXAMPLES

Let us consider some "naturally occurring" representations of G_n and try to compute their characters. First we have the defining representation $\phi:G_n\to G_n$ given by $\phi(A) = A$. If A has eigenvalues $\theta_1,\ldots,\theta_n$ then $\phi(A)$ also has these eigenvalues. Hence $\operatorname{tr}\phi(A) = \theta_1+\cdots+\theta_n$ and $f_\phi(x) = x_1+\cdots+x_n$. Since for each $1\leq i\leq n$ there is exactly one way to put i into the shape $\lambda = (1)$ to form a column-strict plane partition, we have $s_1(x) = x_1+\cdots+x_n$. Hence $f_\phi = s_1$ and $\phi = \phi_1$.

Suppose $\phi:G_n\to G_m = GL(V_m)$ is any representation. Choose a basis $z_1,\ldots,z_m$ for the vector space V_m. Let $S^k(V_m)$ denote the vector space of all homogeneous polynomials of degree k in the variables $z_1,\ldots,z_m$. Thus $\dim S^k(V_m) = \binom{m+k-1}{k}$, and $S^k(V_m)$ is the k-th symmetric power of V_m. Any $B\in G_m$ acts on $S^k(V_m)$ by the rule $B\cdot g(z_1,\ldots,z_m) = g(Bz_1,\ldots,Bz_m)$ so we have a representation of G_m on $S^k(V_m)$, i.e., a homomorphism $G_m\to GL(S^k(V_m)) \cong G_{\binom{m+k-1}{k}}$. Hence G_n acts on $S^k(V_m)$ by composition, i.e., if $A\in G_n$ and $g\in S^k(V_m)$ then $A\cdot g = \phi(A)\cdot g$. The resulting representation is denoted $S^k\phi:G_n\to GL(S^k(V_m))$. It is an important and difficult problem (which comes close to subsuming all of classical invariant theory) to decompose $S^k\phi$ into irreducibles.

The problem of decomposing $S^k\phi$ (up to equivalence) may be stated in combinatorial terms as follows. Let $A = \operatorname{diag}(\theta_1,\ldots,\theta_m)\in G_m$, with respect to the basis $z_1,\ldots,z_m$ of V_m. Write S^kA for the action of A on

S^kV_m, i.e., $S^kA = (S^k\phi_1)(A)$, where $\phi_1 : G_m \to G_m$ is the defining representation. A monomial $z_1^{a_1}\cdots z_m^{a_m} \varepsilon S^kV_m$ is an eigenvector for S^kA with eigenvalue $\theta_1^{a_1}\cdots\theta_m^{a_m}$. Since the monomials $z_1^{a_1}\cdots z_m^{a_m}$ of degree k form a basis for S^kV_m, we have accounted for all the eigenvalues of S^kA. Hence

$$\operatorname{tr} S^kA = \sum_{a_1+\cdots+a_m=k} \theta_1^{a_1}\cdots\theta_m^{a_m}$$

$$= \text{coefficient of } q^k \text{ in } \prod_{i=1}^{m} (1-\theta_i q)^{-1}. \qquad (4)$$

Let M_ϕ be the multiset of monomials for $\phi: G_n \to G_m$ defined above, so $f_\phi(x_1,\ldots,x_n) = \sum_{u\varepsilon M_\phi} u$. It follows from (4) that

$$\sum_{k\geq 0} f_{S^k\phi}(x)q^k = \prod_{u\varepsilon M_\phi} (1-uq)^{-1}. \qquad (5)$$

Thus the problem of decomposing $S^k\phi$ (up to equivalence) is equivalent to the combinatorial problem of expanding the right-hand side of (5) in terms of Schur functions. This is a special case of the notion of plethysm of Schur functions; see Macdonald (1979), p.82.

For now we will be content with decomposing $S^k\phi_1$ (where $\phi_1 : G_n \to G_n$ is the defining representation). By (5) we have

$$\sum_{k\geq 0} f_{S^k\phi_1}(x)q^k = \prod_{i=1}^{n} (1-x_i q)^{-1}$$

$$= \sum_{k\geq 0} h_k(x)q^k,$$

where $h_k(x)$ is the sum of all monomials in $x_1,\ldots,x_n$ of degree k (called the complete (homogeneous) symmetric functions). For any integers $b_1,\ldots,b_n \geq 0$ satisfying $\Sigma b_i = k$ there is a unique tableau of shape $(k) = (k,0,0,\ldots)$ with b_i i's. Hence $f_{S^k\phi_1}(x) = h_k(x) = s_k(x)$, i.e., $S^k\phi_1$ is irreducible with character s_k. Since $f_{\phi_1} = s_1$ we can write $s_k = S^k s_1$.

In an exactly analogous way, given $\phi: G_n \to G_m = GL(V_m)$ we can compute the character of $\Lambda^k\phi: G_n \to GL(\Lambda^kV_m)$, where Λ^k denotes the k-th exterior power, $0\leq k\leq m$. Keeping the same notation as before, a wedge product $z_{i_1}\wedge\cdots\wedge z_{i_k}$, $1\leq i_1<\cdots<i_k\leq m$, is an eigenvector for Λ^kA with eigenvalue $\theta_{i_1}\cdots\theta_{i_k}$. Hence

$$\sum_{k=0}^{m} f_{\Lambda^k\phi}(x)q^k = \prod_{u\in M_\phi} (1+uq). \tag{6}$$

Letting $\phi = \phi_1$ we obtain

$$\begin{aligned}\sum_{k=0}^{m} f_{\Lambda^k\phi_1}(x)q^k &= \prod_{i=1}^{n} (1+x_i q)\\ &= \sum_{k=0}^{m} e_k(x)q^k,\end{aligned}$$

where $e_k(x)$ denotes the k-th <u>elementary symmetric function</u> in $x_1,\ldots,x_n$. For any integers $1\le c_1<\ldots<c_k\le n$ there is a unique tableau of shape $(1^k) = (1,1,\ldots,1)$ (k ones) with parts $c_1,\ldots,c_k$. Hence $f_{\Lambda^k\phi}(x)=e_k(x)=s_{1^k}(x)$, i.e. $\Lambda^k\phi_1$ is irreducible with character s_{1^k}, and we can write $s_{1^k} = \Lambda^k s_1$.

Let us compute one additional example of this nature, which will be of use in Section 4. Let M_n denote the n^2-dimensional vector space of all n×n matrices. Then $A\in G_n$ acts on M_n by the rule $B\to A^{-1}BA$, where $B\in M_n$. This representation of G_n is called the <u>adjoint representation</u>, denoted ad. We now compute its character. Let E_{ij} be the elementary matrix with a one in position (i,j) and zeros elsewhere. Choose $A=\mathrm{diag}(\theta_1,\ldots,\theta_n)$. Then $A^{-1}E_{ij}A = \theta_i^{-1}\theta_j E_{ij}$. Thus

$$\begin{aligned}\mathrm{tr}(\mathrm{ad}\,A) &= \sum_{i,j} \theta_i^{-1}\theta_j\\ &= (\theta_1\ldots\theta_n)^{-1} \sum_{i,j} (\theta_1\ldots\theta_n)\theta_i^{-1}\theta_j.\end{aligned} \tag{7}$$

Consider the partition $(2,1,\ldots,1)\vdash n$. To form a tableau of this shape with largest part $\le n$, choose any (n-1)-element subset S of $\{1,\ldots,n\}$ and insert it (uniquely) into the first column. The additional entry can be any element of $\{1,\ldots,n\}$, with the sole exception that it cannot equal n when $S = \{1,\ldots,n-1\}$. It follows that

$$s_{21^{n-2}}(x_1,\ldots,x_n) = \sum_{i,j} (x_1\ldots x_n)x_i^{-1}x_j - (x_1\ldots x_n).$$

Comparison with (7) yields

$$f_{ad} = (\det)^{-1} s_{21^{n-2}} + s_\emptyset \qquad (\emptyset = \text{null set}).$$

Since $s_\emptyset$ is the character of the trivial representation, this means that M_n has a (unique) one-dimensional subspace W fixed pointwise by G_n. Of course W is just the set of scalar matrices. The complementary invariant subspace M_n^0 to W (the one which affords the character $(\det)^{-1}s_{21^{n-2}}$) consists of the matrices of trace 0. If we restrict the action of G_n on M_n^0 to SL(n,$\mathbb{C}$), then we obtain the <u>adjoint representation</u> of SL(n,$\mathbb{C}$), with character

$$f_{ad} = s_{21^{n-2}}(x_1,\ldots,x_n) = \sum_{i\neq j} (x_1\ldots x_n)x_i^{-1}x_j + (n-1)(x_1\ldots x_n).$$

Since characters of SL(n,$\mathbb{C}$) are defined modulo the relation $x_1\ldots x_n = 1$, we could also write

$$f_{ad} = \sum_{i\neq j} x_i^{-1}x_j + n-1. \tag{8}$$

Though this is not in the "canonical form" given by Theorem 1.3, it is an equally valid expression for f_{ad} (and one which is more natural from the Lie-algebraic viewpoint).

As an exercise, the reader may wish to compute the characters of the actions of G_n on M_n given by BA, $A^{-1}B$, BA^*, A^tB, A^tBA, $A^{-1}BA^*$, and A^tBA^*, where t denotes transpose and $A^* = (A^t)^{-1}$.

Virtually any identity involving symmetric functions can be interpreted in terms of representation theory. We give one such example here.

An elegant combinatorial proof (Knuth 1970, Stanley 1971) can be given of the identity

$$\prod_{i=1}^{n} (1-x_i)^{-1} \prod_{\substack{i,j=1\\ i<j}}^{n} (1-x_ix_j)^{-1} = \sum_{\substack{\lambda\\ \ell(\lambda)\leq n}} s_\lambda(x_1,\ldots,x_n). \tag{9}$$

Now $s_1(x) = \Sigma x_i$ and $s_{11}(x) = \Lambda^2 s_1(x) = \sum_{i<j} x_ix_j$. Thus the left-hand side is the character of the representation $S(\rho_1+\rho_{11})$, i.e., the natural action of $G_n = GL(V_n)$ on the symmetric algebra $S(V_n \oplus \Lambda^2 V_n)$. Thus by (9) we see that in the representation $S(\rho_1+\rho_{11})$, every irreducible polynomial representation of G_n occurs exactly once. A refinement of (9)

(Macdonald 1979, p.46, Ex.7) asserts that

$$\prod_i (1-tx_i)^{-1} \prod_{i<j} (1-x_i x_j)^{-1} = \sum_\lambda t^{c(\lambda)} s_\lambda(x),$$

where $c(\lambda)$ is the number of columns of odd length in λ. From this it is easy to obtain the decomposition of each $S^k(\rho_1+\rho_{11})$, viz., ρ_λ appears in $S^k(\rho_1+\rho_{11})$ (with multiplicity one) if and only if $k = \frac{1}{2}(|\lambda|+c(\lambda))$.

3. UNIMODALITY

Consider the group $SL(2,\mathbb{C})$. By Theorem 1.3 the irreducible characters are just the Schur functions

$$s_m(x,y) = x^m+x^{m-1}y+\cdots+y^m .$$

(Thus the irreducible representations are just $S^m(\rho_1)$.) It is more usual to write this character as

$$s_m(x,x^{-1}) = x^{-m} + x^{-m+2}+\cdots+x^m,$$

which of course is the same as before modulo the relation $xy = 1$. Now suppose ρ is any representation and that ρ_m appears in ρ with multiplicity a_m. Then for sufficiently large k, ρ has the character

$$f_\rho(x,x^{-1}) = \sum_{m=0}^{k} a_m(x^{-m}+x^{-m+2}+\cdots+x^m)$$

$$= \sum_{j=-k}^{k} b_j x^j ,$$

where $b_j = a_j+a_{j+2}+a_{j+4}+\cdots$ for $j\geq 0$, and $b_j = b_{-j}$. It follows that $b_0\geq b_2\geq b_4\geq\cdots$ and $b_1\geq b_3\geq\cdots$. We say that a sequence $c_0,c_1,\ldots,c_r$ is <u>unimodal</u> if $c_0\leq c_1\leq\cdots\leq c_s$ and $c_s\geq c_{s+1}\geq\cdots\geq c_r$ for some s, and is <u>symmetric</u> if $c_i = c_{r-i}$. Thus we have shown:

3.1 <u>Theorem.</u> For any representation ρ of $SL(2,\mathbb{C})$ with character $f_\rho(x,x^{-1}) = \sum_{j=-k}^{k} b_j x^j$, the two sequences $b_{-k},b_{-k+2},\ldots,b_k$ and $b_{-k+1},b_{-k+3},\ldots,b_{k-1}$ are symmetric and unimodal.

Theorem 3.1 can be used as a tool in showing that certain sequences of combinatorial interest are unimodal. For a general discussion of this topic, see Almkvist (1982). Here we present the prototypical case, viz., the action of $SL(V_2)$ on $S^k(S^m V_2)$ or equivalently, the representation

$S^k\rho_m$.

Since $s_m(x,x^{-1}) = x^{-m}+x^{-m+2}+\cdots+x^m$, the character of $\rho = S^k\rho_m$ is given by

$$f_\rho(x,x^{-1}) = \Sigma\ (x^{-m})^{r_0}(x^{-m+2})^{r_1}\cdots(x^m)^{r_m}$$
$$= x^{-mk}\ \Sigma(x^2)^{r_1+2r_2+\cdots+mr_m},$$

where the sums range over all nonnegative integral sequences $(r_0,r_1,\ldots,r_m)$ satisfying $r_0+r_1+\cdots+r_m = k$. Identify $(r_0,r_1,\ldots,r_m)$ with the partition λ with r_i parts equal to i. Then the sum ranges over all partitions λ whose Ferrers diagram (e.g., Andrews 1976, p.6) fits in a $k\times m$ rectangle, and $r_1+2r_2+\cdots+mr_m = |\lambda|$. Hence the coefficient of x^{-mk+2j} in $f_\rho(x,x^{-1})$ is equal to the number $p(j,m,k)$ of partitions of j fitting in a $k\times m$ rectangle (i.e., with $\leqslant k$ parts and largest part $\leqslant m$). It follows from Theorem 3.1 that for fixed m and k, the sequence

$$p(0,m,k),\ p(1,m,k),\ldots,\ p(mk,m,k)$$

is symmetric and unimodal. Although it is possible to prove this result without mentioning SL(2,ℂ) (e.g., Stanley 1982, Cor. 9.6), no simple combinatorial proof is known.

Let us also mention that the polynomial $\sum_j p(j,m,k)q^j$ is the <u>q-binomial coefficient</u> $\left[{m+k \atop k}\right]$, defined by

$$\left[{m+k \atop k}\right] = \frac{[m+k]!}{[m]!\,[k]!},$$

where $[i]! = (1-q)(1-q^2)\cdots(1-q^i)$. (See Andrews 1976, Thm. 3.1) Thus we have shown that the coefficients of $\left[{m+k \atop k}\right]$ are symmetric and unimodal.

4. A LITTLE INVARIANT THEORY

If a group G acts on a ring R, then the fixed ring

$$R^G = \{f\in R \mid Af = f \text{ for all } A\in G\}$$

is called the <u>ring of invariants</u> of G. Suppose we're given a decomposition $R = R_0 \oplus R_1 \oplus \cdots$, where each R_i is a finite-dimensional vector space over a field K and $\oplus$ denotes vector space direct sum. Suppose G acts so that $GR_i = R_i$. Then $R^G = R_0^G \oplus R_1^G \oplus \cdots$, where $R_i^G = R_i \cap R^G$. We then call the power series

$$F(R^G,q) = \sum_{i\geq 0}(\dim R_i^G)q^i$$

the Molien series of R^G (or of G acting on R).

Consider the special case $G = SL(n,\mathbb{C})$ and $R = S(V) = \mathbb{C} \oplus S^1(V) \oplus S^2(V) \oplus \cdots$, where V affords the adjoint representation of $SL(n,\mathbb{C})$ (i.e., V is the space of n×n complex matrices of trace 0, and G acts on V by conjugation). It is easy to see that R^G is generated by n-1 algebraically independent elements $\theta_1,\ldots,\theta_{n-1}$ of degrees 2,3,...,n. Namely, for $B\in V$ take $\theta_i(B)$ to be the coefficient of λ^{n-i-1} in the characteristic polynomial of B. It follows that

$$F(S(V)^G,q) = 1/(1-q^2)\cdots(1-q^n). \tag{10}$$

We will give a combinatorial derivation of (10).

Since we are working with $SL(n,\mathbb{C})$, we deal with the variables $x = (x_1,\ldots,x_n)$ and all our computations are performed modulo the relation $x_1\cdots x_n = 1$. Let $\lambda = (\lambda_1,\ldots,\lambda_{n-1})$ be a partition, and define $\tilde{\lambda} = (\lambda_1,\lambda_1-\lambda_{n-1},\lambda_1-\lambda_{n-2},\ldots,\lambda_1-\lambda_2)$. We claim

$$s_\lambda(1/x_1,\ldots,1/x_n) = s_{\tilde{\lambda}}(x_1,\ldots,x_n) \tag{11}$$

(always modulo $x_1\cdots x_n=1$). While it is easy to give a representation-theoretic proof of (11), a combinatorial approach is instructive. Namely, given a tableau T of shape λ and largest part ≤n, define a new tableau $\tilde{T}$ of shape $\tilde{\lambda}$ by the following condition: if $a_1,\ldots,a_k$ are the elements of the i-th column of T, then the elements $b_1,\ldots,b_{n-k}$ of column λ_1-i+1 of $\tilde{T}$ consist of the elements of the complementary set $\{1,\ldots,n\}-\{a_1,\ldots,a_k\}$. For instance, if n=4, then we have

4 4 3 3 3 2 1 1	4 4 4 4 4 4 3 2
3 2 2 1	3 3 3 2 2 1
1 1	2 2 1 1
T	$\tilde{T}$

This sets up a one-to-one correspondence between the terms of $s_{\tilde{\lambda}}(x_1,\ldots,x_n)$ and $(x_1 \ldots x_n)^{\lambda_1} s_\lambda(1/x_1,\ldots,1/x_n)$, so (11) follows.

Next consider the product

$$(1-q)^{-n+1}\prod_{i\neq j}(1-qx_ix_j^{-1})^{-1} = \sum_{\lambda=(\lambda_1,\ldots,\lambda_{n-1})} P_\lambda(q)s_\lambda(x), \tag{12}$$

where $P_\lambda(q)$ is a formal power series in q. It follows from (5) and (8) that $F_G(q) = P_\emptyset(q)$. In order to expand the left-hand side of (12), we begin with the identity

$$\prod_{i,j=1}^{n} (1-x_iy_j)^{-1} = \sum_{\lambda=(\lambda_1,\ldots,\lambda_n)} s_\lambda(x)s_\lambda(y), \qquad (13)$$

which can be given an elegant combinatorial proof (Knuth 1970, p.726; Stanley 1971, Cor.7.2) similar to that of (9). Make the substitution $y_j \to qx_j^{-1}$ in (13). We obtain

$$\begin{aligned}(1-q)^{-n} \prod_{i\neq j} (1-qx_ix_j^{-1})^{-1} &= \sum_\lambda q^{|\lambda|} s_\lambda(x)s_\lambda(1/x) \\ &= \sum_\lambda q^{|\lambda|} s_\lambda(x)s_{\tilde{\lambda}}(x) \quad \text{(by (11))}. \qquad (14)\end{aligned}$$

(This is similar to Macdonald (1979), ex.5, p.37.) We now appeal to the Littlewood-Richardson rule (Macdonald 1979, I.9) for multiplying Schur functions. It is easy to deduce from this rule (see Stanley 1977, Thm.3.4) that for any partitions μ and ν into $\leq n$ parts, when we expand $s_\mu(x)s_\nu(x)$ in terms of $s_\rho(x)$ for $\ell(\rho) \leq n-1$ (working modulo $x_1\cdots x_n=1$ as always), the coefficient $s_\emptyset(x)$ is $\delta_{\mu\tilde{\nu}}$. In particular, the coefficient of $s_\emptyset(x)$ in $s_\lambda(x)s_{\tilde{\lambda}}(x)$ is one. Thus from (14) we obtain

$$\begin{aligned}(1-q)^{-1}P_\emptyset(q) &= \sum_{\lambda=(\lambda_1,\ldots,\lambda_n)} q^{|\lambda|} \\ &= 1/(1-q)(1-q^2)\cdots(1-q^n) \ ,\end{aligned}$$

so (10) follows.

An analogous argument applies to the case $G = GL(n,\mathbb{C})$ and $R = \Lambda(V) = \mathbb{C} \oplus \Lambda(V) \oplus \cdots \oplus \Lambda^{n^2-1}(V)$, where once again V affords the adjoint representation. Instead of (13) we use

$$\prod_{i,j=1}^{n} (1+x_iy_j) = \sum_{\lambda=(\lambda_1,\ldots,\lambda_n)} s_\lambda(x)s_{\lambda'}(y) \ , \qquad (15)$$

where λ' denotes the conjugate partition to λ (see Knuth 1970, p.726; Stanley 1971, Cor.9.2). Thus we obtain

$$(1+q)^n \prod_{i\neq j} (1+qx_ix_j^{-1}) = \sum_\lambda q^{|\lambda|} s_\lambda(x)s_{\tilde{\lambda}'}(x) \ .$$

The coefficient of $s_\emptyset(x)$ in $s_\lambda(x)s_{\tilde{\lambda}'}(x)$ is one if $\lambda = \lambda'$ and otherwise zero. Hence in this case by (6) and (8) we have

$$(1+q)F(\Lambda(V)^G,q) = \sum_{\substack{\lambda=\lambda' \\ \lambda=(\lambda_1,\ldots,\lambda_n)}} q^{|\lambda|} . \tag{16}$$

It is well-known (and easy to prove combinatorially - see Hardy & Wright (1960), pp.278-9) that the number of self-conjugate partitions of m with $\leq n$ parts is equal to the number of partitions of m into distinct odd parts $\leq 2n-1$. Thus the right-hand side of (16) becomes $(1+q)(1+q^3)\cdots(1+q^{2n-1})$, so

$$F(\Lambda(V)^G,q) = (1+q^3)(1+q^5)\cdots(1+q^{2n-1}). \tag{17}$$

This well-known result is usually proved by much more algebraic means (e.g., Weyl 1946, p.233; Kostant 1958).

The "q-Dyson conjecture" of Andrews (1975), p.216 (see also Macdonald (1981,1982)), in the case $a_1=\cdots=a_n=k$, is equivalent to finding the coefficient of $s_{\emptyset}(x)$ in

$$\prod_{m=1}^{k} \prod_{i,j=1}^{n} (1-q^m x_i x_j^{-1}).$$

Perhaps the combinatorial techniques illustrated here will be of value in resolving the conjecture.

Late note: Our proof of formula (17) is essentially that of D. E. Littlewood (1953), On the Poincaré polynomials of the classical groups, J. London Math. Soc., 28, 494-500.

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