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FACTORIZATION OF PERMUTATIONS INTO n-CYCLES*

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Using the character theory of the symmetric group $\mathfrak{S}_n$, an explicit formula is derived for the number $g_k(\pi)$ of ways of writing a permutation $\pi \in \mathfrak{S}_n$ as a product of k n-cycles. From this the asymptotic expansion for $g_k(\pi)$ is derived, provided that when $k=2$, π has $O(\log n)$ fixed points. In particular, there follows a conjecture of Walkup that if $\pi_n \in \mathfrak{S}_n$ is an even permutation with no fixed points, then $\lim_{n\to\infty} g_2(\pi_n)/(n-2)! = 2$.

1. Introduction

Let π be an element of the symmetric group $\mathfrak{S}_n$ of all permutations of an n-element set. Let $g_k(\pi)$ be the number of k-tuples $(\sigma_1, \ldots, \sigma_k)$ of cycles σ_i of length n such that $\pi = \sigma_1 \cdots \sigma_k$. Thus $g_k(\pi) = 0$ if either

(a) π is an odd permutation and n is an odd integer, or

(b) π is odd, n is even, and k is even, or

(c) π is even, n is even, and k is odd.

Husemoller [6, Proposition 4] attributes to Gleason the result that $g_2(\pi) > 0$ for any even π. The function $g_2(\pi)$ was subsequently considered in [1, 2, 9]. In particular, Walkup [9, p. 316] conjectured that $\lim_{n\to\infty} g_2(\pi_n)/(n-2)! = 2$ where $\pi_1, \pi_2, \ldots$ is any sequence of even permutations without fixed points, with $\pi_n \in \mathfrak{S}_n$. We will use the character theory of $\mathfrak{S}_n$ to derive an explicit expression for $g_k(\pi)$ from which Walkup's conjecture can be deduced. More generally, we can write down the entire asymptotic expansion of the function $g_k(\pi)$ for fixed k (provided the number of fixed points of π remains small when $k=2$). The technique of character theory was also used in [1, Section 3], and some special cases of our results overlap with this paper. In [2, Corollary 4.8] an explicit expression for $g_2(\pi)$ is derived, which is simpler than ours, and which can also be used to prove Walkup's conjecture. I am grateful to the referee for calling my attention to [2].

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2. Character theory

We first review the results from character theory that we will need. Let G be any finite group and $\mathbb{C}G$ its group algebra over $\mathbb{C}$. If C_i, $1\leq i\leq t$, is a conjugacy class of G, then let $K_i=\sum_{g\in C_i} g$ be the corresponding element of $\mathbb{C}G$. If $\chi^1,\ldots,\chi^t$ are the irreducible (ordinary) characters of G with $\deg \chi^i=f^i$, then the elements

$$F_i=\frac{f^i}{|G|}\sum_{i=1}^{t}\chi_i^i K_i,\quad 1\leq j\leq t, \tag{1}$$

are a set of orthogonal idempotents in the center of $\mathbb{C}G$, where χ_i^i denotes χ^i evaluated at any element of C_i. Inverting (1) yields

$$K_j=|C_j|\sum_{i=1}^{t}\frac{\overline{\chi_j^i}}{f^i}F_i, \tag{2}$$

where $|C_j|$ is the number of elements of the class C_j. See, e.g., [3, Section 236]. Since the F_i's are orthogonal idempotents, we have for any integer $k\geq 1$,

$$K_j^k=|C_j|^k\sum_{i=1}^{t}\left[\frac{\overline{\chi_j^i}}{f^i}\right]^k F_i=|C_j|^k\sum_{i=1}^{t}\left[\frac{\overline{\chi_j^i}}{f^i}\right]^k\frac{f^i}{|G|}\sum_{l=1}^{t}\chi_l^i K_l$$

$$=\frac{|C_j|^k}{|G|}\sum_{l=1}^{t}K_l\sum_{i=1}^{t}\left[\frac{\overline{\chi_j^i}}{f^i}\right]^k f^i\chi_l^i. \tag{3}$$

Now let $G=\mathfrak{S}_n$. A *partition of* n may be regarded as a sequence $\rho=\langle a_1,\ldots,a_n\rangle$ of non-negative integers such that $\sum ia_i=n$. We then write $\rho\vdash n$. We also write $\rho=(1^{a_1},2^{a_2},\ldots,n^{a_n})$ where terms i^{a_i} with $a_i=0$ are omitted and where exponents $a_i=1$ are omitted. For instance, $\langle 0,1,0,0,2\rangle=(2,5^2)$ is a partition of 12. For later convenience we also write $(1^{n-1},1)$ for the partition $(1^n)=\langle n,0,\ldots,0\rangle$, and we set $\lambda_i=(1^i,n-i)$ for $0\leq i\leq n-1$. If $\rho=\langle a_1,\ldots,a_n\rangle\vdash n$, then the set of elements of $\mathfrak{S}_n$ with a_i cycles of length i forms a conjugacy class C_ρ of $\mathfrak{S}_n$. The class $C_{(n)}$ of n-cycles is abbreviated C_n, so $|C_n|=(n-1)!$. If $\phi:\mathfrak{S}_n\to\mathbb{C}$ is constant on conjugacy classes and if $\pi\in C_\rho$, then we write interchangeably $\phi(\pi)$ or $\phi(\rho)$ or $\phi(C_\rho)$. Note in particular that $g_k(\pi)$ has this property, so $g_k(\rho)$ denotes $g_k(\pi)$ for any $\pi\in C_\rho$. Recall that for each partition λ of n there is a natural way of associating an irreducible character χ^λ of $\mathfrak{S}_n$ [5, Chapter 7; 7, Chapter 5]. In particular, the partition (n) corresponds to the trivial character $\chi_\rho^n=1$ for all $\rho\vdash n$.

We next state two crucial lemmas involving the characters χ^λ. A proof of Lemma 2.1 is an immediate consequence of the 'graphical method' for determining the characters of $\mathfrak{S}_n$ [5, Chapter 7.4; 7, Chapter 5.3; 8, Chapter 4]. See [5, p. 205; 8, Lemma 4.11] in particular. A proof of Lemma 2.2 essentially appears in [7, p. 139].

Lemma 2.1. *Let* $0 \leq i \leq n-1$ *and* $\lambda \vdash n$. *Then*

$$\chi_n^\lambda = \begin{cases} (-1)^i, & \text{if } \lambda = \lambda_i = (1^i, n-i), \\ 0, & \text{otherwise,} \end{cases}$$

where χ_n^λ *is the value of the character* χ^λ *at any element of* C_n.

Lemma 2.2. *Let* $0 \leq i \leq n-1$ *and* $\rho = \langle a_1, a_2, \ldots, a_n \rangle \vdash n$. *Then*

$$\chi_\rho^{\lambda_i} = \sum \binom{a_1 - 1}{r_1}\binom{a_2}{r_2}\binom{a_3}{r_3} \cdots \binom{a_i}{r_i}(-1)^{r_2+r_4+r_6+\cdots},$$

where the sum is over all partitions $\langle r_1, r_2, \ldots, r_i \rangle$ *of i. In particular,* $\deg \chi^{\lambda_i} = f^{\lambda} = \binom{n-1}{i}$.

3. A formula for $g_k(\pi)$

It is now easy to give a formula for $g_k(\pi)$.

Theorem 3.1. *Let* $\rho = \langle a_1, \ldots, a_n \rangle \vdash n$. *Then*

$$g_k(\rho) = \frac{(n-1)!^{k-1}}{n} \sum_{i=0}^{n-1} \frac{(-1)^{ik}}{\binom{n-1}{i}^{k-1}} \sum_{\langle r_1, \ldots, r_i \rangle} \binom{a_1 - 1}{r_1}\binom{a_2}{r_2}\binom{a_3}{r_3} \cdots \binom{a_i}{r_i}(-1)^{r_2+r_4+r_6+\cdots}$$

where $\langle r_1, \ldots, r_i \rangle$ *ranges over all solutions in non-negative integers to* $\sum jr_j = i$.

Proof. As above, let C_n denote the class of n-cycles in $\mathfrak{S}_n$ and $K_n = \sum_{\pi \in C_n} \pi \in \mathbb{C}\mathfrak{S}_n$. By definition of $\mathbb{C}\mathfrak{S}_n$, we have

$$K_n^k = \sum_{\rho \vdash n} g_k(\rho) K_\rho.$$

Hence by (3) and the fact that the characters of $\mathfrak{S}_n$ are real, there follows

$$g_k(\rho) = \frac{(n-1)!^k}{n!} \sum_{\mu \vdash n} \left(\frac{\chi_n^\mu}{f^\mu}\right)^k f^\mu \chi_\rho^\mu.$$

Then by Lemma 2.1,

$$g_k(\rho) = \frac{(n-1)!^{k-1}}{n} \sum_{i=0}^{n} \left(\frac{\chi_n^{\lambda_i}}{f^{\lambda_i}}\right)^k f^{\lambda_i} \chi_\rho^{\lambda_i} = \frac{(n-1)!^{k-1}}{n} \sum_{i=0}^{n} \frac{(-1)^{ik} \chi_\rho^{\lambda_i}}{(f^{\lambda_i})^{k-1}}.$$

Substituting the values $\chi_\rho^{\lambda_i}$ and f^{λ_i} from Lemma 2.2 completes the proof.

Some special cases of Theorem 3.1 are particularly simple. Putting $\rho=(1^n)$ yields

$$g_k(1^n)=\frac{(n-1)!^{k-1}}{n}\sum_{i=0}^{n-1}(-1)^{ik}\binom{n-1}{i}^{-(k-2)}, \tag{4}$$

the number of ways of writing the identity permutation in $\mathfrak{S}_n$ as a product of k n-cycles. When $k=3$, the sum (4) can be evaluated [4, (2.1); 1, Section 3(ii)]. Namely,

$$g_3(1^n)=\begin{cases}0, & n \text{ even},\\ 2(n-1)!^2/(n+1), & n \text{ odd}.\end{cases}$$

A more combinatorial proof of (5) is essentially given in [1, Corollary 2.2]. It is also clear that $(n-1)!g_k(C_n)=g_{k+1}(1^n)$, since $\pi_1\cdots\pi_k\in C_n$ if and only if there is a (unique) $\pi_{k+1}\in C_n$ satisfying $\pi_1\cdots\pi_k\pi_{k+1}=\varepsilon$. Hence

$$g_k(C_n)=\frac{(n-1)!^{k-1}}{n}\sum_{i=0}^{n-1}(-1)^{i(k+1)}\binom{n-1}{i}^{-(k-1)},$$

$$g_2(C_n)=2(n-1)!/(n+1),\quad n \text{ odd}. \tag{6}$$

This same formula is obtained by setting $\rho=(n)$ in Theorem 3.1. More generally, we have

$$g_k(1^{n-j},j)=\frac{(n-1)!^{k-1}}{n}\sum_{i=0}^{n-1}\frac{\left[\binom{n-j-1}{i}-(-1)^j\binom{n-j-1}{i-j}\right](-1)^{ik}}{\binom{n-1}{i}^{k-1}},$$

for $2\leqslant j\leqslant n$, where we set $\binom{n-j-1}{i-j}=0$ if $i<j$.

As a further special case, if $n=mj+1$, then from Theorem 3.1 we obtain

$$g_k(1,j^m)=\frac{(n-1)!^{k-1}}{n}\sum_{i=0}^{m}\frac{(-1)^{ijk}\binom{m}{i}}{\binom{n-1}{ij}^{k-1}}.$$

In particular, when $m=1$ we get $g_k(1,n-1)=2(n-1)!^{k-1}/n$. Walkup [9, Theorem 1] gives a combinatorial proof that $ng_2(1^{a_1},2^{a_2},\ldots,n^{a_n})=g_2(1^{a_1+1},2^{a_2},\ldots,n^{a_n})$. Thus from $g_2(1,n-1)=2(n-1)!/n$ we get another proof of (6). In effect, we have another proof of the identity [4, (2.1)]. Some other explicit values of $g_2(\rho)$ appear in [1, Corollary 2.2; 2, Example 4.9] and can be deduced from Theorem 3.1 using the appropriate binomial coefficient identity.

4. Asymptotics

We now derive an asymptotic expansion for $g_k(\rho)$, where $\rho = \langle a_1, a_2, \ldots, a_n \rangle$. When $k = 2$, it will be necessary to assume that a_1 is not too large. First we dispose of the easy case $k \geq 3$.

Theorem 4.1. *Fix $k \geq 3$. Let $\rho = \langle a_1, \ldots, a_n \rangle \vdash n$. If $(n-1)k + a_2 + a_4 + \cdots$ is odd, then $g_k(\rho) = 0$. If $(n-1)k + a_2 + a_4 + \cdots$ is even, then for any fixed $j \geq 0$ we have*

$$g_k(\rho) = \frac{2(n-1)!^{k-1}}{n} \left[\sum_{i=0}^{j} \frac{(-1)^{ik} \chi_\rho^{\lambda_i}}{\binom{n-1}{i}^{k-1}} + O(n^{-(j+1)(k-2)}) \right],$$

uniformly in $a_1, a_2, \ldots, a_n$ and $n = \sum i a_i$.

Proof. The assertion for $(n-1)k + a_2 + a_4 + \cdots$ odd is equivalent to (a)–(c) of Section 1. Hence assume $(n-1)k + a_2 + a_4 + \cdots$ is even. Since the partitions λ_i and λ_{n-i-1} are conjugate, we have e.g. by [7, p. 71] that $\chi_\rho^{\lambda_i} = (-1)^{a_2 + a_4 + \cdots} \chi_\rho^{\lambda_{n-i-1}}$. Thus if we set $T_i = (-1)^{ik} \chi_\rho^{\lambda_i} / \binom{n-1}{i}^{k-1}$, then $T_i = T_{n-1-i}$. Hence

$$g_k(\rho) = \begin{cases} \dfrac{2(n-1)!^{k-1}}{n} \displaystyle\sum_{i=0}^{(n-2)/2} T_i, & \text{if } n \text{ is even}, \\[2ex] \dfrac{2(n-1)!^{k-1}}{n} \left(\displaystyle\sum_{i=0}^{(n-3)/2} T_i + \frac{1}{2} T_{(n-1)/2} \right), & \text{if } n \text{ is odd}. \end{cases}$$

Thus

$$\left| \frac{n g_k(\rho)}{2(n-1)!^{k-1}} - \sum_{i=0}^{j} T_i \right| \leq \sum_{i=j+1}^{[n/2]} |T_i| \leq \sum_{i=j+1}^{[n/2]} \frac{|\chi_\rho^{\lambda_i}|}{\binom{n-1}{i}^{k-1}} \tag{7}$$

by Lemma 4.2. For any character χ and element g of any finite group G, we have $|\chi(g)| \leq \deg \chi$, since $\chi(g)$ is the trace of a matrix with $\deg \chi$ rows and columns, whose eigenvalues are roots of unity. Hence by Lemma 2.2, we have $|\chi_\rho^{\lambda_i}| \leq \binom{n-1}{i}$. Thus the error term (7) is bounded by

$$\sum_{i=j+1}^{[n/2]} \frac{1}{\binom{n-1}{i}^{k-2}} \leq \frac{1}{\binom{n-1}{j+1}^{k-2}} + \frac{n}{\binom{n-1}{j+2}^{k-2}} = O(n^{-(j+1)(k-2)}).$$

This completes the proof.

Using Lemma 2.2, we can give the asymptotic expansion of $g_k(\rho)$ as a function of $a_1, a_2, \ldots, a_n$. We expect the $(n-1)!^k$ products $\pi_1 \pi_2 \cdots \pi_k$ to be approximately equidistributed through the $\frac{1}{2}n!$ allowable elements of $\mathfrak{S}_n$. Indeed.

Theorem 4.1, say for $j=2$, asserts that when $k\geqslant 3$,

$$\frac{\frac{1}{2}n!g_k(\rho)}{(n-1)!^k}=1+\frac{(-1)^k(a_1-1)}{(n-1)^{k-1}}+\frac{\binom{a_1-1}{2}-a_2}{\binom{n-1}{2}^{k-1}}+O(n^{-3(k-2)}).$$

When $k=2$, we need a more delicate estimate than $|\chi^{\lambda_i}_\rho|\leqslant\binom{n-1}{i}$. If $F(x)=\sum_{i\geqslant 0}f_ix^i$ and $G(x)=\sum_{i\geqslant 0}g_ix^i$ are power series with real coefficients, write $F(x)\geqslant G(x)$ if $f_i\geqslant g_i$ for all $i\geqslant 0$.

Lemma 4.2. *If $F(x)\geqslant 0$, then*

$$\frac{F(x)(1+x^j)}{1-x}\geqslant\frac{F(x)(1+x^{j+1})}{1-x} \tag{8}$$

for all $j\geqslant 0$.

Proof. We have

$$\frac{F(x)(1+x^j)}{1-x}-\frac{F(x)(1+x^{j+1})}{1-x}=x^jF(x)\geqslant 0,$$

as desired.

Lemma 4.3. *Let $\rho=\langle a_1,\ldots,a_n\rangle\vdash n$, and let $0\leqslant i\leqslant[n/2]$. Then*

$$|\chi^{\lambda_i}_\rho|\leqslant 2^{a_1}\binom{[n/2]}{[i/2]}.$$

Proof. According to Lemma 2.2, we have

$$\sum_{i=0}^{n-1}\chi^{\lambda_i}_\rho x^i=(1+x)^{a_1-1}(1-x^2)^{a_2}(1+x^3)^{a_3}\cdots(1-(-1)^nx^n)^{a_n}.$$

Hence

$$\sum_{i=0}^{n-1}|\chi^{\lambda_i}_\rho|x^i\leqslant\frac{(1+x)^{a_1}(1+x^2)^{a_2}\cdots(1+x^n)^{a_n}}{1-x}.$$

By successive applications of Lemma 4.2, we obtain

$$\sum_{i=0}^{n-1}|\chi^{\lambda_i}_\rho|\,x^i\leqslant\frac{(1+x)^{a_1}(1+x^2)^{a_2+\cdots+a_n}}{1-x}\leqslant\frac{2^{a_1}(1+x^2)^{[n/2]}}{1-x}.$$

Since $\binom{[n/2]}{j-1}\leqslant\binom{[n/2]}{j}$ when $j\leqslant[n/2]$, the proof follows.

Theorem 4.3. *Let $\rho=\langle a_1,a_2,\ldots,a_n\rangle\vdash n$. If $a_2+a_4+\cdots$ is odd (i.e., if ρ is odd), then $g_2(\rho)=0$. If $a_2+a_4+\cdots$ is even (i.e., if ρ is even), then for any fixed $j\geqslant 0$ we* *have*

$$g_2(\rho)=\frac{2(n-1)!}{n}\left[\sum_{i=0}^{j}\frac{\chi_\rho^{\lambda_i}}{\binom{n-1}{i}}+O(2^{a_1}n^{-(j+1)/2})\right]$$

uniformly in $a_1, a_2, \ldots, a_n$ *and* $n=\sum ia_i$.

Proof. As in Theorem 4.1, we may assume $a_2+a_4+\cdots$ is even. Setting $T_i=\chi_\rho^{\lambda_i}/\binom{n-1}{i}$, then as in (7) we obtain

$$\left|\frac{ng_2(\rho)}{2(n-1)!}-\sum_{i=0}^{j}T_i\right|\leq\sum_{i=j+1}^{[n/2]}\frac{\chi_\rho^{\lambda_i}}{\binom{n-1}{i}}.$$

Thus by Lemma 4.3,

$$\left|\frac{ng_2(\rho)}{2(n-1)!}-\sum_{i=0}^{j}T_i\right|\leq\sum_{i=j+1}^{[n/2]}\frac{2^{a_1}i\binom{[n/2]}{[i/2]}}{\binom{n-1}{i}}. \tag{9}$$

Denote the left-hand side of (9) by E_j, and let $t_i=\binom{[n/2]}{[i/2]}/\binom{n-1}{i}$. Then $t_i=O(n^{-(j+1)/2})$ for $i=j+1, j+2, j+3, j+4$ and $[n/2]$. Hence

$$E_j\leq 2^{a_1}n\sum_{i=j+5}^{[n/2]-1}t_i+O(2^{a_1}n^{-(j+1)/2}). \tag{10}$$

We claim that $t_i\geq t_{i+2}$ provided $0\leq i<[n/2]-2$. We will prove only the case $n=2m$, $i=2k-1$ here. The three remaining cases are handled similarly. When $n=2m$ and $i=2k-1$, we have by direct calculation

$$t_i-t_{i+2}=\frac{2m!(2k-1)!(2m-2k-2)!(2m^2-(6k+2)m+4k^2-1)}{(k-1)!(m-k+1)!(2m-1)!}.$$

The largest root of the equation $2x^2-(6k+2)x+4k^2-1=0$ is given by

$$x=\tfrac{1}{2}(3k+1+\sqrt{k^2+6k+3})<\tfrac{1}{2}(3k+1+k+3)=2k+2.$$

Hence if $m>2k+1$, then $2m^2-(6k+2)m+4k^2-1>0$. Since $m>2k+1$ is equivalent to $i<\frac{1}{2}n-2$, the claim is proved.

It follows from (9) and the inequality $t_i\geq t_{i+2}$ that

$$E_j\leq 2^{a_1}n^2(t_{j+5}+t_{j+6})+O(2^{a_1}n^{-(j+1)/2})=O(2^{a_1}n^{-(j+1)/2}),$$

completing the proof.

Thus for instance taking $j=2$ in Theorem 4.3, we obtain that for even ρ,

$$g_2(\rho)=\frac{2(n-1)!}{n}\left[1+\frac{a_1-1}{n-1}+\frac{\binom{a_1-1}{2}-a_2}{\binom{n-1}{2}}+O(2^{a_1}n^{-3/2})\right].$$

Since $a_i=O(n)$, it follows that if $a_1=0$ (or in fact $a_1=o(\log n)$), then $g_2(\rho)/(n-2)!\to 2$ as $n\to\infty$, which is Walkup's conjecture [9, p. 316]. In fact, it suffices to assume only $a_1=O(\log n)$. For assume $a_1\leqslant B(\log n)$ for all n. Take $j>2B(\log 2)-1$ in Theorem 4.3 to obtain

$$g_2(\rho)=\frac{2(n-1)!}{n}\left[\sum_{i=0}^{j}\frac{\chi_\rho^{\lambda_i}}{\binom{n-1}{i}}+o(1)\right]=\frac{2(n-1)!}{n}[1+o(1)],$$

By a more careful analysis, Kleitman has shown (private communication) that $g_2(\rho)$ has the asymptotic expansion

$$g_2(\rho)\sim\frac{2(n-1)!}{n}\sum_{i\geqslant 0}\frac{\chi_\rho^{\lambda_i}}{\binom{n-1}{i}}$$

provided only $a_1=o(n)$. The key step is an improved version of Lemma 4.3, but we will not enter into the details here.

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