© DISCRETE MATHEMATICS 4 (1973) 77–82. North-Holland Publishing Company

A BRYLAWSKI DECOMPOSITION FOR FINITE ORDERED SETS

Richard P. STANLEY

Massachusetts Institute of Technology, Cambridge, Mass., USA

Received 22 November 1971

Abstract. A decomposition is given for finite ordered sets P and is shown to be a unique decomposition in the sense of Brylawski. Hence there exists a universal invariant $g(P)$ for this decomposition, and we compute $g(P)$ explicitly. Some modifications of this decomposition are considered; in particular, one which forms a bidecomposition together with disjoint union.

1. Introduction

Let P be a finite ordered set of cardinality $p > 0$, and let $\mathbf{s}$ denote a chain (totally ordered set) of cardinality s. Johnson [3] considers a polynomial, which we shall denote by $\Lambda(P)$, defined by

$$\Lambda(P) = \sum_{s=1}^{p} e_s n^s ,$$

where $e_s = e_s(P)$ is the number of surjective order-preserving maps $\sigma: P \to \mathbf{s}$ (so $x \leqslant y$ in $P \Rightarrow \sigma(x) \leqslant \sigma(y)$). Johnson's polynomial, called the *representation polynomial* of P, is closely related to the *order polynomial* $\Omega(P)$ of P [4; 5, § 19], defined by

$$\Omega(P) = \sum_{s=1}^{p} e_s \binom{n}{s} .$$

Let x and y be any two incomparable elements of P. Define the ordered sets $P_x{}^y$, $P_y{}^x$ and P_{xy} as follows: $P_x{}^y$ is obtained from P by introducing the new relation $x < y$ (and all relations implied from this by transitivity); $P_y{}^x$ is obtained by introducing $y < x$; and P_{xy} is obtained by identifying x with y. Hence $|P_x{}^y| = |P_y{}^x| = p$, $|P_{xy}| = p-1$. Johnson [3] observes that

$$\Lambda(P) = \Lambda(P_x{}^y) + \Lambda(P_y{}^x) - \Lambda(P_{xy}) .$$

By defining $\Lambda'(P) = (-1)^p \Lambda(P)$, we get

$$(1) \qquad \Lambda'(P) = \Lambda'(P_x{}^y) + \Lambda'(P_y{}^x) + \Lambda'(P_{xy}) .$$

Eq. (1) motivates us to determine every invariant $\Gamma(P)$, defined on all finite ordered sets P, satisfying

$$(2) \qquad \Gamma(P) = \Gamma(P_x{}^y) + \Gamma(P_y{}^x) + \Gamma(P_{xy})$$

for all incomparable $x, y \in P$. This will be done by showing that the decomposition

$$(3) \qquad P \to P_x{}^y + P_y{}^x + P_{xy}$$

forms a unique decomposition in the sense of Brylawski [1; 2]. Basically, this means that by continually applying (3), we can express P in a unique way as a sum of finitely many indecomposables.

We call (3) the A-*decomposition* of P, and we call any function $\Gamma(P)$ satisfying (2) an A-*invariant* of P. It follows from Brylawski's results that there is a universal A-*invariant* $g(P)$ which is a polynomial in variables corresponding to the A-indecomposable elements. Clearly the A-indecomposable elements are just the chains s. Hence $g(P)$ will be a polynomial in infinitely many variables z_s, $s = 1, 2, \ldots$; and any A-invariant $\Gamma(P)$ is obtained from $g(P)$ by setting $z_s = \Gamma(s)$.

Our proof that (3) forms a unique decomposition automatically provides an explicit expression for $g(P)$. This situation differs from Brylawski's decomposition of pregeometries, where the universal invariant (the *Tutte polynomial*) is difficult to give explicitly. We will also consider some modifications of the decomposition (3), in particular one which allows us to introduce disjoint union as a multiplicative decomposition forming a distributive bidecomposition together with the modified form of (3).

2. The A-decomposition

We wish to prove that (3) forms a unique decomposition. All of the properties are trivially verified except for uniqueness, i.e., given any two decompositions of P into indecomposables s (obtained by iterating (3)), the multiplicity of each chain s is the same in both.

Proposition 2.1. *The only way of* A*-decomposing P into indecomposables is*

$$P = \sum_1^p \bar{e}_s \, s \,,$$

where $\bar{e}_s = \bar{e}_s(P)$ is the number of strict surjective order-preserving maps $\tau\colon P \to s$ (so $x < y$ in $P \Rightarrow \tau(x) < \tau(y)$).

Proof. Induction on $p = |P|$ and on the number of incomparable pairs of elements of P. The proposition is clearly true if $P = s$. Now assume it is true for all P' with $|P'| = p-1$, or with $|P'| = p$ but with less incomparable pairs than P. Thus from (3), one A decomposition of P into indecomposables is

$$P = \sum_1^p \bar{e}_s(P_x{}^y)\, s + \sum_1^p \bar{e}_s(P_y{}^x)\, s + \sum_1^{p-1} \bar{e}_s(P_{xy})\, s \,.$$

Hence we need only show

$$\bar{e}_s(P) = \bar{e}_s(P_x{}^y) + \bar{e}_s(P_y{}^x) + \bar{e}_s(P_{xy}) \,. \tag{4}$$

for any incomparable pair x, y of P.

Now the number of surjective strict order-preserving maps $\tau\colon P \to s$ satisfying $\tau(x) < \tau(y)$ is $\bar{e}_s(P_x{}^y)$; satisfying $\tau(x) > \tau(y)$ is $\bar{e}_s(P_y{}^x)$; and satisfying $\tau(x) = \tau(y)$ is $\bar{e}_s(P_{xy})$. From this follows (4).

Corollary 2.2. *The universal* A*-invariant $g(P)$ is given by*

$$g(P) = \sum_1^p \bar{e}_s \, z_s \quad .$$

Hence any A-invariant $\Gamma(P)$ is given by

$$\Gamma(P) = \sum_1^s \bar{e}_s \, \Gamma(s) \, .$$

Example 2.3. The modified representation polynomial $\Lambda'(P) = (-1)^p \, \Lambda(P)$ is an A-invariant and $\Lambda'(s) = (-1)^s \, n \, (n+1)^{s-1}$. Hence we get the identity

$$(5) \qquad \Lambda'(P) = (-1)^p \sum_1^s e_s \, n^s = \sum_1^s \bar{e}_s \, (-1)^s \, n \, (n+1)^{s-1} \, .$$

Example 2.4. It is easily seen that the modified order polynomial $(-1)^p \, \Omega(P)$ is an A-invariant, and $(-1)^s \, \Omega(s) = \binom{-n}{s}$. Hence

$$(-1)^p \, \Omega(P) = (-1)^p \sum_1^p e_s \binom{n}{s} = \sum_1^p \bar{e}_s \binom{-n}{s} \, .$$

This identity is equivalent to (5). For further ramifications of the relation between e_s and $\bar{e}_s$, see [4] or [5, § 19].

Example 2.5. An *order ideal* of P is a subset I of P such that if $x \in I$ and $y < x$ then $y \in I$. Let $j(P)$ denote the number of order ideals of P. Then $(-1)^p \, j(P)$ is an A-invariant, and $(-1)^s \, j(s) = (-1)^s \, (s+1)$. Hence

$$j(P) = (-1)^p \sum_1^p \bar{e}_s \, (-1)^s \, (s+1) = \Omega(P)_{n=2} \, .$$

3. The M-decomposition

Suppose P is a disjoint union (direct sum) of P_1 and P_2. We consider the multiplicative decomposition

$$(6) \qquad P \to P_1 \cdot P_2$$

(not to be confused with the direct product $P_1 \times P_2$), which we call the M-*decomposition.* A function $\Gamma(P)$ satisfying $\Gamma(P) = \Gamma(P_1) \, \Gamma(P_2)$

is called an M-*invariant* of P. For instance, $(-1)^p\ \Omega(P)$ is an M-invariant while $(-1)^p\ \Lambda(P)$ is not.

Note that the ordered sets P which are both A- and M-indecomposable are still the chains s.

Suppose P consists of two disjoint points. Then applying the M-decomposition we get $P = 1 \cdot 1$, while by the A-decomposition, $P = 2 + 2 + 1$. These decompositions differ because the M-decomposition is not *distributive* over the A-decomposition (in the sense of Brylawski). Hence we modify the A-decomposition by requiring that in (3), x and y must belong to the same connected component (or M-indecomposable factor) of P. This new decomposition we call the A′-decomposition. It is easily seen that the A′- and M-decompositions form a *distributive bidecomposition* in the sense of Brylawski. Hence by Brylawski's results there is a universal A′- and M-invariant $t(P)$.

We state the results for $t(P)$ corresponding to those for $g(P)$. The proofs are basically the same and will be omitted.

Proposition 3.1. *Let $P_1, P_2, \ldots, P_c$ be the connected components of P. The only way of bidecomposing P into* A′- *and* M-*indecomposables is*

$$P = \prod_{i=1}^{c} \left(\sum_s \bar{e}_s(P_i)\ s\right) .$$

Corollary 3.2. *The universal* A′- *and* M-*invariant $t(P)$ is given by*

$$t(P) = \prod_{i=1}^{c} \left(\sum_s \bar{e}_s(P_i)\ z_s\right) .$$

4. The E-decomposition

Suppose we modify the A-decomposition by

$$(7) \qquad P \to P_x{}^y + P_y{}^x$$

whenever x and y are incomparable in P. We call (7) the E-*decomposition* of P. Let $e(P)$ be the number of ways of extending P to a total order, so $e(P) = e_p = \bar{e}_p$. Then reasoning as in §2, we obtain:

Proposition 4.1. *The only way of* E*-decomposing P into indecomposables is $P = e(P)\,p$.*

Corollary 4.2. *The universal* E*-invariant $h(P)$ is given by $h(P) = e(P)\,z_p$.*

Some further aspects of the number $e(P)$ are discussed in [5] and [6].

References

[1] T. Brylawski, The Tutte-Grothendieck ring, Ph.D. Thesis, Dartmouth College (1970).
[2] T. Brylawski, A decomposition for combinatorial geometries, Trans. Am. Math. Soc., to appear.
[3] K.L. Johnson, Real representations of finite directed graphs, Ph.D. Thesis, Univ. of Alabama (1971).
[4] R. Stanley, A chromatic-like polynomial for ordered sets, in: Proc. 2nd Chapel Hill Conf. on Combinatorial Mathematics and its Applications (1970) 421–427.
[5] R. Stanley, Ordered structures and partitions, Mem. Am. Math. Soc. 119 (1972).
[6] R. Stanley, The Fibonacci lattice, Fibonacci Quart., to appear.