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BOOK REVIEWS

Principles of Combinatorics by Claude Berge. Academic Press, New York, 1971. 176 pp. First published in the French language under the title *Principes de Combinatoire*, Dunod, Paris, 1968.

The current resurgence of combinatorics (also known as combinatorial analysis and combinatorial theory) is by now recognized by all mathematicians. Scoffers regard combinatorics as a chaotic realm of binomial coefficients, graphs, and lattices, with a mixed bag of *ad hoc* tricks and techniques for investigating them. In reality, there has been a tremendous unifying drive to combinatorics in recent years. We now have a broad and sophisticated understanding of such standard combinatorial concepts as inversion, composition, generating functions, finite differences, and incidence relations.

Another criticism of combinatorics is that it "lacks abstraction." The implication is that combinatorics is lacking in depth and all its results follow from trivial, though possible elaborate, manipulations. This argument is extremely misleading and unfair. It is precisely the "lack of abstraction," i.e., the concrete visualization of the concepts involved, which helps to make combinatorics so appealing to its adherents. On the other hand, the "depth" of the subject is rapidly increasing as it increasingly draws upon more and more techniques and concepts from other branches of mathematics, such as group representation theory, statistical mechanics, harmonic analysis, homological algebra, and algebraic topology, to say nothing of the increasing sophistication of various new purely combinatorial techniques.

The reader whose only contact with combinatorics has been an occasional binomial coefficient summation or a particularly hideous looking graph will want to know where he can get an idea of the current flavor of the subject. I can think of no better way than to recommend Berge's book. Of course only a small selection of topics can be included in a single volume, and none of them can be explored in full generality. Berge manages, however, to choose his topics with impeccable taste, and his exposition is masterful. Particularly noteworthy is the inclusion of some recent results on permutations discovered by the current French school of combinatorialists. Moreover, the use of simple explicit examples throughout greatly facilitates comprehension. The reader's appetite will in all likelihood be whetted for further study.

Berge regards combinatorics as the study of *configurations*, i.e., mappings of a set of objects into a finite abstract set with a given structure. He lists six aspects of studying configurations, viz., study of a known configuration, investigation of an unknown configuration, counting configurations, approximate counting of configurations, enumeration of configurations, and optimization. His book is concerned almost exclusively with the counting aspect, i.e., determining the number of configurations with a given property.

The idea of a configuration is particularly suited to the study of the myriad of classical quantities concerned with problems of distribution and occupancy. The binomial coefficients, Stirling numbers, partition numbers, etc., are much more easily understood from this point of view. Berge adopts from linear algebra, functional analysis, algebraic topology, et al., the important principle that when two structures are isomorphic, it is more meaningful and significant to exhibit explicitly an isomorphism between them, rather than just proving they are isomorphic. In combinatorics, this principle takes the form that the best way to show that two sets have the same number of elements is to construct a bijection between them. For this reason such methods as generating functions are avoided when possible. This is not meant to detract from the method of generating functions, which has been recently greatly clarified. Generating functions, for example, appear to be essential in dealing with Berge's fourth aspect of configurations, viz., approximate counting.

Another unifying thread running through Berge's book is that of partial orderings in general, and lattices in particular. The recent relative stagnation of lattice theory after achieving its heights in the 1930's has ended. Lattice theory has been used in a large number of ways to enhance combinatorics, and its importance is becoming increasingly evident. Though Berge makes no systematic attempt to explain the role of lattice theory in combinatorics, he does give throughout his book examples of lattices which will convince the reader that lattice theory has something to offer combinatorics, and *vice versa*.

Although the main course of the book is superbly developed, one regrets that more time was not devoted to the dressing. The notation frequently does not conform to standard practice, historical references are usually omitted and occasionally incorrect, the bibliographies at the end of each chapter are inadequate, there are no exercises, and the index is quite scanty. Some specific instances of these criticisms (albeit not very significant) are given below, where we discuss the material in more detail.

The book begins with an entertaining Foreword by another distinguished combinatorialist, followed by an excellent introduction entitled "What is Combinatorics," where the six aspects of configurations are discussed. Chapter I, The elementary counting functions, is devoted to a discussion of maps of finite sets and the basic quantities associated with them. Included are such standard topics as the "falling factorial" $[m]_n = m(m-1)(m-2) \cdots (m-n+1)$, the Stirling numbers of the first and second kinds, the "rising factorial"

$$[m]^n = m(m+1)(m+2) \cdots (m+n-1),$$

the binomial coefficients $\binom{n}{m}$ and multinomial coefficients $\binom{n}{n_1, n_2, \cdots, n_p}$, the Fibonacci numbers $F_n = \sum_k \binom{n-k+1}{k}$, the corrected Fibonacci numbers $F_n^* = \sum_k (n/(n-k))\binom{n-k}{k}$, and the Bell numbers $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. The basic properties of these quantities are derived by considering mappings of finite sets. This approach has virtually superseded the old way of looking at "permutations and combinations with and without repetitions," etc., and it is refreshing to see it developed so smoothly and clearly. Somewhat surprisingly, it is never mentioned that the Fibonacci numbers and corrected Fibonacci numbers satisfy the recurrences $F_n = F_{n-1} + F_{n-2}$, $F_n^* = F_{n-1}^* + F_{n-2}^*$. The corrected Fibonacci numbers are more usually called the *Lucas numbers*.

Chapter 2 is entitled Partition problems. In addition to a rather standard treatment of partitions using Ferrer's diagrams and generating functions, there is also a discussion of Young tableaus and the beautiful Robinson-Schensted theorem on the number of permutations of the integers $1, 2, \cdots, n$ which have a longest increasing subsequence of length p and a longest decreasing subsequence of length q. This latter theorem is a generalization of the famous Erdös-Szekeres theorem that a sequence of length $mn+1$ contains either a decreasing subsequence of length $m+1$ or an increasing subsequence of length $n+1$, as proved in Chapter 1. Also considered in Chapter 2 is the relation between standard tableaus and Young's lattice as studied by Germain Kreweras. These results on tableaus provide the reader with a glimpse of the fascinating topic of *plane partitions*, which has been given a systematic combinatorial development only in the past few years.

Chapter 3, Inversion formulas and their applications, sketches some results from an enormously wide area. The first section presents an introduction to a unified treatment of the calculus of finite differences based upon the notion of a *differential operator* on a *normal family* of polynomials. (In the definition of a normal family, the hypothesis that $P_n(x)$ has degree n is inadvertently omitted.) Topics include a generalized binomial theorem and Taylor series expansion. No mention is made of the recent paper of Rota-Mullin, where this approach to finite differences is extensively developed.

§2 of Chapter 3 introduces the reader to the vast and comprehensive theory of Möbius inversion on a locally finite ordered set. Several interesting well-known examples convey the generality and flavor of the subject. It should be mentioned that the group A of arithmetic functions, as defined by Berge, is *not* a ring, contrary to the statement on p. 82. To make A a ring, one must allow $f(x, x)=0$ in the definitions of A. A is then commonly known as the *incidence algebra* of the ordered set X. For some reason, Berge prefers to refer to the *zeta function* ζ of A as the *Riemann function* ξ. Also, the reference given in the footnote on p. 87 is inaccurate. It should be *Baxter algebras and combinatorial identities*. II, Bull. Amer. Math. Soc. **75** (1969), 330–334.

The remaining three sections of Chapter 3 deal basically with more classical material, though the treatment is somewhat modernized, and some recent results are included. Some of the topics considered are (a) Sieve formulas (a superior name for the old "Principle of inclusion-exclusion"), including the "Problème des Rencontres," "Problème des Ménages," and the Euler totient function, (b) the theory of distributions, with a discussion of perfect partitions, and (c) an interesting collection of results on counting labeled trees.

Chapter 4 is devoted to Permutation groups. The basic properties of permutations and finite permutation groups are derived, culminating in a proof of Burnside's lemma, which is essential for the treatment of Pólya's theorem appearing in Chapter 5. The remainder of Chapter 4 contains a beautiful treatment of the relatively inaccessible work of French mathematicians on the analysis of permutations. With the symmetric group S_n is associated a convex polyhedron called by Guilbaud and Rosenstiehl the "permutohedron." In particular, the permutohedron for $n=4$ is the *truncated octahedron*, as sketched on p. 136. Berge proves the basic result that the permutohedron has a natural lattice structure. He then gives some results on expressing permutations as a product of transpositions which are related to the structure of trees. For instance, a necessary and sufficient condition for $n-1$ transpositions to generate the symmetric group S_n is first given. Now suppose $t_1, t_2, \cdots, t_{n-1}$ is a set of transpositions generating S_n, and that $t_1', t_2', \cdots, t_{n-1}'$ is some rearrangement of them. Berge concludes Chapter 4 with a proof (and some consequences) of the Eden-Schützenberger theorem as to when $t_1t_2 \cdots t_{n-1}=t_1't_2' \cdots t_{n-1}'$. It is regrettable that more space was not devoted to a further discussion of the French school of permutation analysis, such as the Cartier-Foata theory of permutations of a multiset, the work of A. Jacques on planar graphs and symmetric groups, and the Foata-Schützenberger theory of Eulerian numbers.

Chapter 5 is devoted to the famous Pólya theorem on enumeration under group action, including the generalization due to deBruijn. Besides the usual applications to counting graphs, coloring cubes, etc., an unusual application is given to the enumeration of knots. There is a minor error on p. 170—$S_n \otimes S_n$ is not the group connected with directed graphs.

The above survey of topics points out the magnificent job Berge has done in sifting out from the vast literature of combinatorics the most interesting, elegant and important results connected with enumeration. Anyone who reads this books will not only derive many hours of fascination and enjoyment, but will also have a much better grasp of the meaning of the current combinatorial revolution. To paraphrase from the Foreword, Berge's book will go a long way toward unknotting the reader from the tentacles of the Continuum and inducing him to join the Rebel Army of the Discrete.

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Modular Lie algebras by G. B. Seligman. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 40. Springer-Verlag Inc., New York, 1967. ix+165 pp. $9.75.

This book is the first to be devoted to Lie algebras over fields of characteristic $p>0$, the so-called modular Lie algebras of the title. Other recent books, such as Jacobson's *Lie Algebras*, are concerned with Lie algebras over an arbitrary field to the extent to which the theory for characteristic 0 may be generalized to arbitrary fields. However, there are significant differences between Lie algebras of characteristic 0 and those of characteristic $p>0$. This is the first book in which the latter are studied in a systematic way.

Complex and real Lie algebras, because of their use in the study of Lie groups, comprise a classical subject with which many mathematicians are acquainted. The extension of classical results to Lie algebras over an arbitrary field began in the 1930's and contributed to the further development of the theory. Crucial differences between Lie algebras of characteristic 0 and of characteristic $p>0$ were recognized early. In the late thirties and early forties a number of significant papers by Jacobson, Zassenhaus and others on Lie algebras of prime characteristic were published. But the difficulties encountered in the study of these algebras appeared intractable, and