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Symmetric functions and Hall polynomials, by I. G. Macdonald, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1979, viii + 180 pp., $34.95.

1. Introduction. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$ be a *partition*, i.e., a (finite or infinite) sequence of nonnegative integers in decreasing order,

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant \cdots,$$

such that only finitely many of the λ_i are nonzero. The number of nonzero λ_i is called the *length* of λ, denoted $l(\lambda)$. If $\Sigma\, \lambda_i = n$, then λ is called a *partition of weight n*, denoted $|\lambda| = n$. Also write $n(\lambda) = \Sigma(i-1)\lambda_i$. Now let G be a finite abelian p-group of type λ, i.e., a direct product of cyclic groups of orders $p^{\lambda_1}, p^{\lambda_2}, \ldots, p^{\lambda_r}, \ldots,$ where p is a prime. If μ and ν are also partitions, then define $g^{\lambda}_{\mu\nu}(p)$ to be the number of subgroups H of G of type μ for which the quotient group G/H is of type ν. (Of course $g^{\lambda}_{\mu\nu}(p) = 0$ unless $|\lambda| = |\mu| + |\nu|$.) $g^{\lambda}_{\mu\nu}(p)$ is a polynomial function of p, called the *Hall polynomial*. Presented in this way, Hall polynomials appear to be of rather limited interest, of use only in dealing with enumerative properties of finite abelian groups. It is remarkable that Hall polynomials occur in many other contexts. The present book contains the first systematic account of their properties (except for the brief summary [32] which is based on some notes of Macdonald which eventually became the book under review).

2. Symmetric functions. The primary reason for the ubiquity of Hall polynomials lies in their close connection with symmetric functions. For this reason the author devotes about half of his book (Chapter I) to the theory of symmetric functions, without reference to Hall polynomials. Just this one chapter is a valuable source of information for anyone working in such fields as combinatorics, algebraic geometry, and representation theory, which frequently impinge on the theory of symmetric functions. Let us elaborate on why mathematicians in these areas should be interested in symmetric functions. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, define the *monomial symmetric function* $m_\lambda = m_\lambda(x)$ to be the formal power series in the infinite set of variables $x = (x_1, x_2, \ldots)$ given by $m_\lambda = \Sigma\, x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, summed over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \ldots)$ of $\lambda = (\lambda_1, \lambda_2, \ldots)$. Let Λ^k be the $\mathbf{Z}$-module spanned (in fact, freely generated) by all m_λ with $|\lambda| = k$, so rank $\Lambda^k = p(k)$, the number of partitions of k. Let $\Lambda = \bigoplus_{k \geq 0} \Lambda^k$. Thus Λ is the free $\mathbf{Z}$-module generated by the m_λ for *all* partitions λ; and Λ has an obvious structure of a graded ring, called the *ring of symmetric functions*. (Macdonald gives a somewhat fancier definition of Λ based on inverse limits.) Chapter I is essentially concerned with the properties of certain bases for Λ^k and the transition matrices between them. This linear algebra approach toward symmetric functions is due to P. Hall [17], and is further developed in [7] and [39]. In addition to the m_λ, there are three other bases of Λ with straightforward definitions, viz., the elementary symmetric functions e_λ, the complete symmetric functions h_λ, and the power sum symmetric functions p_λ. Then there is a less obvious but fundamentally important fifth basis, the *Schur functions* s_λ, which may be defined as follows. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\leqslant n$, define a polynomial $s_\lambda(x_1, \ldots, x_n)$ by

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1 \leqslant i,j \leqslant n}}{\det\left(x_i^{n-j}\right)_{1 \leqslant i,j \leqslant n}}.$$

It is easily seen that $s_\lambda(x_1, \ldots, x_n, 0) = s_\lambda(x_1, \ldots, x_n)$. Hence we can define s_λ to be the unique element of Λ such that for all $n \geqslant l(\lambda)$ we have $s_\lambda(x_1, \ldots, x_n, 0, 0, \ldots) = s_\lambda(x_1, \ldots, x_n)$. (It is interesting to note that the standard tables [**5**] of symmetric functions completely ignore the Schur functions. This gap is remedied by the valuable tables [**20**], [**21**], [**45**, Appendix] and [**4**], among others.) Macdonald also mentions briefly (p. 15) the "forgotten" symmetric functions f_λ of P. Doubilet [**7**]. While his statement that the f_λ "have no particularly simple direct description" cannot be faulted, it should be mentioned that the transition matrices $M(f, m)$ and $M(f, p)$ between the f_λ and the m_λ and p_λ do have simple direct descriptions [**7**, Theorem 8].

Two especially noteworthy topics covered in Chapter I are plethysm (§8) and the Littlewood-Richardson rule (§9). The standard treatment of plethysm in [**29**, Appendix] will not appeal to modern algebraists, and the clear presentation of Macdonald was sorely needed. Plethysm is a certain binary operation on symmetric functions, denoted $\circ$ by Macdonald but more usually $\otimes$. (Macdonald's notation is more logical, since plethysm is a kind of functional composition and does not behave at all like a tensor product.) Macdonald discusses the relationship of plethysm to both the wreath product of symmetric groups and the representation theory of $GL_n(\mathbf{C})$ (pp. 66 and 82). The latter relationship is the easier of the two to understand and will be briefly discussed in the next section.

The Littlewood-Richardson rule, first stated in 1934, is a combinatorial description of the coefficients $c_{\mu\nu}^\lambda$ defined by $s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda$. In view of the importance of this result (e.g., for computing the Clebsch-Gordan series for unitary groups) it is surprising that complete proofs were published only recently, due to Lascoux and Schützenberger [**37**], Thomas [**44**], and now Macdonald.

Let us also mention that Chapter I contains (on pp. 7–9 and various subsequent examples) an exposition of the basic formal properties of the raising operators R_{ij} of Alfred Young. For a discussion of the more subtle combinatorial properties of these operators, see [**14**]. There is also a brief but useful dictionary (Chapter I, (2.15)) for translating formulas and identities involving symmetric functions into the language of λ-rings. See [**19**] for further information on λ-rings.

Regrettably the author has decided to omit ("partly from a desire to keep the size of this monograph within reasonable bounds") the more combinatorial approach to symmetric functions as exemplified by the fundamental one-to-one correspondence associated with the names Robinson, Schensted, Knuth, Schützenberger, ... (e.g., **[12]**, **[39]**) and the closely related "jeu de taquin" of Schützenberger **[37]** and "plactic monoid" of Lascoux and Schützenberger **[26]**–**[28]**. This combinatorial approach transparently explains many seemingly mysterious results involving Schur functions, such as the Littlewood-Richardson rule.

3. Applications of symmetric functions. Let us now consider some uses of the theory of symmetric functions.

(a) *Combinatorics.* When s_λ is expanded in terms of the m_λ's, say $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$, then the $K_{\lambda\mu}$'s (known as the *Kostka numbers*) are the number of ways of inserting μ_1 1's, μ_2 2's, ... into a Ferrers graph (or Young diagram) of shape λ such that the rows are weakly increasing and the columns strictly increasing (Chapter I, (5.13)). For instance, if $\lambda = (4, 2, 1)$ and $\mu = (3, 2, 1, 1)$, then $K_{\lambda\mu} = 4$, corresponding to the tableaux

$$\begin{array}{l} 1112 \\ 23 \\ 4 \end{array} \quad \begin{array}{l} 1112 \\ 24 \\ 3 \end{array} \quad \begin{array}{l} 1113 \\ 22 \\ 4 \end{array} \quad \begin{array}{l} 1114 \\ 22 \\ 3 \end{array}$$

Such tableaux occur in a wide variety of combinatorial situations dealing with lattice paths, ballot problems, trees, languages, partitions, etc. Moreover, the numbers $M(h, m)_{\lambda\mu}$ defined by $h_\lambda = \sum_\mu M(h, m)_{\lambda\mu} m_\mu$ satisfy $M(h, m)_{\lambda\nu} = \sum_\nu K_{\nu\lambda} K_{\nu\mu}$ and are equal to the number of matrices of nonnegative integers with row sums λ_i and column sums μ_j (Chapter I, (6.6)). These two facts should suffice to show the relevance of symmetric functions to combinatorics.

(b) *Representation theory of the symmetric group.* If $f\colon S_n \to \mathbf{Q}$ is a class function on the symmetric group S_n, then we may regard f as defined on partitions ρ of weight n, viz., f_ρ is the value of f on an element of S_n with cycles of length $\rho_1, \rho_2, \ldots$. Let $z_\rho = \prod_{i \geq 1} i^{m_i} m_i!$, where ρ has m_i parts equal to i, and define $\operatorname{ch}(f) = \sum_{|\rho| = n} z_\rho^{-1} f_\rho p_\rho \in \Lambda_{\mathbf{Q}} = \Lambda \otimes_{\mathbf{Z}} \mathbf{Q}$. The symmetric function $\operatorname{ch}(f)$ is called the *characteristic* of f. The reason for dealing with $\operatorname{ch}(f)$ rather than directly with f is that important class functions have simple characteristics, so we may apply our knowledge of symmetric functions to study class functions on S_n. For instance, if $|\lambda| = n$ then the character η_λ induced by the identity character of $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots$, which plays a prominent role in Frobenius' determination of the irreducible characters of S_n, has characteristic $\operatorname{ch}(\eta_\lambda) = h_\lambda$. The main result in Chapter I.7 essentially says that there is a natural indexing of the irreducible characters χ^λ of S_n by partitions λ of weight n such that $\operatorname{ch}(\chi^\lambda) = s_\lambda$. As an immediate consequence (Chapter I, (7.8)), it follows that the character table of S_n is the transition matrix between the power-sum symmetric functions and the Schur functions, i.e., $p_\rho = \sum_\lambda \chi^\lambda_\rho s_\lambda$. See also **[8]** and **[13]** for more combinatorial approaches toward this subject.

(c) *Representation theory of $GL_n(\mathbf{C})$ (and therefore also $SL_n(\mathbf{C})$, $U_n(\mathbf{C})$, $SU_n(\mathbf{C})$, $gl_n(\mathbf{C})$, and $sl_n(\mathbf{C})$).* Suppose that $\phi\colon GL_n(\mathbf{C}) \to GL_N(\mathbf{C})$ is a *polynomial* representation of degree N of $GL_n(\mathbf{C})$ (the group of $n \times n$ invertible matrices with entries in $\mathbf{C}$), so that the entries of $\phi(A)$ are fixed polynomials in the entries of A. For instance,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \overset{\phi}{\mapsto} \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix} \tag{1}$$

is a polynomial representation of $GL_2(\mathbf{C})$ of degree 3. Then there is a fixed symmetric polynomial $\chi_\phi(x_1, \ldots, x_n)$ such that if $\theta_1, \ldots, \theta_n$ are the eigenvalues of $A \in GL_n(\mathbf{C})$, then $\operatorname{tr} \phi(A) = \chi_\phi(\theta_1, \ldots, \theta_n)$. The representation ϕ is uniquely determined (up to equivalence) by χ_ϕ. Moreover, the polynomials χ_ϕ corresponding to *irreducible* polynomial representations of $GL_n(\mathbf{C})$ are precisely the Schur functions $s_\lambda(x_1, \ldots, x_n)$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of length $\leqslant n$. For instance, the representation (1) is irreducible, and $\chi_\phi = s_{(2,0)}(x_1, x_2)$. This theory is developed essentially in the appendix of Chapter I, in the context of polynomial functors. The reader with little background in algebra will undoubtedly regret the author's decision to develop this topic from such an abstract viewpoint.

One simple example will suffice to illustrate the utility of the symmetric function viewpoint toward the representations of $GL_n(\mathbf{C})$. A well-known identity concerning Schur functions (Chapter I, §5, Ex. 4), which can also be given an elegant combinatorial proof [**18**, Theorem 4] once the combinatorial interpretation of the Kostka numbers mentioned above is known, states that

$$\sum_\lambda s_\lambda(x_1, \ldots, x_n) = \prod_{i=1}^{n} (1 - x_i^{-1}) \cdot \prod_{1 < i < j < n} (1 - x_i x_j)^{-1}, \tag{2}$$

where λ ranges over all partitions $(\lambda_1, \ldots, \lambda_n)$ of length $< n$. But if $x_1, \ldots, x_n$ are the eigenvalues of $A \in GL(V)$, where V is an n-dimensional vector space over $\mathbf{C}$, then the right-hand side of (2) is the trace of the action of A on $S(V \oplus \Lambda^2 V)$, where S is the symmetric algebra and Λ^2 the second exterior power. It follows that in the action of $GL(V)$ on $S(V \oplus \Lambda^2 V)$, every irreducible polynomial representation of $GL(V)$ appears exactly once.

As promised above, we can now give the connection between plethysm and the representations of $GL_n(\mathbf{C})$. Let ϕ: $GL_n(\mathbf{C}) \to GL_N(\mathbf{C})$ have character $f(x_1, \ldots, x_n)$ and ψ: $GL_N(\mathbf{C}) \to GL_M(\mathbf{C})$ have character $g(x_1, \ldots, x_N)$. Then $\psi\phi$ is a representation of $GL_n(\mathbf{C})$ of degree M whose character is called the *plethysm* $f \circ g(x_1, \ldots, x_n)$ (with respect to $GL_n(\mathbf{C})$). We can expand f and g as a linear combination of Schur functions. By fixing these linear combinations and taking n sufficiently large, we get $f \circ g(x_1, \ldots, x_n, 0) = f \circ g(x_1, \ldots, x_n)$. Hence we can define the (unrestricted) plethysm $f \circ g \in \Lambda$ whenever $f, g \in \Lambda$ are nonnegative integer linear combinations of Schur functions. (There is then a natural way to define $f \circ g$ for all $f, g \in \Lambda$ which we omit–it should be noted that $(f + g) \circ h \neq f \circ h + g \circ h$.) For instance, a more detailed analysis of (2) (equivalent to I.5, Ex. 7) shows that the irreducible representations of $GL(V)$ appearing in the kth symmetric power $S^k(V \oplus \Lambda^2 V)$ are those whose character s_λ satisfies $\Sigma[\frac{1}{2}(\lambda_i' + 1)] = k$, where $\lambda' = (\lambda_1', \lambda_2', \ldots)$ is the conjugate partition to λ, i.e., λ has λ_i' parts $\geqslant i$. Since the natural representation of $GL(V)$ on $V \oplus \Lambda^2 V$ has character $s_1 + s_{11}$, and of $GL(W)$ on S^kW has character s_k, there follows

$$(s_1 + s_{11}) \circ s_k = \sum s_\lambda,$$

where the sum is over all partitions λ satisfying $\Sigma[\frac{1}{2}(\lambda_i' + 1)] = k$. Note that in the definition of plethysm we could replace $GL_n(\mathbf{C})$ by any group G and ϕ by any representation ϕ: $G \to GL_N(\mathbf{C})$. Thus one can define a more general plethysm $\phi \circ g$ where $g \in \Lambda$. This more general concept has been used by physicists; see [33] for some computations.

(d) *Algebraic geometry*. Let G_{dn} denote the Grassmann variety of all d-dimensional subspaces of an n-dimensional complex vector space. Then there is an isomorphism ρ: $\Lambda/I_{dn} \to H^*(G_{dn}, \mathbf{Z})$ between the quotient of Λ by the ideal I_{dn} generated by all Schur functions s_λ for which either $\lambda_1 > d$ or $\lambda_{n-d+1} > 0$, and the cohomology ring of G_{dn}. A $\mathbf{Z}$-basis for Λ/I_{dn} consists of those Schur functions s_λ for which $\lambda_1 \leqslant d$ and $\lambda_{n-d+1} = 0$, and the images $\rho(s_\lambda)$ of the s_λ are just the *Schubert cycles*. Thus many computations in algebraic geometry are equivalent to computations with symmetric functions, and the theory of symmetric functions can be a great aid in performing these computations. A hint of this appears in Chapter I, §3, Ex. 10, and Chapter I, §4, Ex. 5, where following [25] the Chern classes of the exterior square Λ^2E, symmetric square S^2E, and tensor product $E \otimes F$ of vector bundles E and F are computed. An extensive and systematic application of symmetric functions to algebraic geometry has been undertaken by A. Lascoux (e.g., [22]–[25]). See also [40] for further background.

4. Hall polynomials and *HL*-functions. We hope that the reader is now convinced of the usefulness of symmetric functions. This being the case, he will appreciate their connection with Hall polynomials. Chapter II develops the basic properties of Hall polynomials *per se*, without reference to symmetric functions. In this connection we cannot resist mentioning an obscure but amusing result of P. Hall [16], related to the computation in Chapter II, (1.6), of the number $a_\lambda(p)$ of automorphisms of an abelian p-group of type λ. The result states that

$$\sum_G |G|^{-1} = \sum_G |\mathrm{Aut}\ G|^{-1}$$

(as convergent infinite series), where both sums range over all nonisomorphic finite abelian p-groups G for fixed p, and where Aut G is the automorphism group of G.

Turning now to Chapter III, let t be an indeterminate over $\mathbf{Z}$ and define for a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\leqslant n$ the polynomial $v_\lambda(t) = \prod_{i>0} \prod_{j=1}^{m_i}(1 - t^j)/(1 - t)$, where m_i is the number of λ_k equal to i. Let $x_1, \ldots, x_n$ be additional indeterminates over $\mathbf{Z}$. Define

$$P_\lambda(x_1, \ldots, x_n; t) = \frac{1}{v_\lambda(t)} \sum_{w \in S_n} w\left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\right),$$

where a permutation w of $\{1, 2, \ldots, n\}$ acts on a function $f(x_1, \ldots, x_n)$ by

$wf(x_1, \ldots, x_n) = f(x_{w(1)}, \ldots, x_{w(n)})$. It is not difficult to see that $P_\lambda(x_1, \ldots, x_n; t)$ is a homogeneous symmetric polynomial in $x_1, \ldots, x_n$, of degree $|\lambda|$, with coefficients in $\mathbf{Z}[t]$ (Chapter III, §1). If λ is a partition of length $\leqslant n$, then $P_\lambda(x_1, \ldots, x_n, 0; t) = P_\lambda(x_1, \ldots, x_n; t)$ (Chapter III, (2.5)). Hence there is a unique element $P_\lambda(x; t)$ of $\Lambda[t]$ satisfying $P_\lambda(x_1, \ldots, x_n, 0, 0, \ldots; t) = P_\lambda(x_1, \ldots, x_n; t)$ for all $n \geqslant l(\lambda)$. The symmetric function $P_\lambda(x; t)$ is the *Hall-Littlewood function* or *HL-function* corresponding to the partition λ. It is homogeneous of degree $|\lambda|$. The P_λ serve to interpolate between the Schur functions s_λ and the monomial symmetric functions m_λ, because (Chapter III, (2.3) and (2.4))

$$P_\lambda(x_1, \ldots, x_n; 0) = s_\lambda(x_1, \ldots, x_n), \qquad P_\lambda(x_1, \ldots, x_n; 1) = m_\lambda(x_1, \ldots, x_n).$$

It can also be seen without difficulty that the symmetric functions $P_\lambda(x; t)$ form a $\mathbf{Z}[t]$-basis of $\Lambda[t]$ (Chapter III, (2.7)). Hence the product $P_\mu P_\nu$ of two HL-functions will be a linear combination of the P_λ, where $|\lambda| = |\mu| + |\nu|$. Thus there are polynomials $f^\lambda_{\mu\nu} \in \mathbf{Z}[t]$ such that

$$P_\mu(x; t)P_\nu(x; t) = \sum_\lambda f^\lambda_{\mu\nu}(t)P_\lambda(x; t).$$

A basic result on the Hall polynomials $g^\lambda_{\mu\nu}(t)$ (Chapter III, (3.6)) states that they are the polynomials $f^\lambda_{\mu\nu}(t)$ with coefficients "reversed", i.e.,

$$g^\lambda_{\mu\nu}(t) = t^{n(\lambda) - n(\mu) - n(\nu)} f^\lambda_{\mu\nu}(t^{-1}).$$

Chapter III goes on to develop the basic formal properties of the HL-functions. These include (a) an orthogonality property which plays a key role in the later determination of the irreducible characters of the finite group $GL_n(q)$, (b) the new concept of skew HL-functions, (c) a statement (Chapter III, (6.5)) of the recent remarkable theorem of Lascoux and Schützenberger **[26]** which gives a combinatorial interpretation of the coefficients of the polynomial $K_{\lambda\mu}(t)$ defined by

$$s_\lambda(x) = \sum_\mu K_{\lambda\mu}(t)P_\mu(x; t),$$

and (d) a discussion of the *Green's polynomials* $Q^\lambda_\rho(q) = q^{n(\lambda)}X^\lambda_\rho(q^{-1})$, where $X^\lambda_\rho(t)$ is defined by

$$p_\rho(x) = \sum_\lambda X^\lambda_\rho(t)P_\lambda(x; t).$$

5. Applications of *HL*-functions. The final two chapters represent the culmination of the previously developed theory of HL-functions. Chapter IV is devoted to J. A. Green's theory **[15]** of the characters of GL_n over a finite field, while Chapter V concerns the analogous theory of spherical functions on general linear groups over a nonarchimedean local field. While these two chapters will not be of such general interest as the preceding material, they are essential for understanding the deeper significance of HL-functions.

The details of the computation of the characters of $GL_n(q)$ are rather technical, and indeed even the statement of the main result (Chapter IV, (6.8)) would require too much background for us to include here. Perhaps, however, we can convey some of the flavor of the subject by discussing some very special characters $\chi^{\mu}(q)$ of $GL_n(q)$, first studied by Steinberg [**43**]. (These characters are not mentioned as such by Macdonald, since they play no special role in the theory as he develops it.) Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of weight n and length l. If V is an n-dimensional vector space over $GF(q)$, then the group $GL_n(q)$ acts on the set of all flags $O = V_0 \subset V_1 \subset \cdots \subset V_l = V$ of subspaces of V satisfying $\dim V_i - \dim V_{i-1} = \lambda_i$. This permutation representation is just the induction of the trivial representation from the (parabolic) subgroup P of $GL_n(q)$ fixing one such flag. There then exist irreducible characters $\chi^{\mu}(q)$ of $GL_n(q)$ such that the character $\eta_\lambda(q)$ of the above action decomposes as

$$\eta_\lambda(q) = \sum_{\mu} K_{\lambda\mu}\chi^{\mu}(q).$$

Here $K_{\lambda\mu}$ is the familiar Kostka number. Moreover, the value of the character $\chi^{\lambda}(q)$ at a unipotent element of type μ (i.e., an element of $GL_n(q)$ with eigenvalues 1 and Jordan block sizes $\mu_1, \mu_2, \ldots$) is given by $q^{n(\mu)-n(\lambda)}K_{\lambda\mu}(q^{-1})$. Though we have given only the briefest glimpse into the character theory of $GL_n(q)$, enough was visible to suggest that an important role is played by (a) induction from parabolic subgroups of $GL_n(q)$, and (b) symmetric functions. The reader who assiduously follows Chapter IV will find it much easier to appreciate the deep theory of representations of finite Chevalley groups as developed by Deligne and Lusztig [**6**], and subsequently greatly extended by Lusztig and others.

Chapter V is also about general linear groups, but this time over a nonarchimedean local field F rather than a finite field. Instead of computing characters, Macdonald computes spherical functions. He uses this result to compute the Hecke series and zeta functions for $GL_n(F)$ and the group $GSp_{2n}(F)$ of symplectic similitudes. The computation of the Hecke series for $GSp_{2n}(F)$ completes a calculation started by Satake [**36**], and knowledge of Satake's work is essential for understanding the details of Macdonald's computation.

6. Window-shopping. The value of Macdonald's book is greatly enhanced by the "examples" at the end of many sections. We will discuss a few of them we found especially interesting or enlightening, or for which we can perhaps add some insight.

(a) (Chapter I, §2, Ex. 10) Let l be a prime number and write $n = a_0 + a_1 l + a_2 l^2 + \cdots = a_0 + n_1 l$, with $0 \leqslant a_i \leqslant l - 1$ for all $i \geqslant 0$. Then the number $\mu_l(S_n)$ of conjugacy classes in S_n of order prime to l is equal to $p(a_0)\Pi_{i\geqslant 1}(a_i + 1)$, where $p(a_0)$ is the number of partitions of a_0. In particular, the number of conjugacy classes in S_n of odd order is 2^r, where $[n/2]$ has r ones in its binary expansion. (Macdonald inadvertently defines r incorrectly.) More generally, the author has shown [**30**] that the number of conjugacy classes of odd order in a finite Coxeter group is a power of 2. These results seem to be new, though it should be mentioned that it is not difficult to show directly that the conjugacy class of cycle type $\rho = (\rho_1, \rho_2, \ldots)$ in S_n has order prime to l if and only if ρ has a certain number k of its parts equal to l and at least $(n_1 - k)l$ parts equal to 1, where $\binom{n_1}{k}$ is prime to l. It is well known that the number of binomial coefficients $\binom{n_1}{k}$ prime to l is $\Pi_{i \geqslant 1}(a_i + 1)$. Since the unspecified parts of ρ can form an arbitrary partition of a_0, the value of $\mu_l(S_n)$ follows.

(b) (Chapter I, §3, Ex. 6) This example evaluates $\det(p(i - j + 1))_{1 \leqslant i,j \leqslant n}$, where $p(k)$ is the number of partitions of k (with $p(-k) = 0$ for $k > 0$). This mysterious-looking result becomes quite transparent when one realizes that if f is any function defined on $\mathbf{Z}$ (with values in a commutative ring with identity) satisfying $f(0) = 1$ and $f(-k) = 0$ for $k > 0$, then

$$\det(f(i - j + 1))_{1 \leqslant i,j \leqslant n}$$

is the coefficient of x^n in the power series $(\Sigma_{k \geqslant 0}(-1)^k f(k)x^k)^{-1}$.

(c) (Chapter I, §3, Ex. 7) Let δ be the partition $(n - 1, n - 2, \ldots, 1, 0)$. Then $s_{(m-1)\delta}(x_1, \ldots, x_n) = \Pi_{1 \leqslant i < j \leqslant n}(x_i^{m-1} + x_i^{m-2}x_j + \cdots + x_j^{m-1})$. (There is a rare misprint in the statement of this result.) This formula has the following combinatorial interpretation: Let K_n^m be the graph on the vertex set $\{1, 2, \ldots, n\}$ with exactly m indistinguishable edges between any two distinct vertices. Given a partition $\mu = (\mu_1, \ldots, \mu_n)$ of weight $m\binom{n}{2}$ and length $\leqslant n$, let $t(\mu)$ be the number of ways of orienting the edges of K_n^m so that vertex i has outdegree μ_i, for all $1 \leqslant i \leqslant n$. Then $t(\mu)$ is the Kostka number $K_{m\delta,\mu}$. A purely combinatorial proof of this result appears in **[47]**.

(d) (Chapter I, §5, Ex. 3) For each symmetric function $f \in \Lambda$, let $D(f)$: $\Lambda \to \Lambda$ be the adjoint of multiplication by f with respect to the inner product defined by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$, for all partitions λ and μ, where $\delta_{\lambda\mu}$ is the Kronecker delta. Thus by definition $\langle D(f)u, v \rangle = \langle u, fv \rangle$. (This inner product plays a central role (Chapter I, §4) in the theory of symmetric functions; for instance, the formula $M(h, m)_{\lambda\mu} = \Sigma_\nu K_{\nu\lambda}K_{\nu\mu}$ mentioned earlier is equivalent to the statement that the Schur functions s_λ with $|\lambda| = n$ form an orthonormal basis for Λ^n.) If we write $f \in \Lambda$ as a polynomial $\phi(p_1, p_2, \ldots)$ with rational coefficients, then $D(f) = \phi(\partial/\partial p_1, 2\partial/\partial p_2, 3\partial/\partial p_3, \ldots)$. In particular, $D(e_n)$ and $D(h_n)$ are differential operators introduced by Hammond, and $D(s_\lambda)$ is Foulkes' generalization of the Hammond operators.

(e) (Chapter I, §5, Ex. 9) The three products $\Pi_{i<j}(1 - x_ix_j)$, $\Pi_i(1 - x_i)\Pi_{i<j}(1 - x_ix_j)$, $\Pi_i(1 - x_i^2)\Pi_{i<j}(1 - x_ix_j)$ were first expanded in terms of Schur functions by Littlewood **[29**, p. 238]. Here Macdonald gives a novel proof based on Weyl's identity for the root-systems of types D_n, B_n, C_n, respectively.

(f) (Chapter I, §5, Ex. 13–19) These exercises are devoted to the enumeration of plane partitions. Let $\mathbf{P}$ denote the positive integers. A *plane partition* π of r (written $|\pi| = r$) may be regarded as an r-element subset of $\mathbf{P}^3$ such that if $(i, j, k) \in \pi$ and $(1, 1, 1) \leqslant (i', j', k') \leqslant (i, j, k)$, then $(i', j', k') \in \pi$. It is straightforward to deduce from the basic properties of symmetric functions an elegant formula for the generating function $F(q) = \Sigma_\pi q^{|\pi|}$, where π ranges over all plane partitions contained in the $l \times m \times n$ brick $B = \{(i, j, k)$: $1 \leqslant i \leqslant l, 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant n\}$. MacMahon **[31**, §520] conjectured a simple formula for $G(q) = \Sigma_\sigma q^{|\sigma|}$, where σ ranges over all plane partitions contained in B invariant under the group of order 2 which interchanges the first two coordinates. Macdonald gives a beautiful proof of this conjecture using properties of the root systems A_{n-1} and B_n. (In effect, he is determining the decomposition of the complex Lie algebra $gl_n \subset so_{2n+1}$ in the representation of so_{2n+1} of highest weight $l\lambda_n$, where λ_n is the highest weight of the spin representation of so_{2n+1}.) This conjecture was independently proved by Andrews [**1**]. Recently R. Proctor and the reviewer [**34**] have shown, using Seshadri's standard monomial theory [**38**], that elegant generating functions for counting certain types of plane partitions arise in a simple way from any minuscule representation of a complex semisimple Lie algebra $\mathfrak{g}$. (A representation of $\mathfrak{g}$ is *minuscule* if the Weyl group of $\mathfrak{g}$ acts transitively on the weights.) In particular, the fundamental representations of sl_n lead to $F(q)$, and the spin representation of so_{2n+1} to $G(q)$. (Unfortunately, these are the only infinite classes of minuscule representations which yield interesting combinatorial results.)

Macdonald goes on to consider plane partitions $\pi \subset B$ invariant under the group of order three which cyclically permutes the coordinates. Here he makes an extraordinary conjecture about the generating function $H(q)$ for such plane partitions, recently proved for $q = 1$ by Andrews [**2**]. If I had to single out the most interesting open problem in all of enumerative combinatorics, this would be it. It is natural to consider the full symmetric group S_3 acting on the coordinates, but surprisingly there apparently is no elegant generating function in this case. However, the special case $q = 1$ still seems to be well-behaved [**3**].

(g) (Chapter I, §7, Ex. 3) This exercise is concerned with the interesting "skew characters" $\chi^{\lambda/\mu}$ of S_n, whose characteristics $\mathrm{ch}(\chi^{\lambda/\mu})$ are the skew-Schur functions $s_{\lambda/\mu}$. Some work of Foulkes [**10**], [**11**] carries the combinatorial aspects of this theory further.

(h) (Chapter I, §8, Ex. 4) Given a partition λ of length $\leqslant n$, consider the polynomial $s_\lambda(q^{n-1}, q^{n-2}, \ldots, 1) = q^{n(\lambda)}(a_0 + a_1 q + \cdots + a_d q^d)$, where $a_d \neq 0$. In particular, if $\lambda = (r)$ then $s_{(r)}(q^{n-1}, q^{n-2}, \ldots, 1) = \left[{n+r-1 \atop r}\right]$, the well-known q-binomial coefficient

$$(1 - q^{n+r-1})(1 - q^{n+r-2}) \ldots (1 - q^n)/ (1 - q^r)(1 - q^{r-1}) \ldots (1 - q).$$

It is easily seen that $a_i = a_{d-i}$. Using basic properties of plethysm and polynomial functors Macdonald shows that $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{[d/2]}$ i.e., that the polynomial $s_\lambda(q^{n-1}, q^{n-2}, \ldots, 1)$ is *unimodal*. The proof is actually an elegantly disguised version of a result of Dynkin [**9**, Theorem 0.15, p. 332] applied to the special case $SL_n(\mathbf{C})$. It boils down to defining a certain subgroup G of $SL_n(\mathbf{C})$ isomorphic to $SL_2(\mathbf{C})$ (the so-called "principal *TDS*"), and computing the decomposition of G in the irreducible representation of $SL_n(\mathbf{C})$ with character s_λ. Dynkin's result is stated in an elementary fashion in [**41**]. Recently a completely elementary (but noncombinatorial) proof of the unimodality of the coefficients of $\left[{n+r-1 \atop r}\right]$ has been given by R. Proctor [**35**] and independently by the reviewer [**42**].

(i) (Chapter III, §7, Ex. 7, 8) These examples consider the effect (from a formal viewpoint only) of letting t be a primitive rth root of unity in the HL-function $Q_\lambda(x; t)$. I would have liked to see some discussion of the relationship of the case $t = -1$ to the projective (spin) characters of S_n, and the case where r is prime to the modular representations of S_n. See [**32**, §§3 and 4] for further details.

7. Conclusion. This has been a rather long review of a rather short and seemingly specialized book. I feel such a review is justified because the subject matter of the book deserves to be much more widely known. Despite the amount of material of such great potential interest to mathematicians in so many diverse areas (to say nothing of physicists, chemists, et al., who deal with group representations), the theory of symmetric functions remains all but unknown to the persons it is most likely to benefit. (One notable exception consists of a certain group of physicists, as exemplified by [**46**, Appendix] and the references therein.) Hopefully this beautifully written book will put an end to this state of affairs. Probably the biggest obstacle for many readers will be the rather sophisticated use of algebra, comparable say to Lang's *Algebra*. There is also a somewhat skimpy list of references, which we have tried to extend a little in this review. I have no doubt, however, that this book will become the definitive reference on symmetric functions and their applications. Perhaps the publisher will see fit to introduce a lower priced edition, analogous to the Springer "Study Editions". Otherwise, at close to 20 cents a page, the book may not obtain the distribution it so richly deserves.

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