A Combinatorial Packing Problem[1]

L. D. Baumert, et al.

Introduction. We are concerned with the efficient packing of squares of side two into the $p \times p$ torus. More generally, we are interested in the analogous n dimensional problem: that of packing n dimensional two-cubes efficiently into a $p \times p \times \cdots \times p$ torus. Of course, when p is even, the problem is trivial. (For then, the simplest possible alignment of the cubes completely fills the torus.) Thus, we restrict p to be an odd integer. Further, it should be pointed out that our primary interest is in determining the maximum number $[= \alpha\,(C_p^n)]$ of cubes which can be packed into the torus and that we are only secondarily concerned with the actual structural details of any particular maximal packing. Figure 1a shows that ($p = 9$, $n = 2$) at least 18 such cubes can be placed in the 9×9 torus. Since an odd number of squares (in particular, at least one square) of each row and column must be vacant in any such packing, it follows that Figure 1a exhibits a maximal packing of the 9×9 torus. That is, $\alpha(C_9^2) = 18$.

The notation $\alpha(G)$ is taken from Berge [1] where it is called the *coefficient of internal stability* of the undirected graph G. That is, $\alpha(G)$ denotes the maximum number of vertices of G, no two of which are adjacent. Another function of interest is

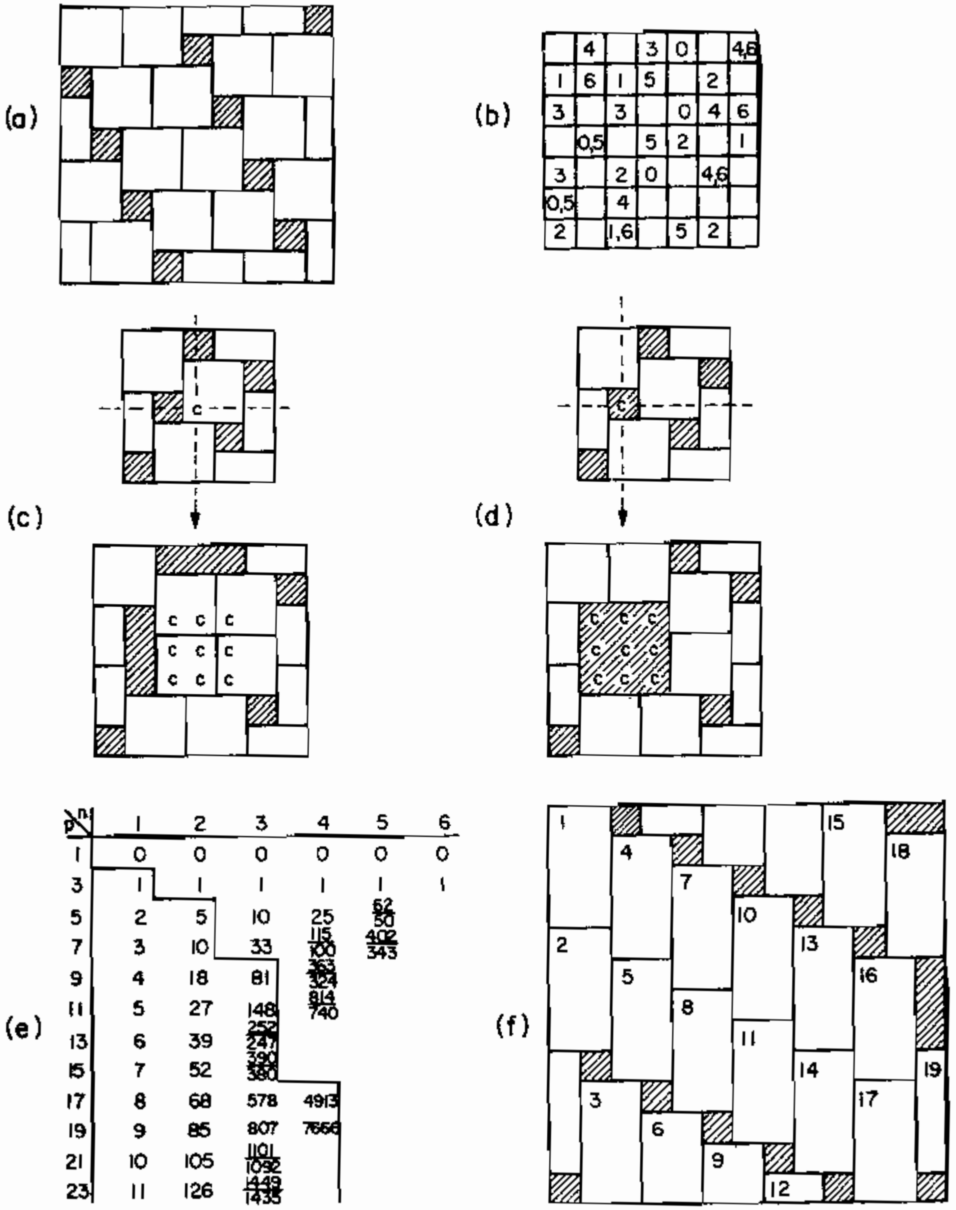
$$\operatorname{cap}(G) = \sup_n \frac{1}{n} \log \alpha(G^n)$$

where G^n denotes a particular n-fold graph theoretic product of G with itself. Cap (G) is called the *capacity* of the graph G and, as its name suggests, is of interest in information theory (see Shannon [2]). The graph C_p is the single

---

AMS 1970 *subject classifications*. Primary 05B40.

[1]This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NAS 7-100, sponsored by the National Aeronautics and Space Administration.



$p \backslash n$	1	2	3	4	5	6
1	0	0	0	0	0	0
3	1	1	1	1	1	1
5	2	5	10	25	62 / 50	
7	3	10	33	115 / 100	402 / 343	
9	4	18	81	363 / 324		
11	5	27	148	814 / 740		
13	6	39	252 / 247			
15	7	52	390 / 380			
17	8	68	578	4913		
19	9	85	807	7666		
21	10	105	1101 / 1092			
23	11	126	1449 / 1435			

FIGURE 1

cycle graph on p vertices, and the known values of $\alpha(C_p^n)$ provide bounds for cap (C_p). However, cap (C_p) remains unknown for every odd value of $p \geqq 5$.

The work reported below on $\alpha(C_p^n)$ was done over a period of time by several people. These include: L. D. Baumert, R. J. McEliece, Eugene Rodemich, Howard C. Rumsey, Jr., Richard Stanley and Herbert Taylor. In addition it has become clear that some of this work was done independently, indeed previously, by R. Stanton Hales [3].

Some bounds for $\alpha(C_p^n)$ and an expansion process. Whenever there is any reason to stress the parameters involved, we shall call the packing cubes, 2^n-*cubes*, and refer to the torus as a p^n-torus. Furthermore, we will use the word *cell* to refer to any n dimensional unit cube.

Let a p^n-torus containing a packing of *size* N_p be given, i.e., a packing which consists of N_p 2^n-cubes. Consider what happens when, with corresponding adjustments in the rest of the torus, an arbitrary preselected cell c is allowed to expand until it becomes a 3^n-cube. That is, when each of the hyperplanes through c perpendicular to a coordinate direction is replaced by three copies of itself. (Figures 1c, 1d give examples of this expansion process applied to the 5^2-torus.) With the convention that every cell created by this process is empty or filled according to whether the cell it is duplicating is empty or filled, this produces a packing of the $(p+2)^n$-torus by 2^n-cubes. Of course, the particular packing obtained depends strongly on the preselected cell c. However, averaging over all the p^n possible expansions (a different expansion for each of the p^n different cells c of the original torus) provides a lower bound on the size of the maximal expansion packing of a $(p+2)^n$-torus derivable from a given packing of a p^n-torus. Thus

THEOREM 1. *If there exists a packing of a p^n-torus of size N_p, then there exists a packing of a $(p+2)^n$-torus of size N_{p+2}, with*

$$N_{p+2} \geqq N_p \cdot ((p+2)/p)^n.$$

PROOF. As noted above, the expansion process takes packings into packings, so only volume arguments need be considered. Expand the p^n-torus in all p^n possible ways getting p^n $(p+2)^n$-tori. Since the expansion process is completely symmetric, each cell of the original p^n-torus will be duplicated the same number of times in this collection of tori. So each cell appears $(p+2)^n$ times, and each time it appears it is filled or vacant according to whether it was filled or vacant in the original packing of the p^n-torus. Thus our collection of $(p+2)^n$-tori contains $N_p(p+2)^n$ 2^n-cubes. So, at least one of them is packed with as many as $N_p((p+2)/p)^n$ such cubes, which completes the proof.

Notice that if (as in Figure 1d) the expansion cell c is empty, it, of course, expands and becomes an empty 3^n-cube, into which one further 2^n-cube can be placed. Since there are $p^n - 2^n N_p$ such vacant cells in any packing of the p^n-torus, our estimate can be increased slightly. That is

COROLLARY 1.

$$N_{p+2} \geqq 1 + N_p(((p+2)^n - 2^n)/p^n).$$

Of course, both estimates, if not already integral, may be replaced by the smallest integer which is larger than them.

COROLLARY 2.

$$\alpha(C_p^n) \geqq 1+\alpha(C_{p-2}^n)\cdot((p^n-2^n)/(p-2)^n).$$

Using the numbers $0, 1, \ldots, p-1$, let us number the p^n-torus in each of its coordinate directions. Thus, each cell of the torus can be designated by an n-tuple with entries from $\{0, 1, \ldots, p-1\}$. A 2^n-cube, then, may be considered to be a set of cells whose coordinates are given by $x+y$, where x is a fixed p-ary n-tuple, y ranges over all 0, 1 n-tuples and the addition is component addition modulo p. Thus x may be considered to be a sort of generalized "upper left-hand corner" for its particular 2^n-cube. It is often convenient to specify packings by designating the cells x which are to be upper left-hand corners in this sense.

Suppose the vectors $x_1, \ldots, x_s$ specify a packing of the p^m-torus and the vectors $y_1, \ldots, y_t$ specify a packing of the p^n-torus. Then the st vectors x_1y_1, $x_1y_2, \ldots, x_1y_t, x_2y_1, \ldots, x_sy_t$ [where the vector x_iy_j is that $(m+n)$-tuple whose first m components are the components of x_i and whose last n components are the components of y_j] specify a packing of the p^{m+n}-torus. This is called a *product packing* of the p^{m+n}-torus. Thus

LEMMA 1. *If a packing of the p^m-torus of size N_p^m and a packing of the p^n-torus of size N_p^n exist, then there exists a packing of the p^{m+n}-torus of size N_p with*

$$N_p \geqq N_p^m \cdot N_p^n.$$

COROLLARY 3.

$$\alpha(C_p^n) \geqq \alpha(C_p^m)\cdot\alpha(C_p^{n-m})$$

for $1 \leqq m \leqq n-1$.

So, by means of Corollaries 2, 3, we have lower bounds on the values taken by $\alpha(C_p^n)$. A study of the known values of $\alpha(C_p^n)$ shows that neither of these lower bounds dominates the other. As far as upper bounds are concerned, consider the following simple volume argument. The percentage of cells vacant in a maximal packing of the p^{n+1}-torus is not less than the percentage of vacant cells in a maximal packing of the p^n-torus. This is because the packing of the p^{n+1}-torus may be considered to be merely the juxtaposition of p packings of the p^n-torus. Allowing for the facts that the 2^{n+1}-cube contains twice as many cells as the 2^n-cube and that packing sizes are integers, yields

LEMMA 2.

$$\alpha(C_p^n) \leqq [p/2\cdot\alpha(C_p^{n-1})]$$

where, as usual, the square brackets denote the greatest integer function.

COROLLARY 4.

$$\alpha(C_p^n) \leqq (p^n-p^{n-1})/2^n.$$

PROOF. Since $\alpha(C_p)=(p-1)/2$ obviously, the corollary follows by neglecting the possible savings offered by the iterated use of the greatest integer function.

Some maximal packings. Figure 1e tabulates many of the known values of $\alpha(C_p^n)$; these results are established in this section. Where Figure 1e contains two entries, these are lower and upper bounds.

THEOREM 2. *For all integers $n>0$, $\alpha(C_1^n)=0$ and $\alpha(C_3^n)=1$. For all odd integers $p>0$, $\alpha(C_p)=(p-1)/2$ and $\alpha(C_p^2)=[(p^2-p)/4]$. Here again the square brackets denote the greatest integer function.*

PROOF. The first three assertions are trivial. The first two are only mentioned because they partially illuminate a conjecture made later. Since $\alpha(C_p)=(p-1)/2$, Lemma 2 shows that the theorem will be proved provided packings of the proper size can be displayed for the p^2-torus. Consider the packings specified by the following upper left-hand corners.

Take $(t, 2t+4s)$, $t=0,1,\ldots,p-1$, $s=0,1,\ldots,a-1$ when $p=4a+1$. Take $(2s, 2t+s)$, $s=0,1,\ldots,2a+1$, $t=0,1,\ldots,a-1$ together with $(2s+1, 2t+s+2a+1)$, $s=0,1,\ldots,2a$, $t=0,1,\ldots,a$ when $p=4a+3$. To verify that these are really packings it is necessary to check in each case that every two vectors of the set differ by as much as 2 modulo p in at least one component; a tedious but straightforward computation which we omit.

THEOREM 3. $\alpha(C_p^n)=(p^n-p^{n-1})/2^n$ *when* $p=k2^n+1$. $\alpha(C_p^n)=(p^{n+1}-3p^n+2^n)/2^n(p-2)$ *when* $p=k2^n+3$. *In both cases these are the upper bounds provided by Lemma* 2.

PROOF. When $p\equiv 1$ modulo 2^n, it is obvious from Corollary 4 that the upper bound is $(p^n-p^{n-1})/2^n=kp^{n-1}$. When $p\equiv 3$ modulo 2^n, it is necessary to investigate the bound of Lemma 2 more closely. Let B_m denote the upper bound on packings of the p^m-torus. Then $B_1=(p-1)/2$ and iterating Lemma 2 we see that to determine $B_m (m\leqq n)$, it is sufficient to show that

$$p^m-p^{m-1}-2p^{m-2}-4p^{m-3}-\cdots-2^{m-1}\equiv 0 \pmod{2^m}.$$

This follows by summing the left side and remembering that p is odd, i.e.,

$$p^m-\left(\frac{p^m-2^m}{p-2}\right)=\frac{k\cdot 2^n p^m+2^m}{p-2}\equiv 0 \pmod{2^m}.$$

So $B_n=(p^{n+1}-3p^n+2^n)/2^n(p-2)$ when $p\equiv 3$ modulo 2^n as claimed, and the theorem will be proved provided we exhibit packings of the proper size.

Let $p=k2^n+1$ first. For $k=1$, consider the packing given by

$$(x_1, x_2, \ldots, x_{n-1}, x_n) \quad \text{where } x_n=2x_1+4x_2+\cdots+2^{n-1}x_{n-1}$$

with $x_1, x_2, \ldots, x_{n-1}$ arbitrary. If x and y represent two 2^n-cubes of this packing, we want to show that they differ by 2 in at least one component. Suppose

$$x-y=(x_1-y_1,\ldots,x_n-y_n)=(Z_1,\ldots,Z_n)$$

has $Z_1,\ldots,Z_{n-1}$ equal to 0, 1 or -1 modulo p. Let P be the set of indices for which $Z_i=+1$ and let M be the set for which $Z_i=-1$. Then, if $2Z_1+\cdots+2^{n-1}Z_{n-1}\equiv 0,\pm 1$ modulo p, we have a congruence of the form

$$\sum_{i \text{ in } P} 2^i \equiv \sum_{j \text{ in } M} 2^j \quad (\text{modulo } p)$$

where, if necessary, P or M has been extended to include the index 0. But, unless both sums are empty, they represent different integers in the range $[0,2^n-1]=[0,p-2]$ and so cannot be congruent modulo p.

Similarly, for $k>1$ a maximal packing is given by

$$(x_1+2j,x_2,\ldots,x_{n-1},x_n) \quad \text{where } x_n=2x_1+4x_2+\cdots+2^{n-1}x_{n-1}$$

with $x_1,x_2,\ldots,x_{n-1}$ arbitrary and $0\leqq j\leqq k-1$.

When $p=k2^n+3$, applying Corollary 2 to these results yields

$$\alpha(C_p^n)\geqq 1+k(p-2)^{n-1}\left(\frac{p^n-2^n}{(p-2)^n}\right)=\frac{p^{n+1}-3p^n+2^n}{2^n(p-2)}=B_n.$$

So packings of this size exist, and our proof is complete.

Note that Corollary 2 only tells the average size of the expanded packing. In the proof above this was shown to be equal to the upper bound B_n. Thus, every maximal packing of the p^n-torus, when $p\equiv 1 \pmod{2^n}$, yields, upon expansion about *any* cell c, a maximal packing for the $(p+2)^n$-torus.

THEOREM 4. $\alpha(C_5^3)=10$, $\alpha(C_5^4)=25$, $\alpha(C_7^3)=33$.

PROOF. Theorem 2 and Corollary 3 show that $10\leqq\alpha(C_5^3)$, whereas Lemma 2 shows that $\alpha(C_5^3)\leqq 12$. Exhaustive search shows that $\alpha(C_5^3)\neq 11$, 12. This was established by P. Slepian and independently confirmed by some others. By Corollary 3, it follows that

$$25=\alpha(C_5^2)\cdot\alpha(C_5^2)\leqq\alpha(C_5^4)\leqq\tfrac{5}{2}\,\alpha(C_5^3)=25$$

this last by Lemma 2. So $\alpha(C_5^4)=25$.

Similarly Corollary 3 and Lemma 2 show that $30\leqq\alpha(C_7^3)\leqq 35$. A computer search, which is discussed in more detail later, showed that $\alpha(C_7^3)\neq 34$, 35 and produced several packings of size 33. One such is indicated in Figure 1b, where the number k in the ith row and jth column indicates that a 2^3-cube has upper left-hand corner (i,j,k).

THEOREM 5. *Let $p=4k+1$ and let $st=k$ be a factorization of k into positive integers s, t with $s\leqq t$. There is a one-to-one correspondence between these factorizations and the essentially distinct maximal packings of the p^2-torus.*

PROOF. Since $\alpha(C_p^2) = (p^2 - p)/4$ here, it follows that each row and column of a maximal packing of the p^2-torus contains exactly one empty square. Thus every column contains exactly $(p-1)/2$ 2-cubes. In the jth column i_j of the 2-cubes join with 2-cubes of the $(j-1)$st column to form 2^2-cubes and $(p-1)/2 - i_j$ join with 2-cubes of the $(j+1)$st column. Continuing this all the way around the torus shows (since p is odd) that $i_j = (p-1)/2 - i_j$, i.e., $i_j = (p-1)/4 = k$. So precisely k 2-cubes of each column extend to the right and the remaining k extend to the left.

Let us look at a particular column of this packing. The 2-cubes which are immediately above and immediately below the empty square of this column must extend in opposite directions; for otherwise, some row of the packing would contain two empty squares, a contradiction. So, without loss of generality, we may assume that the 2-cube immediately above the vacant square extends to the right and that the 2-cube immediately below the vacant square extends to the left. Thus, the structure of the packing in any column is determined by a sequence of l's and r's of length $(p-1)/2$ which specifies the direction each 2-cube extends. Suppose this sequence, for column j, consists of m_1 l's, followed by m_2 r's, ..., followed by m_g r's (g is necessarily even), then (see Figure 2a) the sequence for the next column to the right (column $j+1$) is forced to be m_1 r's, followed by m_2 l's, ..., followed by m_g l's. If the empty space in column j immediately precedes the m_1 l's of that column, then the empty space in column $j+1$ will be forced to immediately precede the m_2 l's of that column. Thus, of the g blocks of squares only the m_1 block changes rows as we move from column j to column $j+1$ and that block moves up exactly one row. Similarly, moving from column $j+1$ to $j+2$ shifts the m_2 block of squares up one row, etc. Thus going from column j to column $j+g$ raises every block exactly one row. So, in completing the circuit from column j all the way around the torus to column $j-1$ (= column $j+p-1$), the m_1 block is moved up precisely N rows, where N is the least integer greater than or equal to $(p-1)/g$. Similarly, the complete tour from column $j+1$ around to column j lifts the m_2 block exactly N rows. In general then, a complete tour of the torus raises every block of squares precisely N rows. But this can only be the case if g divides $p-1$. On the other hand the m_1 block must be raised at least $2m_1$ rows by the time it moves from column j around to column $j-1$, otherwise there is a conflict in column $j-1$. So $(p-1)/g \geqq 2m_1$, whereas $2(m_1 + m_2 + \cdots + m_g) = p-1$, by definition, hence

$$2(m_1 + \cdots + m_g) \geqq 2m_1 g.$$

Since there is no loss of generality in assuming that $m_1 = \max m_i$, it follows that $m_1 = m_2 = \cdots = m_g = (p-1)/2g$, with g even. Clearly these conditions are sufficient for they guarantee that all the blocks will mesh properly after completing a tour of the torus.

Thus, for every even divisor g of $(p-1)/2$, there is a maximal packing of the p^2-torus with uniform block size $m_i = (p-1)/2g = s$, $t = g/2$. That is, a packing

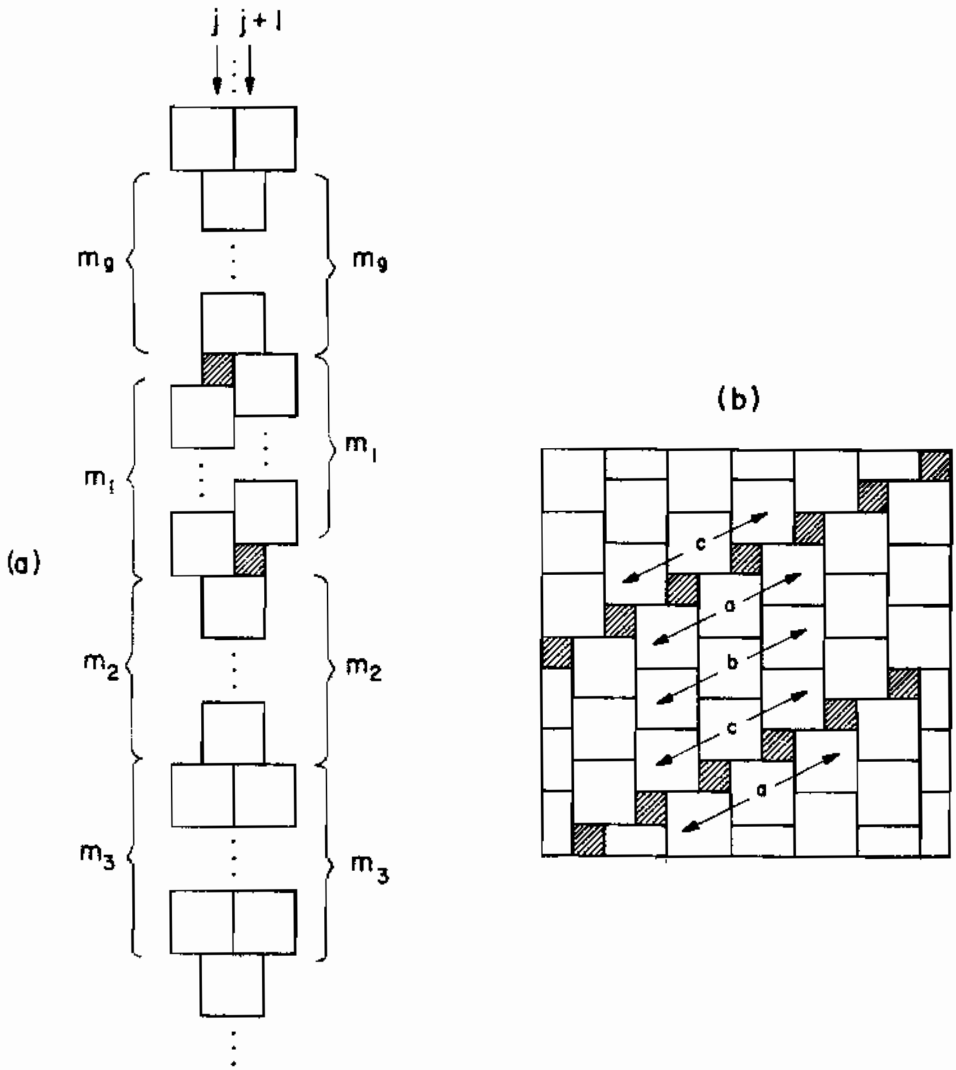


FIGURE 2

which corresponds with the factorization $st = k$. On the other hand, we have seen that in such a packing a particular block of squares only moves up a row once every g columns. This implies that if a packing corresponding to $st = k$ is rotated 90° it yields a packing which corresponds to $s't' = k$ where $s' = t$ and $t' = s$. Thus, all essentially different packings are considered under the requirement $s \leqq t$. So, our proof is complete.

Other packings. Consider the packing of 4×2 rectangles in the 13^2-torus given by Figure 1f. Let the numbered cells designate the upper left-hand corners appearing in the $z = 0$ plane of a 2^3-cube packing of the 13^3-torus.

Further let (x, y, z) with x the row index and y the column index be used to describe the packing. Then the numbered cells of Figure 1f together with the cells derived from them by repeatedly adding (2, 0, 1) modulo 13 provide a packing of the 13^3-torus of size $19 \times 13 = 247$. More generally

THEOREM 6. *Let $p = 8m+5$, then there exists a packing of the p^3-torus of size $(p^3 - p^2 - 4p)/8$.*

PROOF. It is only necessary to describe a 4×2 packing of the p^2-torus of size $(p^2 - p - 4)/8$; for such a packing when repeatedly offset by (2, 0, 1) packs the p^3-torus as claimed. For $p = 5$ only two 4×2 rectangles are required and that is easily achieved. Let $p = 8m+5$ with $m \geqq 1$ and let the 4×2 rectangles have upper left-hand corners

$$\begin{array}{ll} (4j+s, 2s) & j=0,\ldots,m;\ s=0,\ldots,4m+1, \\ (4j+s+4m+4, 2s-1) & j=0,\ldots,m-1;\ s=0,\ldots,4m+2. \end{array}$$

These provide the proper packing. To visualize this, note that Figure 1f consists of 2 bands of 4×2 rectangles, one band being 8 rows wide and the other 4 rows wide. Further, the two bands are offset from each other by one column. The general construction has this same structure with bands of width $4(m+1)$ and $4m$ respectively.

THEOREM 7. $\alpha(C_7^5) \geqq 7^3$.

PROOF. One packing of size 7^3 is given by

$$(x_1, x_2, x_3, 2x_1 + 2x_2 + 2x_3, 2x_1 + 4x_2 + 6x_3)$$

with x_1, x_2, x_3 arbitrary.

LEMMA 3. $\alpha(C_{13}^3) \leqq 252$.

(The main point of this lemma is that $\alpha(C_{13}^3)$ does *not* achieve the upper bound $(=253)$ predicted by Lemma 2. At one time it was considered possible, in view of Theorem 3, that $\alpha(C_p^n)$ achieved the bound of Lemma 2 for all $p \geqq 2^n + 1$.)

PROOF. Any packing of the 13^3-torus may be considered to be the juxtaposition of 13 packings of the 13^2-torus. Since at most 39 2^2-cubes fit in the 13^2-torus, this implies that 253 can only be achieved by using twelve packings of size 39 together with one of size 38. Let $P_0, \ldots, P_{12}$ designate these packings where P_{12} is the one of size 38. Any 2^3-cube appears in two consecutive packings P_i, say P_j and P_{j+1}, and is said to *stick up* from P_j and *stick down* from P_{j+1}. Since p is odd, the number of cubes which stick up (or down) from any particular packing P_i is uniquely determined by the sequence of packing sizes. Here, these numbers are alternately 19 and 20 with, of course, exactly 19 cubes of P_{12} going in each direction.

Consider the maximal packing of the 13^2-torus exhibited in Figure 2b. Note that this packing is the union of three diagonals of cubes, the "a," "b," and "c" diagonals, each of which contains 13 cubes. It is a consequence of Theorem 5 that every maximal packing of the 13^2-torus can be so decomposed into diagonals of 13 cubes each. Furthermore, it follows that two such maximal packings have the same slope to their diagonals if and only if one is a translate of the other. If two such packings of differing slopes are compared, we note that they have precisely 9 cubes in common; each diagonal of the one packing meeting each diagonal of the other packing precisely once. Since our packings $P_0, \ldots, P_{11}$ have at least 19 cubes in common with their immediate neighbors, it follows that $P_0, \ldots, P_{11}$ are all (up to translation) the same packing. In fact, by Theorem 5, there is no loss of generality in assuming that they all are, up to translation, the packing of Figure 2b.

Since P_0 and P_1 share 20 cubes, at least one of the diagonals of P_0 has cubes going in both directions. This further necessitates that particular diagonal appearing in all the packings $P_0, \ldots, P_{11}$. Furthermore, this requires certain 2^3-cubes appearing in that diagonal to be shared simultaneously by P_{11}, P_{12} and by P_{12}, P_0, a contradiction. This contradiction establishes the fact that $\alpha(C_{13}^3) \neq 253$, and our proof is complete.

Whenever $p \equiv 5$ modulo 8, exactly this same process can be used to show that $\alpha(C_p^3)$ never achieves the upper bound of Lemma 2. In fact this result can be generalized further to

$$\alpha(C_p^n) \leqq [p^{n-1}(p-1)/2^n] - 1$$

whenever $p \equiv 2^{n-1}+1$ modulo 2^n, $n \geqq 3$.

A conjecture. As a result of trying to fit a formula to the known data (Figure 1e) using various ad hoc methods, the following rather nice conjecture evolved:

$$\alpha(C_p^n) = \left[\frac{p-1}{2^n}\right] \cdot \left(\frac{p^n - \sigma^n}{p-\sigma}\right) + \alpha(C^n_{p \bmod 2^n})$$

where σ is the residue of $p-1$ modulo 2^n.

Note that this formula implies that the first 2^{n-1} entries in each column of Figure 1e are initial conditions which determine the behavior of the function in that column. Thus all the entries above the staircase line of Figure 1e are initial conditions. In these terms then perhaps the nice packings of Theorem 3 are merely reflections of the fact that $\alpha(C_1^n)$ and $\alpha(C_3^n)$ are trivial.

This formula predicts $\alpha(C_{13}^3)$ and $\alpha(C_{15}^3)$ as 247 and 384 respectively.

Computation. The bounds of Corollary 3 and Lemma 2 place $\alpha(C_7^3)$ between 30 and 35. A computer was used to establish that $\alpha(C_7^3) = 33$. The first step was to show that no packing of size 35 could exist. Of course, any packing of the 7^3-torus may be considered as the juxtaposition of 7 packings for the 7^2-torus. Since $\alpha(C_7^2) = 10$, these 7 packings $P_0, \ldots, P_6$ must all be of maximal size in order for their union to constitute a packing of size 35. So a straightforward approach to the problem is merely to list all the maximal packings of the 7^2-torus and try to find 7 of them that fit together properly.

Finding all the maximal packings of the 7^2-torus is not too difficult a project. A few minutes' hand computation shows that, up to the automorphisms of the 7^2-torus, there are only three such packings. Furthermore, the automorphisms of the torus take packings into packings. Thus any automorphism of the 7^2-torus applied to all the packings $P_0, \ldots, P_6$ leads to a problem equivalent to the original one. So P_0 may be assumed to be one of these 3 packings, without loss of generality. However, $P_1, \ldots, P_6$ cannot then be so restricted. Indeed they must be allowed to range over all the maximal packings; there are 980 of these. Again, these are not difficult to generate (inside a computer) for the original 3 packings correspond to 20 which are inequivalent under translations. Thus, it is a simple matter of entering the 20 packings and letting the computer produce the full 980 from these.

Since the sequence of packings sizes corresponding to $P_0, \ldots, P_6$ is $10, \ldots, 10$ it follows that P_i and P_{i+1} share exactly five 2^3-cubes. A preliminary calculation was performed which showed that relatively few of the 980 packings agreed with any particular one in as many as five cubes. So a fairly simple computer search program was written which first selected five cubes from P_0 and then in turn inserted as P_1 each of the 980 packings having those five cubes. Of course, for any particular P_1, knowing the five cubes shared with P_0 determines the five cubes shared by P_1, P_2, etc. Once P_5 has been selected the structure of P_6 is completely determined, since five of its cubes are determined by P_0 and the remainder are determined from P_5. However, the configuration forced on P_6 at this point never was a 10 packing of P_6 (the cubes coming from P_0 always intersected those coming from P_5). In this manner it was shown that $\alpha(C_7^3) \neq 35$.

If $\alpha(C_7^3)$ were 34 then, without loss of generality, the packing size sequence could be restricted to one of

$$(10, 10, 10, 10, 10, 10, 8), (10, 10, 10, 10, 10, 9, 9),$$
$$(10, 10, 10, 10, 9, 10, 9), (10, 10, 10, 9, 10, 10, 9).$$

Corresponding to each of these sequences is the sequence showing the number of 2^3-cubes shared by P_i, P_{i+1}; these are:

$$(6, 4, 6, 4, 6, 4, 4), (5, 5, 5, 5, 5, 4, 5), (6, 4, 6, 4, 5, 5, 4), (5, 5, 5, 4, 6, 4, 5).$$

Here, again, there is no loss of generality in restricting P_0 to be one of our original three packings. Having picked five or six cubes, as the case may be, from P_0, the computer program proceeded as before, with the added difficulty that the packings of size 9 were not at hand. Where necessary the relevant packings of this size were generated by extending the known partial packing in all possible ways to the proper size. This process allowed the computer to notice the size of the packings it was constructing. In the process of deciding that 34 cubes could not be packed into the 7^3-torus, it produced many packings of size 33. One such is indicated in Figure 1b.