EQUIVARIANT COHOMOLOGY AND RESOLUTION

PIERRE ALBIN AND RICHARD MELROSE

Abstract. The 'Folk Theorem' that a smooth action by a compact Lie group can be (canonically) resolved, by iterated blow up, to have unique isotropy type is proved in the context of manifolds with corners. This procedure is shown to capture the simultaneous resolution of all isotropy types in a 'resolution tower' which projects to a resolution, with iterated boundary fibration, of the quotient. Equivariant K-theory and the Cartan model for equivariant cohomology are tracked under the resolution procedure as is the delocalized equivariant cohomology of Baum, Brylinski and MacPherson. This leads to resolved models for each of these cohomology theories, in terms of relative objects over the resolution tower and hence to reduced models as flat-twisted relative objects over the resolution of the quotient. As a result an explicit equivariant Chern character is constructed, essentially as in the non-equivariant case, over the resolution of the quotient.

Contents

1. Introduction	1
2. Manifolds with corners	3
3. Blow up	6
4. Iterated fibration structures	7
5. G-actions	11
6. Invariant tubes and collars	13
7. Boundary resolution	14
8. Resolution of G-actions	15
9. De Rham cohomology and blow-up	18
10. Equivariant deRham cohomology	19
11. Borel-Cartan isomorphism	20
12. Borel Bundle	22
13. Reduced Cartan model	25
14. Equivariant K-theory	26
15. Representation bundle	28
16. Delocalized equivariant cohomology	30
17. Equivariant Chern character	31
Appendix. The circle action on the sphere	32
References	33

1. Introduction

In this paper the 'Folk Theorem' that the smooth action of a general compact Lie group, G, on a compact manifold, M, can be resolved to have components with unique isotropy type is proved and various cohomological consequences of this construction are derived. In particular, directly on the resolution, a 'delocalized' equivariant cohomology is defined, and shown to reduce to the cohomology of Baum, Brylinski and MacPherson in the Abelian case. The equivariant Chern character, giving an isomorphism to the equivariant K-theory with complex coefficients is then obtained from the usual Chern character by twisting with flat coefficients.

The resolution of a smooth Lie group action is discussed by Duistermaat and Kolk [5] (which we follow quite closely), by Kawakubo [11] and by Wasserman [15] but goes back at least as far as Jänich [10] and Hsiang [8]. In these approaches however there are residual finite group actions, particular reflections, due to the use of real projective blow up. Using radial blow up, and hence working in the category of manifolds with corners, these problems disappear and it is shown below that there is a canonical resolution which simultaneously resolves all isotropy types. This results in what we call a *resolution structure* (Y_*, ϕ_*). Here Y_I is a collection of compact manifolds with corners labelled by the set, $\mathcal{I}$, of conjugacy classes of isotropy groups, i.e. isotropy types, of the action. There are always minimal 'open' isotropy types for which the corresponding manifold, Y, (possibly not connected) gives a resolution of the action on M. That is, there is a smooth G-action on Y with unique isotropy type on each component and a smooth G-equivariant map

$$\beta : Y \longrightarrow M \tag{1}$$

which is a diffeomorphism of the interior of Y to the minimal isotropy type(s). Here, β is the iterated blow-down map for the resolution. This manifold carries a resolution tower in the sense that there is a G-invariant partition of the boundary hypersurfaces of Y into manifolds with corners, H_I, labelled by the non-minimal isotropy types, carrying G-equivariant fibrations

$$\phi_I : H_I \longrightarrow Y_I. \tag{2}$$

Here Y_I resolves the, generally singular, closure $M_I \subset M$, of the corresponding isotropy type M^I,

$$\beta_I : Y_I \longrightarrow M_I,\ \beta\big|_{H_I} = \beta_I \circ \phi_I. \tag{3}$$

Thus the resolution procedure resolves the inclusion relation between the M_I corresponding to the stratification of M by isotropy types, into the intersection relation between the H_I. The resolution tower for M naturally induces a resolution tower for each Y_I. The quotient of a group action with fixed isotropy type is smooth and the resolution tower induces a resolution, Z, of the quotient $G\backslash M$ in a similar form, as a compact manifold with corners with iterated fibrations of the boundary hypersurfaces over the quotients of the resolutions of the isotropy types. For the convenience of the reader a limited amount of background information on manifolds with corners and blow up is included below.

Once the resolution is constructed, the Cartan model for the equivariant cohomology, $H^*_G(M)$, is lifted to it and then projected to the quotient. In the free case, Borel's theorem identifies this localized equivariant cohomology with the cohomology of the quotient. In the case of a group action with unique isotropy type we show that the equivariant cohomology reduces to the cohomology over the quotient with coefficients in a flat bundle of algebras, which we call the *Borel bundle*, which is modelled on the invariant polynomials on the Lie algebra of the isotropy group – or equivalently the symmetric part of the total tensor product of the dual. In the general case the equivariant cohomology is identified with the relative cohomology, with respect to the tower of fibrations, twisted at each level by this flat coefficient bundle; the naturality of the bundle ensures that there are pull-back maps under the boundary fibrations induced on the twisted forms. Thus the Borel bundle represents the only equivariant information over the resolution of the quotient needed to recover the equivariant cohomology. In this construction we adapt Cartan's form of the isomorphism in the free case as presented by Guillemin and Sternberg [6] to the case of a fixed isotropy group.

Using the approach through G-equivariant bundles to equivariant K-theory, $\mathrm{K}_G(M)$, as discussed by Atiyah and Segal, ([1], [14]), we give a similar lift of it to the resolution of the action and then project to the resolution of the quotient. This results in a closely analogous reduced model for equivariant K-theory where the Borel bundles are replaced by what we term the 'representation bundles', which are flat bundles of rings modelled on the representation ring of the isotropy group over each resolved isotropy type. Cartan's form of the Borel-Weil construction gives a map back to the Borel bundle.

The representation bundles over the resolution tower amount to a resolution (in the Abelian case where it was initially defined) of the sheaf used in the construction of the delocalized equivariant cohomology of Baum, Brylinski and MacPherson ([2], see also [4]). The close parallel between the reduced models for equivariant K-theory and equivariant cohomology allow us to introduce, directly on the resolution, a *delocalized* deRham cohomology $H^*_{\mathcal{R},G}(M)$ generalizing their construction to the case a general compact group action. As in the Abelian case, the Chern character gives an isomorphism,

$$\mathrm{Ch}_G : K_G(M) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^{\mathrm{even}}_{\mathcal{R},G}(M) \tag{4}$$

to equivariant K-theory with complex coefficients. These results are also related to the work of Rosu, [13], and earlier work of Illman, [9], in the topological setting and likely carry over to other cohomology theories.

The authors are grateful to Eckhard Meinrenken for very helpful comments on the structure of group actions with unique isotropy type and to Haynes Miller for topological advice.

2. Manifolds with corners

By a *manifold with corners, M*, we shall mean a topological manifold with boundary with a covering by coordinate charts

$$M = \bigcup_j U_j,\ F_j : U_j \longrightarrow U'_j \subset \mathbb{R}^{m,\ell} = [0,\infty)^\ell \times \mathbb{R}^{m-\ell}, \tag{2.1}$$

where the U_j and U'_j are (relatively) open, the F_j are homeomorphisms and the transition maps

$$F_{ij} : F_i(U_j) \longrightarrow F_j(U_i),\ U_i \cap U_j \neq \emptyset \tag{2.2}$$

are required to be smooth in the sense that all derivatives are bounded on compact subsets; an additional condition is imposed below. The ring of smooth functions $\mathcal{C}^\infty(M) \subset \mathcal{C}^0(M)$ is fixed by requiring $(F_j^{-1})^*(u\big|_{U_j})$ to be smooth on U'_j, in the sense that it is the restriction to U'_j of a smooth function on an open subset of $\mathbb{R}^m$. The part of the boundary of smooth codimension one, which is the union of the inverse images under the F_i of the corresponding parts of the boundary of the $\mathbb{R}^{m,\ell}$, is dense in the boundary and the closure of each of its components is a *boundary hypersurface* of M. Subsequently we also describe the finite union of non-intersecting boundary hypersurfaces, in this sense, as a boundary hypersurface. We shall insist, as part of the definition of a manifold with corners, that these boundary hypersurfaces each be *embedded,* meaning near each point of each of these closed sets, the set itself is given by the vanishing of a local smooth defining function x which is otherwise positive and has non-vanishing differential at the point. In the absence of this condition M is a *tied manifold.* It follows that each such boundary hypersurface (in either sense), H, of a manifold with corners is globally the zero set of a smooth, otherwise positive, *boundary defining function* $\rho_H \in \mathcal{C}^\infty(M)$ with differential non-zero on H; conversely H determines ρ_H up to a positive smooth multiple. The set of connected boundary hypersurfaces is denoted $\mathcal{M}_1(M)$ and the *boundary faces* of M are the *components* of the intersections of elements of $\mathcal{M}_1(M)$. We denote by $\mathcal{M}_k(M)$ the set of boundary faces of codimension k. Thus if $F \in \mathcal{M}_k(M)$ and $F' \in \mathcal{M}_{k'}(M)$ then $F \cap F'$ can be identified with the union over the elements of a subset (possibly empty of course) which we may denote $F \cap F' \subset \mathcal{M}_{k+k'}(M)$. Once again it is convenient, and consistent, to describe a subset of $\mathcal{M}_k(M)$ with non-intersecting elements as a boundary face, and then the intersection of two boundary faces (even in this broader sense) *is* a boundary face.

By a manifold from now on we shall mean a manifold with corners, so the qualifier will be omitted except where emphasis seems appropriate. The traditional object will be called a boundaryless manifold.

As a consequence of the assumption that the boundary hypersurfaces are embedded, each boundary face of M is itself a manifold with corners (for a tied manifold the boundary hypersurfaces are more general objects, namely *articulated manifolds* which have boundary faces identified). At each point of a manifold with corners there are, by definition, *local product coordinates* $x_i \ge 0$, y_j where $1 \le i \le k$ and $1 \le j \le m-k$ (and either k or $m-k$ can be zero) and the x_i define the boundary hypersurfaces through the point. Unless otherwise stated, by local coordinates we mean local product coordinates in this sense. The local product structure near the boundary can be globalized:-

Definition 1. On a compact manifold with corners, M, a *boundary product structure* constitutes a choice $\rho_H \in \mathcal{C}^\infty(M)$ for each $H \in \mathcal{M}_1(M)$, of defining functions for the boundary hypersurfaces, an open neighborhood $U_H \subset M$ of each $H \in \mathcal{M}_1(M)$ and smooth vector fields V_H defined in each U_H such that

$$V_H \rho_K = \begin{cases} 1 & \text{in } U_H \text{ if } K = H \\ 0 & \text{in } U_H \cap U_K \text{ if } K \ne H, \end{cases} \tag{2.3}$$
$$[V_H, V_K] = 0 \text{ in } U_H \cap U_K \ \forall\ H, K \in \mathcal{M}_1(M).$$

Integration of each V_H from H gives a product decomposition of a neighborhood of H as $[0, \epsilon_H] \times H$, $\epsilon_H > 0$ in which V_H is differentiation in the parameter space on which ρ_H induces the coordinate. Shrinking U_H allows it to be identified with such a neighborhood without changing the other properties (2.3). Scaling ρ_H and V_H allows the parameter range to be taken to be $[0, 1]$ for each H.

Proposition 2.1. *Every compact manifold has a boundary product structure.*

Proof. The construction of the neighborhoods U_H and normal vector fields V_H will be carried out inductively. For the inductive step it is convenient to consider a strengthened hypothesis. Note first that the data in (2.3) induces corresponding data on each boundary face F of M – where the hypersurfaces containing F are dropped, and for the remaining hypersurfaces the neighborhoods are intersected with F and the vector fields are restricted to F – to which they are necessarily tangent. It may be necessary to subdivide the neighborhoods if the intersection $F \cap H$ has more than one component. In particular this gives data as in (2.3) but with M replaced by F. So such data, with M replaced by one of its hypersurfaces, induces data on all boundary faces of that hypersurface. Data as in (2.3) on a collection of boundary hypersurfaces of a manifold M, with the defining functions ρ_H fixed, is said to be consistent if all restrictions to a given boundary face of M are the same.

Now, let $\mathcal{B} \subset \mathcal{M}_1(M)$ be a collection of boundary hypersurfaces of a manifold M, on which boundary defining functions ρ_H have been chosen for each $H \in \mathcal{M}_1(M)$, and suppose that neighborhoods U_K and vector fields V_K have been found satisfying (2.3) for all $K \in \mathcal{B}$. If $H \in \mathcal{M}_1(M) \setminus \mathcal{B}$ then we claim that there is a choice of V_H and U_H such that (2.3) holds for all boundary hypersurfaces in $\mathcal{B} \cup \{H\}$, with the neighborhoods possibly shrunk. To see this we again proceed inductively, by seeking V_H only on the elements of a subset $\mathcal{B}' \subset \mathcal{B}$ but consistent on all common boundary faces. The subset $\mathcal{B}'$ can always be increased, since the addition of another element of $\mathcal{B} \setminus \mathcal{B}'$ to $\mathcal{B}'$ requires the same inductive step but in lower overall dimension, which we can assume already proved. Thus we may assume that V_H has been constructed consistently on all elements of $\mathcal{B}$. Using the vector fields V_K, each of which is defined in the neighborhood U_K of K, V_H can be extended, locally uniquely, from the neighborhood of $K \cap H$ in K on which it is defined to a neighborhood of $K \cap H$ in M by demanding

$$\mathcal{L}_{V_K} V_H = [V_K, V_H] = 0. \tag{2.4}$$

The commutation condition and other identities follow from this and the fact that they hold on K. Moreover, the fact that the V_K commute in the intersections of the U_K means that these extensions of V_H are consistent for different K on their common domains. In this way V_H satisfying all conditions in (2.3) has been constructed in a neighborhood of the part of the boundary of H in M corresponding to $\mathcal{B}$. In the complement of this part of the boundary one can certainly choose V_H to satisfy $V_H \rho_H = 1$ and combining these two choices using a partition of unity (with two elements) gives the desired additional vector field V_H once the various neighborhoods U_K are shrunk.

Thus, after a finite number of steps the commuting normal vector fields V_K are constructed near each boundary hypersurface. □

Note that this result is equally true with the wider definition of boundary hypersurface above, however it is crucial that the different 'hypersurfaces' do not have self-intersections.

The existence of such normal neighborhoods of the boundary hypersurfaces ensures the existence of 'product-type' metrics. That is, one can choose a metric g globally on M which near each boundary hypersurface H is of the form $d\rho_H^2 + \phi_H^* h_H$ where $\phi_H : U_H \longrightarrow H$ is the projection along the integral curves of V_H and h_H is a metric, inductively of the same product-type, on H. Thus near a boundary face

$F \in \mathcal{M}_k(M)$, which is defined by ρ_{H_i}, $i = 1, \dots, k$, the metric takes the form

$$g = \sum_{i=1}^{k} d\rho_{H_i}^2 + \phi_F^* h_F \tag{2.5}$$

where ϕ_F is the local projection onto F with leaves the integral surfaces of the k commuting vector fields V_{H_i}. In particular

Corollary 2.2. *On any manifold with corners there exists a metric g, smooth and non-degenerate up to all boundary faces, for which the boundary faces are each totally geodesic.*

A diffeomorphism of a manifold sends connected boundary faces to boundary faces – which is to say there is an induced action on $\mathcal{M}_1(M)$.

Definition 2. A diffeomorphism F of a manifold M is said to be *boundary intersection free* if for each $H \in \mathcal{M}_1(M)$ either $F(H) = H$ or $F(H) \cap H = \emptyset$. More generally a collection $\mathcal{G}$ of diffeomorphisms is said to be boundary intersection free if $\mathcal{M}_1(M)$ can be partitioned into subsets $B_i \subset \mathcal{M}_1(M)$ such that the elements of each B_i are disjoint and the induced action of each $F \in \mathcal{G}$ preserves the partition, i.e. maps each B_i to itself.

Thus a diffeomorphism is boundary intersection free if and only if it sends boundary faces in the wider sense to boundary faces.

A manifold with corners, M, can always be realized as an embedded submanifold of a boundaryless manifold. As shown in [12], if $\mathcal{F} \subset \mathcal{M}_1(M)$ is any disjoint collection of boundary hypersurfaces then the 'double' of M across $\mathcal{F}$, meaning $2_{\mathcal{F}}M = M \sqcup M / \cup\mathcal{F}$ can be given (not however naturally) the structure of a smooth manifold with corners and if $\mathcal{F}$ is a maximal disjoint subset then the number of boundary hypersurfaces of $2_{\mathcal{F}}M$ is strictly smaller than for M. After a finite number of steps, the iteratively doubled manifold is boundaryless and M may be identified with the image of one of the summands.

3. Blow up

A subset $X \subset M$ of a manifold (with corners) is said to be a *p-submanifold* if at each point of X there are local (product) coordinates for M such that $X \cap U$, where U is the coordinate neighborhood, is the common zero set of a subset of the coordinates. An *interior p-submanifold* is a p-submanifold no component of which is contained in the boundary of M. A p-submanifold of a manifold is itself a manifold with corners, and the collar neighborhood theorem holds in this context. Thus the normal bundle to X in M has (for a boundary p-submanifold) a well-defined inward-pointing subset, forming a submanifold with corners $N^+X \subset NX$ (defined by the non-negativity of all $d\rho_H$ which vanish on the submanifold near the point) and, as in the boundaryless case, the exponential map, but here for a product-type metric, gives a diffeomorphism of a neighborhood of the zero section with a neighborhood of X :

$$T : N^+X \supset U' \longrightarrow U \subset M. \tag{3.1}$$

The radial vector field on N^+X induces a vector field R near X which is tangent to all boundary faces.

Proposition 3.1. *If X is a closed p-submanifold in a compact manifold then the boundary product structure in Proposition 2.1, for any choice of boundary defining functions, can be chosen so that V_H is tangent to X unless X is contained in H.*

Proof. The condition that the V_H be tangent to X can be carried along in the inductive proof in Proposition 2.1, starting from the smallest boundary face which meets X. □

If $X \subset M$ is a closed p-submanifold then the radial blow-up of M along X is a well-defined manifold with corners $[M; X]$ obtained from M by replacing X by the inward-pointing part of its spherical normal bundle. It comes equipped with the blow-down map

$$[M; X] = S^+X \sqcup (M \setminus X), \ \beta : [M; X] \longrightarrow M. \tag{3.2}$$

The preimage of X, S^+X, is the 'front face' of the blow up, denoted $\mathrm{ff}([M; X])$. The natural smooth structure on $[M; X]$, with respect to which β is smooth, is characterized by the additional condition that a radial vector field R for X, as described above, lifts under β (i.e. is β-related) to $\rho_{\mathrm{ff}} X_{\mathrm{ff}}$ for a defining function ρ_{ff} and normal vector field X_{ff} for the new boundary introduced by the blow up.

Except in the trivial cases that $X = M$ or $X \in \mathcal{M}_1(M)$ the front face is a 'new' boundary hypersurfaces of $[M; X]$ and the preimages of the boundary hypersurface of M are unions of the other boundary hypersurfaces of $[M; X]$; namely the lift of H is naturally $[H; X \cap H]$. So, in the non-trivial cases and unless X separates some boundary hypersurface into two components, there is a natural identification

$$\mathcal{M}_1([M; X]) = \mathcal{M}_1(M) \sqcup \{\mathrm{ff}([M; X])\} \tag{3.3}$$

which corresponds to each boundary hypersurface of M having a unique 'lift' to $[M; X]$, as the boundary hypersurface which is the closure of the preimage of its complement with respect to X. In local coordinates, blowing-up X corresponds to introducing polar coordinates around X in M.

Lemma 3.2. *If X is a closed interior p-submanifold and M is equipped with a boundary product structure in the sense of Proposition 2.1 the normal vector fields of which are tangent to X then the radial vector field for X induced by the exponential map of an associated product-type metric commutes with V_H near any $H \in \mathcal{M}_1(M)$ which intersects X and on lifting to $[M; X]$, $R = \rho_{\mathrm{ff}} X_{\mathrm{ff}}$ where X_{ff} and together with the lifts of the ρ_H and V_H give a boundary product structure on $[M; X]$.*

Proof. After blow up of X the radial vector field lifts to be of the form $a\rho_{\mathrm{ff}} V_{\mathrm{ff}}$ for any normal vector field and defining function for the front face, with $a > 0$. The other product data lifts to product data for all the non-front faces of $[M; X]$ and this lifted data satisfies $[R, V_H] = 0$ near ff. Thus it is only necessary to show, using an inductive argument as above, that one can choose ρ_{ff} to satisfy $V_H \rho_{\mathrm{ff}} = 0$ and $R\rho_{\mathrm{ff}} = \rho_{\mathrm{ff}}$ in appropriate sets to conclude that $R = \rho_{\mathrm{ff}} V_{\mathrm{ff}}$ as desired. □

4. Iterated fibration structures

Recall that a fibration is a surjective smooth map $\phi : H \longrightarrow Y$ between manifolds with the property that for each component of Y there is a manifold Z such that each point p in that component has a neighborhood U for which there is a diffeomorphism giving a commutative diagram with the projection onto U :

$$
\begin{tikzcd}
\phi^{-1}(U) \arrow[rr, "F_U"] \arrow[dr, "\phi"'] & & Z \times U \arrow[dl, "\pi_U"] \\
& U. &
\end{tikzcd}
\tag{4.1}
$$

Set $\operatorname{codim}(\phi) = \dim Z$, which will be assumed to be the same for all components of Y. The image of a boundary face under a fibration must always be a boundary face (including the possibility of a component of Y).

The restriction of the blow-down map to the boundary hypersurface introduced by the blow up of a p-submanifold is a fibration, just the bundle projection for the (inward-pointing part of) the normal sphere bundle. In general repeated blow up will destroy the fibration property of this map. However in the resolution of a G-action the fibration condition persists. We put this into a slightly abstract setting as follows.

Definition 3. An iterated fibration structure on a manifold M is a fibration, $\phi_H : H \longrightarrow Y_H$ for each $H \in \mathcal{M}_1(M)$ with the consistency properties that if $H_i \in \mathcal{M}_1(M)$, $i = 1, 2$, and $H_1 \cap H_2 \neq \emptyset$ then $\operatorname{codim}(\phi_{H_1}) \neq \operatorname{codim}(\phi_{H_2})$ and

$$
\begin{gathered}
\operatorname{codim}(\phi_{H_1}) < \operatorname{codim}(\phi_{H_2}) \Longrightarrow \\
\phi_{H_1}(H_1 \cap H_2) \in \mathcal{M}_1(Y_{H_1}),\ \phi_{H_2}(H_1 \cap H_2) = Y_{H_2} \text{ and } \exists \text{ a fibration} \\
\phi_{H_1H_2} : \phi_{H_1}(H_1 \cap H_2) \longrightarrow Y_{H_2} \text{ giving a commutative diagram:}
\end{gathered}
$$

$$
\begin{tikzcd}
H_1 \cap H_2 \arrow[rr, "\phi_{H_1}"] \arrow[dr, "\phi_{H_2}"] & & \phi_{H_1}(H_1 \cap H_2) \arrow[dl, "\psi_{H_1H_2}"'] \\
& Y_{H_2}. &
\end{tikzcd}
\tag{4.2}
$$

Lemma 4.1. *An iterated fibration structure induces iterated fibration structures on each of the manifolds Y_H.*

Proof. Each boundary hypersurface F of Y_H is necessarily the image under ϕ_H of a unique boundary hypersurface of H, therefore consisting of a component of some intersection $H \cap K$ for $K \in \mathcal{M}_1(M)$. The condition (4.2) ensures that $\operatorname{codim}(\phi_H) < \operatorname{codim}(\phi_K)$ and gives the fibration $\phi_{HK} : F \longrightarrow Y_K$. Thus for Y_H the bases of the fibrations of its boundary hypersurfaces are all the Y_K's with the property that $H \cap K \neq \emptyset$ and $\operatorname{codim}(\phi_H) < \operatorname{codim}(\phi_K)$ with the fibrations being the appropriate maps ϕ_* from (4.2).

Similarly the compatibility maps for the boundary fibration of Y_H follow by the analysis of the intersection of three boundary hypersurfaces H, K and J where $\operatorname{codim}(\phi_H) < \operatorname{codim}(\phi_K) < \operatorname{codim}(\phi_J)$. Any two intersecting boundary hypersurfaces of Y_H must arise in this way, as $\phi_H(H \cap K)$ and $\phi_H(H \cap J)$ and the compatibility map for them is ϕ_{JK}. □

Proposition 4.2. *If M carries an iterated fibration structure and X is a closed interior p-submanifold which is transversal to the fibers of ϕ_H for each $H \in \mathcal{M}_1(M)$ then $[M; X]$ has an iterated fibration structure given by the $\beta_X^* \phi_H$ and $\phi_{\mathrm{ff}} = \beta_X\big|_{\mathrm{ff}}$:* $\mathrm{ff}([M; X]) \longrightarrow X$.

Recall that submanifolds which do not intersect are included in the notion of transversal intersection.

Proof. If $H \cap X \neq \emptyset$ the transversality condition ensures that $\phi_H(X) = Y_H$ and then $\phi_H\big|_X$ is itself a fibration. At each point $p \in Y_H$ there is a neighborhood U and a diffeomorphism, F, as in (4.1) such that

$$F(\phi^{-1}(U)) = U \times Z_H,\ F(\phi^{-1}(U) \cap X) = U \times Z_{H\cap X}.$$

The lift of H to $[M; X]$ is $[H; H \cap X]$ and

$$[U \times Z_H; U \times Z_{H\cap X}] = U \times [Z_H; Z_{H\cap X}],$$

so the diffeomorphism F induces a diagram

$$\begin{tikzcd}
\widetilde{\phi}_H^{-1}(U) \arrow[rr, "\widetilde{F}_U"] \arrow[dr, "\widetilde{\phi}_H"'] & & U \times [Z_H; Z_{H\cap X}] \arrow[dl, "\pi_U"] \\
& U &
\end{tikzcd}$$

which shows $\widetilde{\phi}_H = \phi_H \circ \beta_{\mathrm{ff}}$ to be a fibration.

In this way, each boundary hypersurface of $[M; X]$ has a fibration. Namely, a boundary hypersurface H is either the front face of the blow-up, and hence $\phi_{\mathrm{ff}} = \beta_X$ is the restriction of the blow-down map, or is the lift (or possibly a component of the lift) of a boundary hypersurface from M with the blow-down map as fibration. It therefore only remains to check the compatibility conditions.

The compatibility maps for the fibrations of the hypersurfaces of M clearly lift to give compatibility maps for the lifts. Thus it is only necessary to check compatibility between the fibrations on these lifted boundary hypersurfaces of $[M; X]$ and that of the front face. So, let H be a hypersurface of M that intersects X. In terms of the the notation above, the codimension of $\widetilde{\phi}_H$ is the equal to $\dim Z_H$ while the codimension of ϕ_{ff} is equal to $\dim Z_H - \dim Z_{H\cap X}$. The diagram (4.2) in this case is

$$\begin{tikzcd}
\mathrm{ff} \cap [H; H \cap X] \arrow[rr, "\beta_X"] \arrow[dr, "\widetilde{\phi}_H"'] & & H \cap X \arrow[dl, "\phi_H"] \\
& \phi_H(H \cap X) = Y_H. &
\end{tikzcd}$$

and so the requirements of Definition 3 are met. $\square$

Definition 4. If M carries an iterated fibration structure as in Definition 3 then a boundary product structure is said to be *compatible* with the fibration structure if for each pair of intersecting boundary faces H_1 and H_2 with $\operatorname{codim}(\phi_{H_1}) < \operatorname{codim}(\phi_{H_2})$

$$\rho_{H_2}\big|_{H_1} \in \phi_{H_1}^* \mathcal{C}^\infty(Y_{H_1}) \text{ near } H_2, \tag{4.3}$$

$$V_{H_2}\big|_{H_1} \text{ is } \phi_{H_1}\text{-related to a vector field on } Y_{H_1} \text{ near } H_2 \text{ and} \tag{4.4}$$

$$V_{H_1}\big|_{H_2} \text{ is tangent to the fibers of } \phi_{H_2}. \tag{4.5}$$

Proposition 4.3. *For any iterated fibration structure on a compact manifold, M, there is a compatible boundary product structure.*

Proof. We follow the proof on Proposition 2.1. In particular, we will use the notion of consistent boundary data on a collection of boundary hypersurfaces.

First, choose boundary defining functions satisfying (4.3). Let $H \in \mathcal{M}_1(M)$ and define $\mathcal{H} \subset \mathcal{M}_1(M)$ to consist of those boundary hypersurfaces $K \in \mathcal{M}_1(M)$ which intersect H and satisfy $\operatorname{codim}(\phi_K) < \operatorname{codim}(\phi_H)$. If $L \in \mathcal{H}$, we may assume inductively that we have chosen $\rho_H\big|_K$ for all boundary hypersurfaces $K \in \mathcal{H}$ with $\operatorname{codim}(\phi_K) < \operatorname{codim}(\phi_L)$, and then choose an extension to $H \cap L$ as a lift of a boundary defining function for the boundary face $\phi_L(H \cap L)$. This allows ρ_H to be defined on a neighborhood of $H \cap K$ in K for all $K \in \mathcal{H}$; extending it to a boundary defining function of H in M fulfills the requirements.

Next, suppose normal vector fields consistent with the iterated fibration structure and associated collar neighborhoods have been found for some subset $\mathcal{B} \subseteq \mathcal{M}_1(M)$ with the property that $H \in \mathcal{M}_1(M) \setminus \mathcal{B}$ and $K \in \mathcal{B}$ implies that $H \cap K = \emptyset$ or $\operatorname{codim}(\phi_H) < \operatorname{codim}(\phi_K)$. Let $H \in \mathcal{M}_1(M) \setminus \mathcal{B}$ be such that ϕ_H has maximal codimension among the boundary hypersurfaces of M that are not in $\mathcal{B}$. We show that there is a choice of V_H and U_H such that (2.3) and the conditions of Definition 4 hold for all boundary hypersurfaces in $\mathcal{B} \cup \{H\}$.

As before an inductive argument allows us to find V_H in a neighborhood of all intersections $H \cap K$ with $K \in \mathcal{B}$ with the property that $V_H\big|_K$ is tangent to the fibers of ϕ_K. Then V_H can be extended into U_K using the vector fields V_K by demanding that

$$\mathcal{L}_{V_K} V_H = [V_K, V_H] = 0$$

thus determining V_H locally uniquely in a neighborhood of $H \cap K$ in M for all $K \in \mathcal{B}$.

If $K \in \mathcal{M}_1(M) \setminus \mathcal{B}$ intersects H, then Y_K is itself a manifold with an iterated fibration structure and $\phi_H(H \cap K)$ is one of its boundary hypersurfaces. We can choose boundary product data on Y_K – since it has smaller dimension than M we may assume that the proposition has been proven for it. Under a fibration there is always a smooth lift of vector fields, a connection, so V_H on $\phi_H(H \cap K)$ may be lifted to a vector field V_H on $H \cap K$. In this way V_H may be chosen on the intersection of H with any of its boundary faces. Then V_H may be extended into a neighborhood U_H of H in M in such a way that $V_H \rho_H = 1$. By construction the commutation relations with all the previously constructed vector fields are satisfied and V_H is compatible with the iterated fibration structure at all boundary hypersurfaces in $\mathcal{B} \cup \{H\}$. Thus the inductive step is justified. □

Using Proposition 3.1 and Lemma 3.2 we see that iterated fibration structures and boundary product structures are preserved when blowing up appropriately placed p-submanifolds.

Proposition 4.4. *If M is a manifold with an iterated boundary fibration structure and $X \subset M$ is a closed interior p-submanifold which is transversal to the fibers of the fibration of each boundary hypersurface then $[M; X]$ has a boundary product structure which is compatible with the iterated fibration structure on $[M; X]$ given by Proposition 4.2, is such that the normal vector fields to boundary hypersurfaces other than the front face are β-related to a boundary product structure on M and is such that $\rho_{\mathrm{ff}} V_{\mathrm{ff}}$ is β-related to a radial vector field for X.*

5. G-actions

Let G be a compact Lie group and M a compact manifold (with corners). An action of G on M is a smooth map $A : G \times M \longrightarrow M$ such that $A(\mathrm{Id}, \zeta) = \zeta$ for all $\zeta \in M$ and

$$\begin{tikzcd} & G \times M \arrow[dr, "A"] & \\ G \times G \times M \arrow[ur, "\cdot \times \mathrm{Id}"] \arrow[dr, "\mathrm{Id} \times A"'] & & M \\ & G \times M \arrow[ur, "A"'] & \end{tikzcd} \tag{5.1}$$

commutes; here $\cdot$ denotes the product in the group.

We will usually denote $A(g, \zeta)$ as $g \cdot \zeta$. Since each element $g \in G$ acts as a diffeomorphism on M, it induces a permutation of the boundary hypersurfaces of M. If g is in the connected component of the identity of G, this is the trivial permutation.

Our convention is to assume, as part of the definition, that the action of G is boundary intersection free in the sense of Definition 2. That is, the set $\mathcal{M}_1(M)$ of boundary hypersurfaces can be partitioned into disjoint sets

$$\begin{gathered} \mathcal{M}_1(M) = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_l,\ H, H' \in B_i \Longrightarrow H \cap H' = \emptyset, \\ \text{and s.t. } g \cdot H \in B_i \text{ if } H \in B_i. \end{gathered} \tag{5.2}$$

The contrary case will be referred to as a G-action *with boundary intersection* – it is shown below in Proposition 7.2 that by resolution the boundary intersection can be removed. As justification for our convention, note that the G-actions which arise from the resolution of a G-action on a manifold without boundary are always boundary intersection free.

For a given G-action, the *isotropy* (or *stabilizer*) subgroup of G at $\zeta \in M$ is

$$G_\zeta = \{g \in G; g \cdot \zeta = \zeta\}. \tag{5.3}$$

It is a closed, and hence Lie, subgroup of G.

The action of G on M induces a pull-back action on $\mathcal{C}^\infty(M)$. The differential of this action at $\mathrm{Id} \in G$ induces the action of the Lie algebra $\mathfrak{g}$ on $\mathcal{C}^\infty(M)$ where $V \in \mathfrak{g}$ is represented by a vector field $\alpha(V) \in \mathcal{V}_{\mathrm{b}}(M)$, the Lie algebra of smooth vector fields on M tangent to all boundary faces, given by

$$\alpha(V) f(\zeta) = \frac{d}{dt} f\left(e^{-tV}\zeta\right)\big|_{t=0}, \text{ for all } f \in \mathcal{C}^\infty(M). \tag{5.4}$$

Since $[\alpha(V), \alpha(W)] = \alpha([V, W])$, this is a map of Lie algebras, $\alpha : \mathfrak{g} \longrightarrow \mathcal{V}_{\mathrm{b}}(M)$. The differential at $\zeta \in M$ will be denoted

$$\alpha_\zeta : \mathfrak{g} \longrightarrow T_\zeta M. \tag{5.5}$$

The image always lies in $T_\zeta F$ where $F \in \mathcal{M}_k(M)$ is the smallest boundary face containing ζ.

Proposition 5.1. *For any compact group action on a compact manifold, satisfying* (5.2)*, the elements $H \in B_i$ for each i have a collective defining function $\rho_i \in \mathcal{C}^\infty(M)$ which is G-invariant, there is a corresponding G-invariant product structure near* *the boundary consisting of smooth G-invariant vector fields V_i and neighborhoods U_i of* $\operatorname{supp}(B_i) = \cup\{H \in B_i\}$ *for each i such that*

$$V_i\rho_j = \begin{cases} 1 & \text{in } U_i \text{ if } i = j \\ 0 & \text{in } U_i \cap U_j \text{ if } i \neq j. \end{cases} \tag{5.6}$$

Furthermore there is a G-invariant product-type metric on M.

Proof. Any collection of non-intersecting boundary hypersurfaces has a common defining function, given by any choice of defining function near each boundary hypersurface in the set extended to be strictly positive elsewhere. If ρ_i' is such a defining function for $\operatorname{supp}(B_i)$ then so is $g^*\rho_i$ for each $g \in G$, since by assumption it permutes the elements of B_i. Averaging over G gives a G-invariant defining function. Similarly each of the vector fields V_H in (2.3) is only restricted near H so these can be combined to give collective normal vector fields V_i which then have the properties in (5.6). Since the commutation conditions are bilinear they cannot be directly arranged by averaging, but the normal vector fields can be constructed, and averaged, successively.

A product-type metric made up (iteratively) from this invariant data near the boundary can similarly be averaged to an invariant product-type metric. In fact the average of any metric for which the boundary faces are all totally geodesic has the same property. □

One direct consequence of the existence of an invariant product structure near the boundary is that, as noted above, a smooth group action on a manifold with corners can be extended to a group action on a closed manifold. This allows the consideration of the standard properties of group actions to be extended trivially from the boundaryless case to the case considered here of manifolds with corners.

Theorem 5.2. *Suppose M is a compact manifold with corners with a smooth action by a compact Lie group G – so assumed to satisfy* (5.2) *– then if M is doubled successively, as at the end of §2, across the elements of a partition into l non-intersecting but G-invariant subsets of the boundary hypersurfaces, to a manifold without boundary, $\widehat{M}$, then there is a smooth action of $\mathbb{Z}_2^l \times G$ on $\widehat{M}$ such that M embeds G-equivariantly into $\widehat{M}$ as a fundamental domain for the $\mathbb{Z}_2^l$-action.*

Proof. (See [12, Chapter 1] and [3, §II.1]) A partition of $\mathcal{M}_1(M)$ of the stated type does exist, as in (5.2). Proposition 5.1 shows the existence of a G-invariant product-type metric, collective boundary defining functions and product decompositions near the boundary hypersurfaces. First consider the union of two copies of M, denoted $M^{\pm}$, with all points in $\operatorname{supp}(B_1)$, i.e. all points in the boundary hypersurfaces in B_1, identified

$$M_1 = (M^+ \sqcup M^-)/\simeq_1,\ p \simeq_1 p',\ p = p' \text{ in } H \in B_1. \tag{5.7}$$

Now, the local product decompositions near each element of B_1 induce a $\mathcal{C}^\infty$ structure on M_1 making it again a manifold with corners. Thus ρ_1, the collective defining function for B_1 on $M = M^+$ can be extended to the smooth function

$$\rho_1' = \begin{cases} \rho_1 & \text{on } M^+ \\ -\rho_1 & \text{on } M^-. \end{cases} \tag{5.8}$$

Similarly the corresponding normal vector field V_1 extends to be smooth when defined as $-V_1$ on M^-. The action of G on M gives actions on $M^\pm$ which are consistent on $\operatorname{supp}(B_1)$ and the product decomposition of the group action shows that the combined action on M_1 is smooth. The boundary hypersurfaces of M_1 fall into two classes. Those arising from boundary hypersurfaces of M which meet one of the elements of B_1, these appear as the doubles of the corresponding hypersurfaces from M. The boundary hypersurfaces of M which do not meet an element of B_1 contribute two disjoint boundary hypersurfaces to M_1. It follows that the decomposition of $\mathcal{M}_1(M)$ in (5.2) induces a similar decomposition of $\mathcal{M}_1(M_1)$ in which each B_i, $i = 2, \dots, l$ contains the preimages of the boundary hypersurfaces of M, other than the elements of B_1, under the natural projection $M_1 \longrightarrow M$. The $\mathbb{Z}_2$ action on M_1 given by exchanging signs is smooth, by construction, and commutes with the G-action.

Thus this procedure can be repeated l times finally giving a manifold without boundary with smooth G-action as desired. □

6. Invariant tubes and collars

As note above the doubling construction allows the standard properties of group actions on boundaryless manifolds to be transferred to the context of manifolds with corners. In fact the standard proofs may also be extended directly.

If $\zeta \in M$ then the stabilizer G_ζ acts on $T_\zeta M$ and on the metric balls, of an invariant product-type metric, in $T_\zeta M$. If ζ is contained in a corner of codimension $k \geq 0$, then the exponential map for the metric identifies a G_ζ-invariant neighborhood of ζ in M with the inward-pointing vectors in the small ball in $T_\zeta M$, and hence establishes the basic linearization result.

Proposition 6.1 (Bochner)**.** *If $\zeta \in M$ is contained in a corner of codimension $k \geq 0$ then there is a G_ζ-invariant neighborhood $\mathcal{U}_\zeta$ of ζ in M, a linear action α_ζ of G_ζ on $\mathbb{R}^{m,k}$, and a G_ζ-equivariant diffeomorphism $\chi_\zeta : \mathcal{U}_\zeta \longrightarrow B^+$ to (the inward-pointing part of) a ball $B^+ \subset \mathbb{R}^{m,k}$.*

Corollary 6.2. *If G is a compact Lie group acting smoothly on a manifold M, then $M^G = \{\zeta \in M; g \cdot \zeta = \zeta \text{ for all } g \in G\}$ is an interior p-submanifold of M.*

A *slice at* $\zeta \in M$ for the smooth action of G is a p-submanifold, S, of M through ζ such that

i) $T_\zeta M = \alpha_\zeta(\mathfrak{g}) \oplus T_\zeta S$,
ii) $T_{\zeta'} M = \alpha_{\zeta'}(\mathfrak{g}) + T_{\zeta'} S$ for all $\zeta' \in S$,
iii) S is G_ζ-invariant,
iv) If $g \in G$ and $\zeta' \in S$ are such that $g \cdot \zeta' \in S$ then $g \in G_\zeta$.

For $\varepsilon \in (0, 1)$, set

$$S_\varepsilon = \chi_\zeta^{-1}(\alpha_\zeta(\mathfrak{g})^\perp \cap B^+(\varepsilon))$$

where $B^+(\epsilon) \subset T_\zeta M$ is the set of inward-pointing vectors of length less than ε. Since the vector fields in the image of α are tangent to all of the boundary faces, S_ε is necessarily a p-submanifold of M through ζ. Elements $k \in G_\zeta$ satisfy $T_\zeta A(k)(\alpha_\zeta(X)) = \alpha_\zeta(\mathrm{Ad}k(X))$, so the tangent action of G_ζ preserves $\alpha_\zeta(\mathfrak{g})$ and hence S_ε is G_ζ-invariant. The Slice Theorem for boundaryless manifolds [5, Theorem 2.3.3], applied to $\widehat{M}$, shows that S_ε is a slice for the G-action at ζ if ε is small enough.

Similarly, the following result is [5, Theorem 2.4.1] applied to $\widehat{M}$.

Proposition 6.3 (Tube Theorem)**.** *If G acts smoothly on a manifold M and $\zeta \in M$, then there is a representation space V of G_ζ with G_ζ-invariant subset V^+ of the form $\mathbb{R}^{\ell,k}$, a G-invariant neighborhood U of $\zeta \in M$, a G_ζ-invariant neighborhood, V, of the origin in V^+ and a G-equivariant diffeomorphism*

$$\phi : G \times_{G_\zeta} V^+ \longrightarrow U, \quad s.t.\ \phi(0) = \zeta.$$

It is straightforward to check (see [11, Lemma 4.16]) that the G-isotropy group of $[(g, v)] \in G \times_{G_\zeta} V$ is conjugate (in G) to the G_ζ isotropy group of v in V. Thus, if U is a neighborhood of ζ as in Proposition 6.3 and $\zeta' \in U$, then

$$G_{\zeta'} \text{ is conjugate to a subgroup of } G_\zeta. \tag{6.1}$$

Exponentiation using a product-type G-invariant metric allows a neighborhood of a G-invariant p-submanifold $X \subseteq M$ to be identified with a neighborhood of the zero section of its normal bundle.

Proposition 6.4 (Collar Theorem)**.** *If G acts smoothly on a manifold M and $X \subseteq M$ is a G-invariant interior p-submanifold, then there exists a G-invariant neighborhood U of X in M and a G-invariant diffeomorphism from the normal bundle NX of X to U that identifies the zero section of NX with X and for all sufficiently small $\varepsilon > 0$ the submanifolds*

$$\mathcal{S}_\varepsilon(X) = \{\zeta \in M; d(\zeta, X) = \varepsilon\}$$

are G-invariant and the G-actions on $S_\varepsilon(X)$ and $S_{\varepsilon'}(X)$ are intertwined by translation along geodesics normal to X.

Proof. As a p-submanifold, X has a tubular neighborhood in M, which by exponentiating we can identify with

$$U_\varepsilon = \{\zeta \in M; d(\zeta, X) \le \varepsilon\}. \tag{6.2}$$

For ε small enough, each $\zeta \in U_\varepsilon$ is connected to X by a unique geodesic of length less than ε, γ_ζ. Since the G-action is distance preserving and short geodesics are the unique length-minimizing curves between their end-points,

$$g \cdot \gamma_\zeta = \gamma_{g\cdot\zeta}, \text{ for every } g \in G,\ \zeta \in U_\varepsilon.$$

It follows that G preserves $S_{\varepsilon'}(X)$ for all $\varepsilon' < \varepsilon$ and that translation along geodesics normal to X intertwines the corresponding G-actions, as claimed. □

7. Boundary resolution

In this section the first steps towards resolution of a group action by radial blow-up are taken. Namely it is shown that on the blow-up of a G-invariant closed p-submanifold, X, the group action extends smoothly, and hence uniquely, from $M \setminus X$ to $[M; X]$; the blow-down map is then equivariant. Using this it is then shown that any smooth action, not requiring (5.2), on a manifold with corners lifts to a boundary intersection free action, i.e. one which does satisfy (5.2), after blowing-up appropriate boundary faces.

Let $\mathcal{J}(M)$ be the set of isotropy groups for a smooth action of G on M.

Proposition 7.1. *If $X \subseteq M$ is a G-invariant closed p-submanifold for a smooth action by a compact Lie group, G, on M then $[M;X]$ has a unique smooth G-action such that the blow-down map $\beta : [M;X] \longrightarrow M$ is equivariant and*

$$\mathcal{J}([M;X]) = \mathcal{J}(M \setminus X). \tag{7.1}$$

Proof. The blown-up manifold is

$$[M;X] = N^+X \sqcup (M \setminus X)$$

with smooth structure consistent with the blow up of the normal bundle to X along its zero section. Thus $[M;X]$ is diffeomorphic to $M \setminus U_\varepsilon$ with U_ε as in (6.2). This diffeomorphism induces a smooth G-action on $[M;X]$ with respect to which the blow-down map is equivariant. The result for isotropy groups, (7.1), follows from (6.1), namely the isotropy groups away from the front face of $[M;X]$ are certainly identified with those in $M \setminus X$ and the isotropy groups on $\mathrm{ff}([M;X])$ are identified with those in S_ε for small $\varepsilon > 0$. □

A general smooth group action will lift to be boundary intersection free on the *total boundary blow-up* of M. This manifold M_{tb}, discussed in [7, §2.6], is obtained from M by blowing-up all of its boundary faces, in order of increasing dimension. Blowing up all of the faces of dimension less than k separates all of the faces of dimension k so these can be blown-up in any order without changing the final space which is therefore well-defined up to canonical diffeomorphism.

Proposition 7.2. *If G acts smoothly on a manifold M, without necessarily satisfying* (5.2)*, the induced action of G on M_{tb} is boundary intersection free, i.e. does satisfy* (5.2)*.*

Proof. Let $\beta : M_{\mathrm{tb}} \longrightarrow M$ be the blow-down map. Any boundary hypersurface Y of M_{tb} is the lift of a boundary face $F = \beta(Y)$ of M. Since each element G acts on M by a diffeomorphism it sends $\beta(Y)$ to a boundary face of M of the same dimension as F, say $F' = \beta(Y')$. The induced action on M_{tb} will send the boundary face Y to Y' and, from the definition of M_{tb}, Y' is either equal to Y or disjoint from Y. Hence the action of G on M_{tb} is boundary intersection free. □

In fact it is generally possible to resolve an action to be boundary intersection free by blowing up a smaller collection of boundary faces. Namely, consider all the boundary faces which have the property that they are a component of an intersection $H_1 \cap \cdots \cap H_N$ where the $H_i \in \mathcal{M}_1(M)$ are intertwined by G, meaning that for each $1 \le i < j \le N$ there is an element $g_{ij} \in G$ such that $g_{ij}(H_j) = H_i$. This collection of boundary faces satisfies the chain condition that if F is an element and $F' \supset F$ then F' is also an element. In fact this collection of boundary faces is divided into transversal subcollections which are closed under intersection and as a result the manifold obtained by blowing them up in order of increasing dimension is well-defined. It is straightforward to check that the lift of the G-action to this partially boundary-resolved manifold is boundary intersection free.

8. Resolution of G-actions

The set, $\mathcal{J}(M)$, of isotropy groups which occur in a smooth G-action is necessarily closed under conjugation, since if $G_\zeta \in \mathcal{J}$ then $G_{g\zeta} = gG_\zeta g^{-1}$. Let $\mathcal{I} = \mathcal{J}/G$

be the set of conjugacy classes of isotropy groups for the action of G on M and for each $I \in \mathcal{I}$ let

$$M^I = \{\zeta \in M; G_\zeta \in I\}, \tag{8.1}$$

be the corresponding *isotropy type.* Proposition 6.3 shows these to be smooth p-submanifolds and they stratify M, with a natural partial order

$$I' \preccurlyeq I \text{ or } M^I \preccurlyeq M^{I'} \text{ if } K \in I \text{ is conjugate to a subgroup of an element of } I'.$$

Thus minimal elements with respect to $\preccurlyeq$ are the ones with the largest isotropy groups. We also set

$$M_I = \mathrm{cl}(M^I) \subset \bigcup_{I' \preccurlyeq I} M^{I'} \tag{8.2}$$

Proposition 8.1. *The isotropy types $M^I \subset M$ for a smooth action by a compact group G form a finite collection of p-submanifolds each with finitely many components.*

Proof. In [5, Proposition 2.7.1], this result is shown for boundaryless manifolds. By passing from M to $\widehat{M}$ as in Theorem 5.2, the same is true for manifolds with corners with the local product condition implying that M^I is a p-submanifold following from Proposition 6.2. □

Definition 5. A *resolution* of a smooth G-action on a compact manifold M (with corners) is a manifold, Y, obtained by the successive blow up of closed G-invariant p-submanifolds of M to which the G-action lifts to have a unique isotropy type.

Proposition 7.1 shows that there is a unique lifted G-action such that the iterated blow-down map is G-equivariant.

Such a resolution is certainly not unique – as in the preceding section, in the case of manifolds with corners, it is always possible to blow up a boundary face in this way, but this is never required for the resolution of an action satisfying (5.2). We show below that there is a canonical resolution obtained by successively blowing up minimal isotropy types. To do this we note that the blow-ups carry additional structure.

Definition 6. A *full resolution* for a G-action on a manifold, M, is a resolution in the sense of Definition 5 where Y carries an iterated fibration structure in the sense of Definition 3 with the fibration of each boundary hypersurface, $\phi_H : H \longrightarrow Y_H$, G-equivariant for a smooth G-action on Y_H which has a unique isotropy type not present in the action on Y. A *partial resolution structure* on a manifold Y is such an iterated fibration structure, with G-equivariant projections to bases with unique isotropy type not present in the manifold but where the action on Y may itself not have unique isotropy type.

Thus a full resolution is a resolution which has partial resolution structure.

Proposition 8.2. *Let M be a smooth manifold with a smooth boundary intersection free action by a compact Lie group G and partial resolution structure then any minimal isotropy type $X = M^I$ is a closed interior p-submanifold and if it is transversal to the fibers of all the boundary fibrations then $[M; X]$ has an induced partial resolution structure.*

Proof. As for a boundaryless manifold the minimal isotropy type is closed in M since its closure can only contain points with smaller isotropy group. It is an interior p-submanifold by Proposition 8.1, thus the blow up $[M;X]$ is well-defined. The G-action lifts smoothly to $[M;X]$ by Proposition 7.1 and the defining isotropy type I is not present in the resolved action. The assumed transversality allows Proposition 4.4 to be applied to conclude that the iterated fibration structure lifts to $[M;X]$ and so gives a partial resolution structure. □

Theorem 8.3. *A compact manifold (with corners), M, with a smooth, boundary intersection free, action by a compact Lie group, G, has a canonical full resolution, obtained by iterative blow-up of minimal isotropy types.*

Proof. In view of Proposition 8.2 it only remains to show, iteratively, that at each stage of the resolution any minimal isotropy type is transversal to the fibers of the earlier blow ups.

At the first step there is no required transversality condition and subsequent blow-ups can be carried out. Thus we can assume, inductively, that the partial resolution structure exists at some level and then we simply need to check that any minimal isotropy type for the lifted action is transversal to the fibers of each of the fibrations. Transversality is a local condition and at a point of boundary codimension greater than one the compatibility condition for an iterated fibration structure ensures that the fibration of one of the boundary hypersurfaces through that point has smallest leaves and it is necessarily the 'most recent' blow up. Thus we need only consider the case of a point of intersection of the minimal isotropy type and the front face produced by the blow up of an earlier minimal isotropy type in which there are (locally) no intermediate blow ups. Working locally, in the manifold before the earlier of the two blow ups, we simply have a manifold with a G-action and two intersecting isotropy types, one of which is locally minimal.

Now, by Proposition 6.3, if ζ is such a point of intersection, with isotropy group H, it has a neighborhood, U, with a G-equivariant diffeomorphism to $L = G \times_H V^+$ with V^+ the inward-pointing unit ball in a representation space V for H. The points in V with isotropy group H form a linear subspace and H acts on the quotient. Thus the action is locally equivariantly diffeomorphic to $G \times_H W^+ \times B$ where the action is trivial on B and $W^+ \subset W$ is a ball around the origin in a vector space W with linear H-action such that $W^H = \{0\}$. Thus any isotropy type meeting M^H at ζ is represented in twisted product by $G \times_H (V^+)^I \times B$ where I is an isotropy class in H. In particular such a neighboring isotropy type is a bundle over the minimal isotropy type and meets the fibers of a normal sphere bundle of small radius transversally. Thus, on blow-up it meets the fibers of the front face, which are these spheres, transversally.

Thus in fact the successive blow-ups are always transversal to the fibers of the early ones and hence the successive partial resolution structures lift and finally give a full resolution.

The uniqueness of this full resolution follows from the fact that at each stage the alternative is to blow up of one of a possibly finite set of minimal isotropy types. Since these are disjoint the order at this stage does not matter and hence, inductively, any such order produces a canonically diffeomorphic full resolution. □

We will see below (Lemma 11.2) that the orbit space $G\backslash Y$ of the full resolution of a G-action on M is a manifold with an iterated fibration structure representing the resolution of the quotient $G\backslash M$.

9. De Rham cohomology and blow-up

Consider a manifold H and the product $U = [0,1) \times H$. The form bundle decomposes as

$$\Lambda^k_p U = \Lambda^k_{p'} H + \Lambda^{k-1}_{p'} H,\ p = (x, p'), \tag{9.1}$$

corresponding to the decomposition into tangential and normal parts

$$u = u_t + dx \wedge u_n. \tag{9.2}$$

If $\chi \in \mathcal{C}^\infty([0,1))$ is a cut-off function, with $\chi(x) = 1$ for $x < \frac{1}{2}$ and $\chi(x) = 0$ for $x > \frac{3}{4}$ then the 'normal retraction operator'

$$T_H u = \chi(x) \int_0^x u_n(t, \cdot) dt \in \mathcal{C}^\infty(U; \Lambda^* H) \tag{9.3}$$

has the basic property that $u - dT_H u$ is purely tangential in $x < \frac{1}{2}$.

The definition of T can be written more invariantly in terms of contraction with ∂_x, since $u_n(x) = \iota_{\partial_x} u$ and the integral in (9.3) is along the fibers of ∂_x. In consequence, if V_H is a normal vector field for a boundary hypersurface $H \in \mathcal{M}_1(M)$, satisfying $V_H \rho_H = 1$ in a product neighborhood $U_H = \{\rho_H < 1\}$, then the retraction T depends only on the choice of χ and defines

$$T_H : \mathcal{C}^\infty(M; \Lambda^* M) \longrightarrow \mathcal{C}^\infty(M; \Lambda^* H),\ \iota_{V_H}(u - dTu) = 0 \text{ in } \{\rho_H < \tfrac{1}{2}\}. \tag{9.4}$$

Lemma 9.1. *If $\phi_K : K \longrightarrow Y_K$ is a fibration of a boundary hypersurface $K \neq H$ and V_H is tangent to the fibers of ϕ_K then for $u \in \mathcal{C}^\infty(M; \Lambda^* M)$*

$$i^*_K u = \phi^*_K v_K,\ v_k \in \mathcal{C}^\infty(K; \Lambda^* K) \Longrightarrow i^*_K(u - dTu) = \phi^*_K v_K. \tag{9.5}$$

Proof. By assumption, u pulls back to a basic form on K with respect to the fibration ϕ_K. Since V_H is, by assumption, tangent to the fibers of ϕ_K it follows that

$$\iota_{V_H} i^*_K u = \iota_{V_H} \phi^*_K v_K = 0 \text{ in } U_H \cap K \tag{9.6}$$

so $i^*_K T_H u = 0$ and hence $i^*_K(u - dTu) = \phi^*_K v_K$. □

On a manifold, Y, an iterated boundary fibration, $\Phi = \{\phi_H, Y_H\}$, induces a subcomplex of the deRham complex. This may be represented as a subcomplex of the direct sum of the deRham complexes for Y and all the bases:

$$\mathcal{C}^\infty_\Phi(Y; \Lambda^* Y) = \mathcal{C}^\infty(Y; \Lambda^* Y) \oplus \bigoplus_{H \in \mathcal{M}_1(Y)} \mathcal{C}^\infty(Y_H; \Lambda^* Y_H) \tag{9.7}$$

fixed by the constraints that $(u, v_{H_1}, \dots, v_{H_N}) \in \mathcal{C}^\infty_\Phi(Y; \Lambda^* Y)$ if and only if

$$i^*_H u = \phi^*_H v_H\ \forall\ H \in \mathcal{M}_1(Y). \tag{9.8}$$

Thus the v_H are determined by u since each ϕ_H is a fibration. Written out in this form the deRham differential acts diagonally on u and the v_H. The cohomology of this complex will be called the iterated fibration cohomology of Y.

Proposition 9.2. *If $X \subset Y$ is a closed interior p-submanifold which is transversal to the fibers of each ϕ_H and $\widetilde{\Phi} = \{\Phi, \beta_X\}$ is the induced iterated boundary fibration on $[M; X]$ from Proposition 4.2 then pull back under the blow-down map*

$$\beta_X^* : \mathcal{C}^\infty_\Phi(Y; \Lambda^* Y) \longrightarrow \mathcal{C}^\infty_{\widetilde{\Phi}}([Y; X]; \Lambda^*[Y; X]) \tag{9.9}$$

induces an isomorphism in cohomology.

Proof. Another basic property of the normal retraction operator is that, in terms of the inclusions $i_0 : \{x = 0\} \longrightarrow U$, $i_\chi : \{\chi = 1\} \longrightarrow U$ and the projection $\pi : \{\chi = 1\} \longrightarrow H$, any closed form u on U satisfies

$$i_\chi^*(u - dT_H u) = \pi^* i_0^* u. \tag{9.10}$$

Choosing a boundary product structure and radial vector field in accordance with Proposition 4.4 this property implies that, for any closed form

$$\tilde{u} \in \mathcal{C}^\infty_{\widetilde{\Phi}}([Y, X]; \Lambda^*[Y; X]),\ \tilde{u} - dT_H \tilde{u} = \beta_X^* u$$

where $u \in \mathcal{C}^\infty_\Phi(Y; \Lambda^* Y)$ is closed.

It follows that the iterated fibration complex of $[Y; X]$ retracts onto the lift of the iterated fibration complex of Y, and so pull-back induces an isomorphism in cohomology. □

Corollary 9.3. *Under iterated blow up of interior p-submanifolds which are successively transversal to the fibers of the fibrations of the blow-ups of previous centers the deRham cohomology lifts canonically to the iterated fibration cohomology of the blown up space.*

10. Equivariant deRham cohomology

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and let M be a manifold on which G acts smoothly. We recall Cartan's model for the equivariant cohomology of M.

The differential of the action of G on itself by conjugation is the *adjoint representation* of G on its Lie algebra, $\mathfrak{g}$,

$$\mathrm{Ad}_G : G \longrightarrow \mathrm{End}(\mathfrak{g}).$$

An inner product on $\mathfrak{g}$ is Ad_G-invariant precisely when the induced metric on G is invariant under left and right translations; by the usual averaging argument, such a metric exists if G is compact.

The action of g^{-1} on M is a diffeomorphism $M \longrightarrow M$ and so induces a left action on forms, elements of $\mathcal{C}^\infty(M; \Lambda^*)$, by pull-back. Recall that whenever G acts on two spaces X_1 and X_2, there is an induced action on $F \in \mathrm{Map}(X_1, X_2)$ by

$$(g \cdot F)(\zeta) = g \cdot \left(F(g^{-1} \cdot \zeta)\right), \text{ for every } \zeta \in X_1. \tag{10.1}$$

An *equivariant differential form* is a polynomial map

$$\omega : \mathfrak{g} \longrightarrow \mathcal{C}^\infty(M, \Lambda^* M)$$

which is invariant in that it intertwines the actions of G. This can also be thought of as a section of the bundle $S(\mathfrak{g}^*) \otimes \Lambda^*$ where $S(\mathfrak{g}^*)$, the symmetric part of the tensor powers of the dual $\mathfrak{g}^*$ may be identified with the ring of polynomials on $\mathfrak{g}^*$. We denote the set of these smooth equivariant forms by $\mathcal{C}^\infty_G(M; S(\mathfrak{g}^*) \otimes \Lambda^*)$; they form an algebra with respect to the usual wedge product. This algebra is graded by defining

$$\operatorname{degree}(\omega) = \text{differential form degree}(\omega) + 2\left(\text{polynomial degree}(\omega)\right).$$

The *equivariant differential*

$$d_{\mathrm{eq}} : \mathcal{C}^\infty_G(M; S(\mathfrak{g}^*) \otimes \Lambda^*) \longrightarrow \mathcal{C}^\infty_G(M; S(\mathfrak{g}^*) \otimes \Lambda^*(M)),$$
$$(d_{\mathrm{eq}}\omega)(v) = d(\omega(v)) - \mathfrak{i}_{\alpha(v)}(\omega(v)),\ \forall\ v \in \mathfrak{g},$$

then increases the degree by one and $d_{\mathrm{eq}}^2 = 0$. The resulting equivariant cohomology groups will be denoted $H^q_G(M)$. A theorem of Cartan shows that, if G and M are compact,

$$H^q_G(M) = H^q\left((M \times EG)/G\right)$$

where EG is a contractible space on which G acts freely.

For a manifold Y with a smooth G-action and an iterated fibration structure all of whose fibrations are equivariant, we define the equivariant iterated fibration complexes

$$\mathcal{C}^\infty_{G,\Phi}(Y; S(\mathfrak{g}^*) \otimes \Lambda^*) = \left(S(\mathfrak{g}^*) \otimes \mathcal{C}^\infty_\Phi(Y; \Lambda^* M)\right)^G$$

with differential d_{eq} acting diagonally. The resulting cohomology groups, $H^*_{G,\Phi}(Y)$ will be called the fibered equivariant cohomology of Y.

Theorem 10.1. *The complex $\mathcal{C}^\infty_{G,\Phi}(Y; S(\mathfrak{g}^*)\otimes\Lambda^*)$ for a full resolution of the smooth G-action on a manifold with corners retracts into the pull-back of the Cartan complex and hence the fibered equivariant cohomology of the resolution is naturally isomorphic to the equivariant cohomology of M.*

Proof. As explained in Theorem 8.3 the full resolution is obtained from M by iterative blow up of interior p-submanifolds of M and, at each step, the p-submanifold is transversal to the fibers of the fibrations of the boundary fibrations. Thus it suffices to generalize Corollary 9.3 to the equivariant setting.

For a product $U = [0,1)_x \times H$ with a G-action induced from a G-action on H, an equivariant form u as in (9.2) decomposes

$$d^U_{\mathrm{eq}} u = d^H_{\mathrm{eq}} u_t + dx \wedge (\partial_x u_t - d^H_{\mathrm{eq}} u_n).$$

It follows that, with T_H the normal retraction operator (9.3) and π, i_0 and i_χ as in (9.10), $u - d_{\mathrm{eq}} T_H u$ restricted to $x < \frac{1}{2}$ is always purely tangential and is equal to $\pi^* i_0^* u$ near H if u is equivariantly closed. Thus the proof of Proposition 9.2 extends to the equivariant context by replacing d by d_{eq}. □

11. Borel-Cartan isomorphism

For a free action by a compact group on a compact manifold, Borel showed that

$$H^*_G(M) = H^*(G\backslash M) \tag{11.1}$$

where there is a natural pull-back from right to left. An alternative proof due to Cartan, exhibits this isomorphism as an extension of Chern-Weil theory. In this section, apart from working on manifolds with corners (for boundary-intersection free compact group actions) we extend Cartan's argument to the setting, also analyzed by Borel, where there is a unique isotropy type.

If G acts smoothly and freely on a manifold M then the quotient is a smooth manifold and $\pi : M \longrightarrow G\backslash M$ is a principal bundle. Forms on M can be interpreted as sections of $S(\mathfrak{g}^*) \otimes \Lambda^*$, corresponding to the constant polynomials and pull-back of forms then gives a map

$$\pi^\# : \mathcal{C}^\infty(G\backslash M; \Lambda^*) \longrightarrow (\mathcal{C}^\infty(M; S(\mathfrak{g}^*) \otimes \Lambda^*))^G \tag{11.2}$$

which induces an isomorphism between ordinary and equivariant cohomology. We briefly recall the treatment of Cartan's map, which is a left inverse of this map, by Guillemin and Sternberg in [6]. This arises from the choice of a connection on M as a principal G-bundle. Such a connection can be thought of as an Ehresmann connection, which is to say a smooth ('horizontal') subbundle $\Upsilon \subset TM$ which is G-invariant and transversal to the group action. The corresponding connection form, $\theta : TM \longrightarrow \mathfrak{g}$, is determined by

$$\begin{gathered} \theta_m(v) = 0 \ \forall \ v \in \Upsilon_m, \\ \theta(\alpha(v)) = v \text{ for all } v \in \mathfrak{g} \end{gathered} \tag{11.3}$$

where α is the differential of the group action from (5.4). The curvature of the connection is the G-equivariant $\mathfrak{g}$-valued two form, Ω_θ, defined in terms of the projection h onto Υ along $\alpha(\mathfrak{g})$ by

$$\Omega_\theta(X, Y) = -\theta([hX, hY]).$$

Theorem 11.1. *If M is a manifold on which G acts freely, then for any G-connection the map*

$$S(\mathfrak{g}^*) \otimes \mathcal{C}^\infty(M; \Lambda^*) \ni f \otimes \omega \mapsto f(\Omega_\theta) \wedge \omega_{\text{hor}} \in \mathcal{C}^\infty(M; \Lambda^*)_{\text{hor}}$$

projects to

$$\mathrm{CW}_\theta : (S(\mathfrak{g}^*) \otimes \mathcal{C}^\infty(M; \Lambda^*))^G \longrightarrow \mathcal{C}^\infty(G\backslash M; \Lambda^*)$$

which is a left inverse to $\pi^\#$ in (11.2) *and induces an isomorphism from equivariant to deRham cohomology.*

Proof. Directly from the definition it follows that $\mathrm{CW}_\theta \circ \pi^\# = \mathrm{Id}$.

The G-action on M makes $\mathcal{C}^\infty(M, \Lambda^*)$ into a G^* algebra in the sense of [6, Definition 2.3.1] and the connection induces the structure of a $W(\mathfrak{g})^*$-module in the sense of [6, Definition 3.4.1]. For a $W(\mathfrak{g})^*$-module 'Cartan's formula' is analyzed in [6, Chapter 5] where it is shown that $\pi^\# \circ \mathrm{CW}_\theta$ is cohomologous to the identity. □

For manifolds with corners, Theorem 11.1 gives a result in the setting of absolute equivariant and deRham cohomology, but the relative version also follows from the same argument. There are several complexes realizing relative cohomology. One of the standard complexes is forms with compact support in the interior, to which this proof applies unchanged. More naturally in the present context one can consider smooth forms which pull back to zero on each boundary face. Then, provided the connection is given by projection onto the orthocomplement of the image of the Lie algebra using a G-invariant product-type metric on M the argument carries over.

If the action of G on M is not free but has a unique isotropy type, then the quotient $G\backslash M$ is still smooth and M fibers smoothly over it but in general it is not a principal bundle, as shown by Borel – see [5, Theorem 2.6.7].

Proposition 11.2 (Borel)**.** *Let M be a manifold with a (boundary intersection free) G-action with a unique isotropy type, if $N(K)$ is the normalizer of an isotropy* *group K then M is G-equivariantly diffeomorphic to $G\times_{N(K)} M^K$ and the inclusion $M^K \hookrightarrow M$ induces a diffeomorphism*

$$(N(K)/K)\backslash M^K = N(K)\backslash M^K \cong G\backslash M.$$

Proof. We follow the proof of [5, Theorem 2.6.7] in the boundaryless case. It is shown in Proposition 8.1 that for a fixed isotropy group M^K is a smooth interior p-submanifold. The normalizer $N(K)$ acts on M^K with isotropy group K so the quotient group $W(K) = N(K)/K$ acts freely on M^K. Thus the quotient $W(K)\backslash M^K$ is smooth. The diagonal action of $N(K)$ on the product

$$N(K) \times (G \times M^K) \ni (n,(g,m)) \longmapsto (gn^{-1}, nm) \in G \times M^K \tag{11.4}$$

is free, so the quotient $G \times_{N(K)} M^K$ is also smooth. Moreover the action of G on M factors through the quotient, $gm = gn^{-1} \cdot nm$, so defines the desired smooth map

$$G \times_{N(K)} M^K \longrightarrow M. \tag{11.5}$$

This is clearly G-equivariant for the left action of G on $G \times_{N(K)} M^K$ and is the identity on the image of $\{\mathrm{Id}\} \times M^K$ to M^K. The Slice Theorem shows that the inverse map, $m \longmapsto [(g,m')]$ if $m' \in M^K$ and $gm' = m$ is also smooth, so (11.5) is a G-equivariant diffeomorphism.

The quotient $G\backslash(G \times_{N(K)} M^K) = N(K)\backslash M^K$ is smooth and the smooth structure induced on $G\backslash M$ is independent of the choice of K. □

12. Borel Bundle

Thus for manifolds M with a unique isotropy type the quotient $G\backslash M$ is a manifold and it is therefore natural to hope for an extension of (11.1) to this more general setting. We will see that this is possible provided the deRham cohomology of the quotient is twisted by a flat bundle, which we now define and call the Borel bundle.

From Proposition 11.2 it follows that, in this case of unique isotropy type, the Lie algebras of the isotropy groups form a smooth bundle over M on which the adjoint action of G is equivariant. The bundle of polynomials on the fibers of this bundle is similarly equivariant for the adjoint action – although infinite dimensional this is just the direct sum of the finite dimensional bundles homogeneous of each degree. This action descends to the subbundle of invariant polynomials, with fibre $S(\mathfrak{k}_m^*)^{K_m}$, for the adjoint action of the isotropy group at each point. As an equivariant bundle this descends to the quotient, giving, by definition, the *Borel bundle, B*, over $G\backslash M$.

This bundle is non-trivial in general. For instance, let M be the connected double cover of the circle with the associated free $\mathbb{Z}_2$ action. This action can trivially be extended to an action of the twisted product $G = \mathbb{S}^1 \rtimes \mathbb{Z}_2$ (where commuting the non-trivial $\mathbb{Z}_2$ element past an element $z \in \mathbb{S}^1 \subseteq \mathbb{C}$ replaces z with $\overline{z}$) since the twisting does not affect the $\mathbb{Z}_2$ product. Now M has a unique isotropy group, $K = \mathbb{S}^1$, which is normal in G. The Weyl group of K in G is $\mathbb{Z}_2$ and it acts non-trivially on $\mathbb{R}^1$, the Lie algebra of K. The Borel bundle will thus be a non-trivial line bundle on $M/\mathbb{Z}_2 = \mathbb{S}^1$.

Lemma 12.1. *For an action with unique isotropy type, the Borel bundle over the quotient has a natural flat structure induced by any choice of isotropy group, K,* *as the quotient of the trivial bundle* $S(\mathfrak{k}^*)^K$ *over* M^K *by the action of* $W(K) = N(K)/K$, *in which the connected component of the identity acts trivially.*

Proof. By Borel's theorem above, $G\backslash M$ is diffeomorphic to $N(K)\backslash M^K$ and hence to $W(K)\backslash M^K$. Let $W_0(K)$ be the connected component of the identity in $W(K)$ and let $N'(K) \subset N(K)$ be the inverse image of $W_0(K)$ in $N(K)$. Then $N'(K)$ is a normal subgroup of $N(K)$ with finite quotient.

The Lie algebra $\mathfrak{k}$ of K is a Lie subalgebra of $\mathfrak{n}$, the Lie algebra of $N(K)$. Choosing an Ad invariant metric on $\mathfrak{n}$, the Lie algebra,$\mathfrak{w}$, of $W(K)$ may be identified with $\mathfrak{k}^\perp$. The Ad-invariance of $\mathfrak{k}$ implies the Ad-invariance of $\mathfrak{w}$; thus $\mathfrak{k}$ and $\mathfrak{w}$ Lie commute. Exponentiating the Lie algebra $\mathfrak{w}$ into $N'(K)$ gives a subgroup $W'(K)' \subset N'(K)$ which is a finite cover of $W_0(K)$ and which commutes with K.

It follows that the adjoint action of $N'(K)$ on $S(\mathfrak{k}^*)^K$ is trivial and that the trivial bundle $S(\mathfrak{k}^*)^K$ over M^K descends to be flat over $N(K)\backslash M^K$ as the quotient of the flat bundle $S(\mathfrak{k}^*)^K \times N'(K)\backslash M^K$ under the finite group $N(K)/N'(K)$. If $\widetilde{K}$ is another choice of isotropy group and $g \in G$ is such that $gKg^{-1} = \widetilde{K}$, then $gN(K)g^{-1} = N(\widetilde{K})$ so the diffeomorphism $g\cdot : M^K \longrightarrow M^{\widetilde{K}}$ intertwines the $N(K)$ and $N(\widetilde{K})$-actions and descends to a diffeomorphism

$$W(K)\backslash M^K \longrightarrow W(\widetilde{K})\backslash M^{\widetilde{K}}. \tag{12.1}$$

The adjoint action of g sends $\mathfrak{k}$ to $\widetilde{\mathfrak{k}}$, so it identifies the Borel bundle B over $W(K)\backslash M^K$ with the Borel bundle $\widetilde{B}$ over $W(\widetilde{K})\backslash M^{\widetilde{K}}$ and pulling back via the diffeomorphism $G\backslash M \longrightarrow W(K)\backslash M^K$ gives the Borel bundle B. $\square$

Given the natural diffeomorphism between $G\backslash M$ and $W(K)\backslash M^K$, a smooth section $\gamma \in \mathcal{C}^\infty(G\backslash M; B \otimes \Lambda^*)$ is naturally identified with an element of

$$\mathcal{C}^\infty(W(K)\backslash M^K; B) \otimes_{\mathcal{C}^\infty(G\backslash M)} \mathcal{C}^\infty(G\backslash M; \Lambda^*).$$

Thus, it is equivalently thought of as a smooth, G-invariant, basic, differential form on M with coefficients in the bundle of invariant polynomials on the Lie algebra of the istotropy groups. Given a fixed choice of Ad-invariant metric on $\mathfrak{g}$, there is therefore a well-defined 'pull-back' map which we denote

$$\pi^\# : \mathcal{C}^\infty(G\backslash M; B \otimes \Lambda^*) \longrightarrow \left(\mathcal{C}^\infty(M; S(\mathfrak{g}^*) \otimes \Lambda^*)\right)^G. \tag{12.2}$$

Here, an Ad-invariant polynomial on the Lie algebra of G_ζ is extended to be constant on the orthocomplement. By the Ad-invariance of the metric, this gives a G-invariant form on M.

Theorem 12.2. *For a smooth action, with unique isotropy type, by a compact Lie group* G *on a compact manifold (with corners)* M, $\pi^\#$ *in* (12.2) *is a map of complexes and induces an isomorphism between* B*-twisted deRham cohomology on* $G\backslash M$ *and* G*-equivariant cohomology on* M.

Proof. The image under $\pi^\#$ of an element of $\mathcal{C}^\infty(G\backslash M; B \otimes \Lambda^*)$ is basic as a form, and hence, for any choice of isotropy group K can be regarded as a basic form on each M^K. It follows that the action of d_{eq} on these G-invariant forms reduces to the action of d_{eq} for the $W(K)$-action on M^K which on these basic $W(K)$-invariant forms is just the action of d.

Thus it suffices to show that the subspace of $N(K)$-invariant forms on M^K which are basic with coefficients in $S(\mathfrak{k}^*)^K$ generates the $N(K)$-equivariant cohomology of M^K. This is a form of Borel's theorem for free actions.

Thus we may just assume that the action of G on M has a fixed, normal, isotropy group K. In the normal case we may use the notation from the proof of Lemma 12.1. With $G' \subset G$ the lift of the connected component W_0 of $W = G/K$ the G-equivariant cohomology of M is, essentially by definition of the Cartan complex, the G/G'-invariant part of the G'-equivariant cohomology of M. Since the Lie algebra splits $\mathfrak{g} = \mathfrak{k} + \mathfrak{w}$, with $\mathfrak{w}$ the Lie algebra of W, the polynomial coefficients may be identified with $S(\mathfrak{k}^*) \otimes S(\mathfrak{w}^*)$. Since the adjoint action of K on $\mathfrak{w}$ is trivial, as is the action of W_0 on $\mathfrak{k}$, the Cartan forms for the action of G' on M are simply the elements of

$$S(\mathfrak{k}^*)^K \otimes \mathcal{C}^\infty_{W_0}(M; S(\mathfrak{w}^*) \otimes \Lambda^*). \tag{12.3}$$

The equivariant differential is just the W_0-equivariant differential on the second factor, acting trivially on the first. Thus, Cartan's isomorphism applied to the second factor here shows that the G'-equivariant cohomology is precisely given by the action of d on basic forms:

$$S(\mathfrak{k}^*)^K \otimes \mathcal{C}^\infty(W_0\backslash M; \Lambda^*). \tag{12.4}$$

As noted above, the G-equivariant cohomology in this (normal) case is the G/G' invariant part of the G'-equivariant cohomology, which is precisely the B-twisted cohomology of $G\backslash M$, in the normal and hence the general case. □

Suppose H and Y are manifolds with smooth G-actions with unique isotropy types and $\phi : H \longrightarrow Y$ is a G-equivariant map. If $\zeta \in H$ then the isotropy group at ζ, G_ζ, is a closed subgroup of the isotropy group $G_{\phi(\zeta)}$ at $\phi(\zeta)$. The invariant polynomials on the Lie algebra of a group restrict to be invariant polynomials (for the action of the subgroup) on the Lie algebra of any subgroup. Thus it follows that any section of the bundle of invariant polynomials on the Lie algebras of the isotropy groups on Y pulled back to H and restricted at each point, is a section of the corresponding bundle on H. Thus the projection of the map to the quotients, which is necessarily smooth, induces a pull-back map on sections of the Borel bundles tensored with forms which we denote

$$\phi^\# : \mathcal{C}^\infty(G\backslash Y; B \otimes \Lambda^*) \longrightarrow \mathcal{C}^\infty(G\backslash H; B \otimes \Lambda^*). \tag{12.5}$$

Note that it involves both pull-back and restriction to the Lie subalgebra at each point.

Lemma 12.3. *Let H and Y be manifolds with smooth G-actions, each of which has a unique isotropy type, and suppose that $\phi : H \longrightarrow Y$ is a G-equivariant smooth map then $\phi^\#$ in* (12.5) *is flat – that is, the pull-back of a closed (local) section is closed.*

Proof. As noted above, a (local) section of $B \otimes \Lambda^*$ on $G\backslash Y$ may be identified with its pull-back which is a basic G-invariant form with coefficients in the corresponding bundle of invariant polynomials on the isotropy Lie algebras. When the polynomial coefficients are extended, using an Ad-invariant metric, to polynomials on $\mathfrak{g}$, the equivariant differential reduces to the standard differential since the forms are basic. Thus, a closed local B-twisted form on $G\backslash Y$ lifts to an equivariantly closed, basic, G-invariant form (on the preimage) with values in the polynomials on $\mathfrak{g}$ which vanish on the orthocomplement of the isotropy algebra at each point. Restricting this form to a submanifold H^K for any choice of isotropy group gives a closed form in the ordinary sense, with polynomial coefficients. Since the bundle of isotropy algebras for the action of G on H^K is trivial, the restriction of the polynomial coefficients to $\mathfrak{k}$ gives a basic closed form. This form does not depend on the earlier choice of Ad-invariant metric since $\mathfrak{k}$ is a subspace of the isotropy Lie algebra for the image point. Since the form is basic, this lifts to a unique form on H along H^K. Moveover, the G-invariance of the lifted form implies that the forms for different choices of the isotropy group combine to give a G-invariant form which is basic and equivariantly closed, since this follows from its G-invariance and the fact that it is closed on each H^K. Moreover, it is the image of the lift of a section of $B \otimes \Lambda^*$ from (the preimage of the open set in) $G\backslash H$. Thus in fact the map $\phi^\#$ does map sections to sections, is defined on the fibres and intertwines the (twisted) differentials. □

13. Reduced Cartan model

The resolution of a compact group action on a manifold X in §8 gives a resolution of the quotient as a manifold with corners with iterated fibration structure. With the resolution denoted by Y, let $Z = G\backslash Y$ be the resolution of the quotient; this is smooth since G acts with unique isotropy type on Y. The boundary hypersurfaces of the quotient may be identified with the equivalence classes under the action of G of the boundary hypersurfaces of Y and the boundary fibrations of Y, being G-equivariant, descend to give an iterated fibration structure on Z, $\psi_H : H \longrightarrow Z_H$ and consistency maps ψ_{HK} on the boundary faces of the Z_H. Each $Z_H = G\backslash Y_H$ carries a natural Borel bundle and by Lemma 12.3 there are pull-back maps (involving restriction to a bundle of subalgebras) between sections of these bundles, covering the ψ_H and ψ_{HK}. We will denote the iterated fibration structure on Z by (Z_*, ψ_*) or simply Ψ.

Definition 7. The reduced Cartan model, $\mathcal{C}^\infty_{B,\Psi}(Z;\Lambda^*)$, on the resolution of the action of a compact group G on a compact manifold M is the relative flat-twisted deRham complex consisting of the subcomplex of the direct sum

$$\mathcal{C}^\infty(Z; B \otimes \Lambda^*) \oplus \bigoplus_{H \in \mathcal{M}_1(Z)} \mathcal{C}^\infty(Z_H; B \otimes \Lambda^*) \tag{13.1}$$

which satisfies the natural consistency conditions under pull-back under all ψ_H (and hence ψ_{HK}.)

Theorem 13.1. *The cohomology of the reduced Cartan complex for the resolution is naturally isomorphic to the equivariant cohomology of the space, with the pull-back under $\pi^\#$ inducing the isomorphism.*

Proof. To prove that $\pi^\#$ induces an isomorphism in cohomology we pass to a relative form of both the lifted Cartan complex of §10 and of the reduced Cartan complex above. Thus, let $\mathcal{B} \subseteq \mathcal{M}_1(Z)$ be a collection of boundary faces which is closed below, in the sense that it contains any boundary face which corresponds to an isotropy type containing the isotropy group of an element of $\mathcal{B}$. Thus $\mathcal{B}$ can also be identified with the image of a G-invariant subset $\mathcal{B}_Y \subset \mathcal{M}_1(Y)$ with the same property. Then consider the subcomplex of (13.1)

$$\mathcal{C}^\infty_{B,\Psi}(Z;\Lambda^*;\mathcal{B}) = \{(u, v_1, \ldots, v_N) \in \mathcal{C}^\infty_{B,\Psi}(Z;\Lambda^*); v_i = 0 \text{ for all } H_i \in \mathcal{B}\}$$

and correspondingly for the resolved Cartan complex on Y. Again the pull-back map $\pi^\#$ acts from the reduced complex to the resolved complex.

In the case that $\mathcal{B} = \mathcal{M}_1(Z)$ we already know that $\pi^\#$ induces an isomorphism in cohomology since the cohomology is simply the relative cohomology of the main manifold with its group action with unique isotropy type and hence Theorem 12.2 applies.

Now, consider two such subsets $\mathcal{B} \subset \mathcal{B}'$ which differ by just one element $H \in \mathcal{M}_1(Z)$. This gives a two short exact sequence of complexes with maps induced by $\pi^\#$:

$$\begin{tikzcd} \mathcal{C}^\infty_{B,\Psi}(Z;\Lambda^*;\mathcal{B}') \arrow[r] \arrow[d, "\pi^\#"] & \mathcal{C}^\infty_{B,\Psi}(Z;\Lambda^*;\mathcal{B}) \arrow[r] \arrow[d, "\pi^\#"] & \mathcal{C}^\infty_{B,\Psi}(H;\Lambda^*;\mathcal{B}) \arrow[d, "\pi^\#"] \\ \mathcal{C}^\infty_{G,\Phi}(Y;\Lambda^*;\mathcal{B}'_Y) \arrow[r] & \mathcal{C}^\infty_{G,\Phi}(Y;\Lambda^*;\mathcal{B}_Y) \arrow[r] & \mathcal{C}^\infty_{G,\Phi}(H_Y;\Lambda^*;\mathcal{B}_Y). \end{tikzcd} \tag{13.2}$$

Here of course H is really the orbit of one hypersurface in $\mathcal{M}_1(M)$ under the G-action. Since the action is always assumed to be boundary intersection free, the elements are disjoint.

Now, proceeding by induction over the dimension of M we may assume that $\pi^\#$ induces an isomorphism on cohomology when acting on H. Also inductively, starting from $\mathcal{M}_1(Y)$, we may assume that it induces an isomorphism for the cohomology relative to $\mathcal{B}'$. Thus the Fives Lemma applies to the long exact sequence in cohomology to show that it also induces an isomorphism on cohomology relative to $\mathcal{B}$ and hence in general. □

Note that the reduced Cartan cohomology can also be identified directly with a cohomology theory over the quotient $G\backslash M$ for any smooth G action by a compact group on a compact manifold. Namely the Borel bundle induces a sheaf over $G\backslash M$ where sections over an open set are precisely sections over the preimages of the open set in the resolution (Z_*, ψ_*) with the compatibility condition under lifting and restriction. Then the (Čech) cohomology with coefficients in this sheaf can be identified, by standard arguments, with the cohomology of the reduced Cartan complex.

14. Equivariant K-theory

Next we turn to an analysis of the lifting and identification of equivariant K-theory under resolution in parallel to the discussion of cohomology above.

We use the model for equivariant K-theory as the Grothendieck group based on equivariant (complex) vector bundles over the manifold. Thus if M is a compact manifold with corners with a smooth action of a compact Lie group G then an (absolute) equivariant K-class on M is fixed by a pair of vector bundles $E^\pm$ each of which has an action of G as bundle isomorphisms covering the action of G on the base:

$$L^\pm_E(g) : g^*E^\pm \longrightarrow E^\pm. \tag{14.1}$$

The equivalence relation fixing a class from such data is stable G-equivariant isomorphism so $(E^\pm, L^\pm_E)$ and $(F^\pm, L^\pm_F)$ are equivalent if there exist G-equivariant vector bundles A and B and bundle isomorphisms

$$T^\pm : E^\pm \oplus A \longrightarrow F^\pm \oplus B \tag{14.2}$$

commuting with the G-actions. The *regularity* of the data is an issue here, the standard choice is to take all data to be continuous but the alternative of smooth data is more relevant for the construction of the Chern character. In fact these two choices give the same theory:-

Proposition 14.1. *For a smooth compact group action on a compact manifold the inclusion of smooth data, with smooth equivalences, induces an isomorphism of equivariant K-theory based on smooth and continuous data.*

Both theories are therefore denoted $K_G(M)$.

Proof. This is a standard result but a proof actually follows from the discussion below. □

On the resolved manifold, with its resolution structure, Y, $\phi_H : H \longrightarrow Y_H$ for each $H \in \mathcal{M}_1(Y)$ there is a 'relative' version of equivariant K-theory. Namely, a K-class is represented by a family of pairs of G-equivariant vector bundles, $E^\pm$ over Y and $E^\pm_H$ over each Y_H with the compatibility conditions that

$$E^\pm\big|_H = \phi_H^* E^\pm_H \text{ with their } G\text{-actions.} \tag{14.3}$$

Looking at intersections of hypersurfaces, (14.3) implies the corresponding compatibility conditions between $E^\pm_H$ restricted to a boundary hypersurface of H and $E^\pm_K$ under the fibration ϕ_{HK} to the appropriate Y_K. There is a natural notion of equivalence as in (14.2) where the bundles A and B are of this same form and all maps factor through the fibrations ϕ_H over boundary faces. We denote the corresponding Grothendieck group as $K_{G,\Phi}(Y)$ – of course it depends on the resolution data and not just the manifold Y.

Proposition 14.2. *For K-groups, based on either continuous or smooth data, lifting from M to Y induces a canonical isomorphism*

$$K_{G,\Phi}(Y) = K_G(M). \tag{14.4}$$

Proof. For continuous data this is immediate since the lift of a G-equivariant bundle to each Y_H gives a bundle satisfying the compatibility conditions (14.3). Conversely such data defines a *continuous* bundle over M by taking the quotients under all the ϕ_H; thus there is an isomorphism at the level of the bundle data.

For smooth data this is not quite the case, a smooth bundle certainly lifts to give smooth compatible data on the resolution but the converse does not hold. Nevertheless, normal retraction easily shows that any smooth compatible data on the resolution can be deformed by G-equivariant homotopy, and hence G-equivariant isomorphism, to be the lift of a smooth G-bundle over M. The same argument applies to equivalence so the smooth equivariant K-theory groups for M and the relative smooth K-group for the resolution are again canonically isomorphic. □

We do not need to analyze odd K-theory separately since it may be defined, as in the untwisted case, as the null space of the pull-back homomorphism corresponding to $\iota : M \hookrightarrow \mathbb{S} \times M$, $m \longmapsto (1, m)$:

$$\mathrm{K}^1_G(M) \longrightarrow \mathrm{K}^1_G(\mathbb{S} \times M) \xrightarrow{\iota^*} K_G(X) \tag{14.5}$$

where G acts trivially on $\mathbb{S}$. As in the standard case, pull-back under the projection $\mathbb{S} \times M \longrightarrow M$ induces a decomposition

$$K_G(\mathbb{S} \times M) = \mathrm{K}^1_G(M) \oplus \mathrm{K}^0_G(M). \tag{14.6}$$

This discussion carries over directly to the relative K-theory of the resolution to define $\mathrm{K}^1_{G,\Phi}(Y)$ and gives the analogue of (14.6).

As in the case of G-equivariant cohomology we wish to pass from this 'resolved' model to a reduced model over the quotient of the resolution by the G-action, which is the resolution of the quotient. Again we start with the easy cases.

Lemma 14.3. *For a free action by a compact group on a compact manifold the equivariant K-theory is canonically isomorphic, under the pull-back map, to the K-theory of the quotient.*

Proof. In the free case $M \longrightarrow G\backslash M$ is a principal G-bundle and an equivariant bundle over M projects to a bundle over $G\backslash M$ with any G-equivariant bundle isomorphism projecting to a bundle isomorphism. This identifyies the G-equivariant K-theory of the total space with standard K-theory of the base, in both odd and even cases. □

15. Representation bundle

As is clear from the discussion above, the fundamental case needed to reduce K-theory to a model on the quotient is that of an action with unique isotropy type. As before we first consider the case of a fixed, hence normal, isotropy group, $K \subset G$. Thus G acts through a free action of the quotient $W(K) = G/K$. As discussed in §11, if $W(K)$ is connected then it is the image under the projection of a connected group in G which commutes with K. In the general case this is true of $W_0(K)$, the component of the identity in $W(K)$.

So, consider the special case in which G acts through a free action of $W_0(K) = G/K$ which is assumed to be connected. Then K acts fiberwise on any G-equivariant bundle over M. The Peter-Weyl Theorem shows that a G-equivariant vector bundle over M decomposes as a direct sum of tensor products of chosen irreducible representations forming the representation ring, $W_i \in \mathcal{R}(K)$, of K and of vector bundles V_i over M :

$$E = \sum_i W_i \otimes V_i. \tag{15.1}$$

Here the fiber of V_i at $m \in M$ can be fixed by choosing a non-zero matrix element for W_i in $\mathcal{C}^\infty(G)$ and taking the K-span of the image of the average of the K-action against it. The action of G induces an action of $W_0(K)$ on this bundle covering the free action on M; thus V_i projects to a bundle over $G\backslash M$.

Lemma 15.1. *If G acts on M through a free action of a connected quotient $W_0(K) = G/K$ with respect to a normal subgroup then*

$$K_G(M) = \mathcal{R}(K) \otimes_{\mathbb{Z}} K(G\backslash M) \tag{15.2}$$

is the tensor product of the representation ring of K with the K-theory of the quotient.

Proof. This can be seen in close parallel with the discussion of deRham cohomology above. □

As noted earlier, if the G-action has a fixed isotropy group K even if $W(K) = G/K$ is not connected, there is a subgroup $G' \subset G$, generated by K and a lift of the connected component $W_0(K)$ of $W(K)$ such that $G'/K = W_0(K)$. In this case G/G' acts on $\mathcal{R}(K)$ through the adjoint action of a lift of $W(K)$ into G in which the component of the identity commutes with K and since the adjoint action of K on $\mathcal{R}(K)$ is trivial. Moreover $G/G' = W(K)/W_0(K)$ is normal in $W(K)$ and so acts on $M/W_0(K)$. The natural group-theoretic extension of Lemma 15.1 then follows.

Lemma 15.2. *If G acts on M with fixed isotropy group K then setting $W(K) = G/K$ with $W_0(K)$ as component of the identity*

$$K_G(M) = (\mathcal{R}(K) \otimes K(M/W_0(K)))^{W(K)/W_0(K)} \tag{15.3}$$

is the $W(K)/W_0(K)$-invariant part under the adjoint action of $W(K)/W_0(K)$ on $\mathcal{R}(K)$.

To get a more geometric formulation of this result observe that the Borel bundle discussed in §11 above, has an analogue in terms of the representation ring.

Lemma 15.3. *In the setting of Lemma 15.2 the quotient by the adjoint action of $W(K)/W_0(K)$ on $\mathcal{R}(K)$ induces a flat bundle, which we denote $\mathcal{R}(G\backslash M)$ over the quotient, $G\backslash M$, with fiber $\mathcal{R}(K)$.*

Proof. This is essentially a tautology since we have a free action of the finite group $W(K)/W_0(K)$ on $\mathcal{R}(K) \times M/W_0(K)$ covering its free action on $M/W_0(K)$. □

Now, standard complex K theory over a manifold can be extended to the case of coefficients in a ring, resulting in the tensor product of the cohomology, but can also be extended to the case of a flat bundle of rings over the space.

So, instead of the formulation (15.3) we can say

Corollary 15.4. *For a group action with fixed isotropy group, the equivariant K-theory of the space is canonically isomorphic to the K-theory with coefficients in the flat bundle of representation rings for the isotropy group over the quotient.*

The general case of a group action with fixed isotropy type is similar. In that case we can consider the representation rings of the isotropy groups as a bundle, with discrete fiber, $\mathcal{R}(K_m)$ over M. Borel's isomorphism from Proposition 11.2, shows that this bundle is trivial and moreover the quotient, $\mathcal{R}(G\backslash M)$ under the adjoint action of G is a flat bundle over $G\backslash M$. In fact, for any choice, K, of isotropy group, it is naturally isomorphic to the quotient of $\mathcal{R}(K) \times M^K$ by $N(K) \subset G$ and hence is the same bundle as discussed above. Thus,

Proposition 15.5. *For the action of a compact Lie group with fixed isotropy type on a compact manifold M, Corollary 15.4 extends to identify the equivariant K-theory of M with the $\mathcal{R}(G\backslash M)$-twisted K-theory of $G\backslash M$.*

In case of two G-actions with fixed isotropy groups and a fibration $\phi : M \longrightarrow Y$ intertwining them there is a natural 'Peter-Weyl' pull-back map. Namely for each point $m \in M$ the isotropy group at m is identified with a subgroup of the isotropy group at $\phi(m)$. This induces a decomposition of an element $W \in \mathcal{R}(K_{\phi(m)})$ as a finite sum of elements of $\mathcal{R}(K_m)$. By the equivariance of ϕ the induced map on the quotients

$$\tilde{\phi} : G\backslash M \longrightarrow G\backslash Y \tag{15.4}$$

induces an identification of each element of $\mathcal{R}(G\backslash Y)_{\phi(\tilde{m})}$ with a finite sum of elements in the fiber $\mathcal{R}(G\backslash M)_m$. This induces a pull-back construction, so that if

$$\begin{gathered} V \text{ a bundle with coefficients in } \mathcal{R}(G\backslash Y) \Longrightarrow \\ \tilde{\phi}^{\#}V \text{ is a bundle with coeffiencients in } \mathcal{R}(G\backslash M). \end{gathered} \tag{15.5}$$

This allows reduced K-theory to be defined on the quotient of the resolution in close analogy with the definition of the relative K-theory for the resolution.

Definition 8. The reduced K-theory group $K_{\mathcal{R},\Psi}(Z)$ for a compact group action on a compact manifold is the Grothendieck group constructed from pairs of objects, each consisting of a collection of vector bundles with coefficients in the representation bundles of the isotropy groups, one over the resolution $Z = G\backslash Y$ and one over each $Z_H = G\backslash Y_H$ such that restricted to each boundary hypersurface the bundle is the pull-back under $\tilde{\psi}_H^{\#}$ (or more generally $\tilde{\psi}_{HK}^{\#}$) of the bundle over the image. The equivalence is stable isomorphism over each space consistent under these pull-back maps.

Theorem 15.6. *For any compact group action on a compact manifold M with resolution Y and $Z = G\backslash Y$ with induced iterated boundary fibration Ψ and representation bundle $\mathcal{R}$, there is a natural isomorphism*

$$K_G(M) = K_{\mathcal{R},\Psi}(Z). \tag{15.6}$$

Proof. This amounts to a repetition of the proof of Theorem 11.1. □

16. Delocalized equivariant cohomology

The reduced Cartan model for (localized) equivariant cohomology in §13 and the corresponding model for equivariant K-theory in §14 above are directly comparable and immediately suggest a model for *delocalized* equivariant cohomology as introduced by Baum, Brylinski and MacPherson [2]. Namely, in Lemma 15.3 the flat bundle of (discrete) rings $\mathcal{R}(G\backslash M)$ is defined for any smooth group action with unique isotropy type and we may therefore consider in that case the space of smooth deRham forms

$$\mathcal{C}^\infty(Z; \mathcal{R} \otimes \Lambda^*) \tag{16.1}$$

twisted by the representation ring – this can always be constructed as a quotient by a finite Weyl group action.

In the case of the quotient of the full resolution of a G-action on a manifold with corners, all the quotients $Z = G\backslash Y$ and $Z_H = G\backslash Y_H$ carry such representation bundles and there is a natural pull-back map generated by the ψ_H. Thus the relative deRham complex is well-defined as the subcomplex of the direct sum satisfying the compatibility conditions:

$$\begin{aligned} \mathcal{C}^\infty_{\mathcal{R},\Psi}(Z; \Lambda^{\text{even}}) = \big\{ &(u, v_*),\ u \in \mathcal{C}^\infty(Z; \mathcal{R} \otimes \Lambda^{\text{even}}), \\ &v_H \in \mathcal{C}^\infty(Z_H; \mathcal{R} \otimes \Lambda^{\text{even}}),\ \psi_H^{\#} v_H = u_H\big|_H\ \forall\ H \in \mathcal{M}_1(Y) \big\} \end{aligned} \tag{16.2}$$

and similarly $\mathcal{C}^\infty_{\mathcal{R},\Psi}(Z; \Lambda^{\text{odd}})$. Since the coefficient bundle is flat the deRham differential projects under the quotient in (16.1) and is intertwined by the pull-back maps $\psi_H^{\#}$.

We shall denote the cohomology defined by this complex as

$$H^{\text{even}}_{\mathcal{R},\Psi}(M) \text{ and } H^{\text{odd}}_{\mathcal{R},\Psi}(M). \tag{16.3}$$

Although defined directly on the quotient these rings are well defined since the resolution and other structures used to defined them have been shown to be canonical above.

Theorem 16.1. *In the case of a smooth Abelian group action the cohomology groups $H^*_{\mathcal{R},\Psi}(M)$ are naturally isomorphic to the delocalized cohomology groups of Baum, Brylinsky and MacPherson.*

Proof. In the Abelian case there are no issues with Weyl group quotients and the definition of delocalized cohomology given above just amounts to twisting the deRham complex by the representation ring on each of the manifolds Y and Y_H. Following the arguments above it is straightforward to check that this is precisely the resolution of the coefficient sheaf used in the definition in [2]. □

The identity of the cohomology groups also follows from the discussion of the Chern character in the next section.

17. Equivariant Chern character

It is not immediately apparent that $H^*_{\mathcal{R},\Psi}(Z)$ fixes a contravariant functor for smooth G-action, since in general a smooth G-equivariant map between manifolds does not lift to a smooth map between the resolutions of the quotients as defined above. Nevertheless this follows immediately since we can identify these rings with G-equivariant K-theory with complex coefficients.

Theorem 17.1. *The Chern character, defined locally by a choice of compatible connections, defines a map*

$$\mathrm{Ch}_G : \mathrm{K}_{\mathcal{R},\Psi}(Z) \longrightarrow H^{even}_{\mathcal{R},\Psi}(Z) \tag{17.1}$$

for any smooth action of a compact Lie group on a manifold and this map induces a (Baum-Brylinski-MacPherson) isomorphism

$$\mathrm{Ch}_G : \mathrm{K}_{\mathcal{R},\Psi}(Z) \otimes \mathbb{C} \longrightarrow H^{even}_{\mathcal{R},\Psi}(Z). \tag{17.2}$$

Proof. A compatible connection on the component bundles of a pair defining an element of $K_G(M)$ in the reduced model discussed in §14 can be introduced by starting from the 'bottom' of the resolution tower and successively extending. Since the coefficient bundles are flat rings, or by lifting to the finite cover by $W(K)/W_0(K)$ at each level, the Chern character is then given by the standard formula

$$v_h = \exp(\nabla^2/2\pi i) \in \mathcal{C}^\infty(Z_H; \mathcal{R} \otimes \Lambda^{\mathrm{even}}). \tag{17.3}$$

These forms are clearly compatible so define the class $\mathrm{Ch}_G \in H^{\mathrm{even}}_{\mathcal{R},\Psi}(M)$. The standard arguments in Chern-Weil theory show that the resulting class is independent of choice of connection. Thus the Chern character (17.1) is defined as in the setting of smooth manifolds.

In the case of a manifold with unique isotropy type, Lemma 15.2 allows this map to be derived from the standard, untwisted, Chern character. Namely the quotient is then a single manifold with corners and the Chern character as defined above is simply the quotient under the finite group action by $W(K)/W_0(K)$ of the standard Chern character

$$K(M^K/W_0(K)) \longrightarrow H^{\mathrm{even}}(M^K/W_0(K)). \tag{17.4}$$

It therefore follows that it induces an isomorphism as in (17.2) in that case. Moreover, this is equally true for absolute and relative K-theory and cohomology.

The proof that (17.2) holds in general follows the same pattern as the proofs above of the identity of G-equivariant K-theory and cohomology with the reduced models. Namely, for K-theory and cohomology the partially relative rings can be defined with respect to any collection of boundary $\mathcal{B} \subset \mathcal{M}_1(G\backslash M)$ which contains all hypersurfaces smaller than any element. In the corresponding long exact sequences in K-theory and delocalized cohomology, which in the second case either can be deduced by analogy from the case of coefficient rings or else itself can be proved inductively, the Chern character then induces a natural transformation by the Fives Lemma. □

Appendix. The circle action on the sphere

According to Guillemin and Sternberg [6, §11.7], whenever a torus acts on a surface with non-empty fixed point set, the surface is diffeomorphic to the sphere and action is effectively the rotation of the sphere around the z-axis. Their subsequent computation of the equivariant cohomology makes use of equivariant formality and we now show that it is straightforward to carry out this computation, even for non-commutative groups, by resolving the sphere.

Thus consider a compact group G (not necessarily Abelian) acting smoothly on $M = \mathbb{S}^2$. Suppose that G has a codimension one normal subgroup H that acts trivially on M, and that the quotient $\mathbb{S}^1 = G/H$ acts on M by rotating around the z-axis (in the usual embedding of M into $\mathbb{R}^3$).

Thus the G-action has two isotropy types: the 'north and south poles', $\{N, S\}$ constitute an isotropy type corresponding to G, and their complement constitutes an isotropy type corresponding to H. This action is resolved by lifting to

$$Y = [M; \{N, S\}].$$

The boundary of Y is the disjoint union of two circles and the boundary fibration is the map from each circle to the corresponding pole. In this case

$$\mathcal{C}^\infty_{G,\Phi}(Y; S(\mathfrak{g}^*) \otimes \Lambda^*) = \\ \{(\omega, f_N, f_S) \in \mathcal{C}^\infty_G(Y; S(\mathfrak{g}^*) \otimes \Lambda^*) \oplus \bigoplus_{N,S} S(\mathfrak{g}^*)^G ; i_N^*\omega = f_N, i_S^*\omega = f_S\}$$

where we are identifying f_N with $f_N \otimes 1 \in \mathcal{C}^\infty_G(\mathbb{S}^1; S(\mathfrak{g}^*) \otimes \Lambda^*)$ and similarly with f_S.

We can identify the quotient Y/G with the unit interval, and in this case the Borel bundle is the trivial $S(\mathfrak{h}^*)^H$ bundle. Thus the reduced Cartan complex is

$$\mathcal{C}^\infty_{B,\Psi}([0,1]; \Lambda^*) = \Big\{(\omega, f_N, f_S) \in \big(S(\mathfrak{h}^*)^H \otimes \mathcal{C}^\infty([0,1]; \Lambda^*)\big) \oplus \bigoplus_{N,S} S(\mathfrak{g}^*)^G ; \\ i_0^*\omega = r(f_N), i_1^*\omega = r(f_S)\Big\}$$

where $r : S(\mathfrak{g}^*)^G \longrightarrow S(\mathfrak{h}^*)^H$ is the natural restriction map, and the differential is the exterior derivative on the first factor. Since the interval is contractible, we find

$$H^*_G(\mathbb{S}^2) = H^*(\mathcal{C}^\infty_{G,\Phi}(Y; S(\mathfrak{g}^*) \otimes \Lambda^*), d_{\mathrm{eq}}) = H^*(\mathcal{C}^\infty_{B,\Psi}([0,1]; \Lambda^*), d) \\ = \{(f_N, f_S) \in S(\mathfrak{g}^*)^G \oplus S(\mathfrak{g}^*)^G ; r(f_N) = r(f_S)\}$$

and so $H^q_G(\mathbb{S}^2)$ is trivial if q is odd and is non-trivial for all even $q \geq 0$.

The representation bundle is also trivial in this case, so the same reasoning shows that

$$K^0_G(\mathbb{S}^2) = \{(\tau_N, \tau_S) \in R(G) \oplus R(G); \rho(\tau_N) = \rho(\tau_S)\}, \quad K^1_G(\mathbb{S}^2) = 0$$

where $\rho : R(G) \longrightarrow R(H)$ is the restriction map. Indeed, classes in $K_{\mathcal{R},\Psi}([0,1])$ consist of vector bundles over the interval and its end points with, respectively, coefficients in $R(H)$ and $R(G)$ and the compatibility condition is induced by the restriction map.

Finally note that the complex (16.2) in this case is given by

$$\mathcal{C}^\infty_{\mathcal{R},\Psi}([0,1];\Lambda^*) = \\ \{(\omega, \tau_N, \tau_S) \in (R(H) \otimes \mathcal{C}^\infty([0,1];\Lambda^*)) \oplus \bigoplus_{N,S} R(G); i_0^*\omega = \rho(\tau_N), i_1^*\omega = \rho(\tau_S)\}$$

with differential given by the exterior derivative on the first factor. Thus the delocalized equivariant cohomology in this case is

$$H^{\mathrm{even}}_{\mathcal{R},\Psi}(M) = \{(\tau_N, \tau_S) \in R(G) \oplus R(G); \rho(\tau_N) = \rho(\tau_S)\}, \quad H^{\mathrm{odd}}_{\mathcal{R},\Psi}(M) = 0.$$

The Chern character from equivariant K-theory to delocalized equivariant cohomology is the identity, while the Chern character into (localized) equivariant cohomology is localization at the identity in G.

References

[1] M. F. Atiyah and G. B. Segal, *Equivariant K-theory and completion*, J. Differential Geometry **3** (1969), 1–18.

[2] Paul Baum, Jean-Luc Brylinski, and Robert MacPherson, *Cohomologie équivariante délocalisée*, C. R. Acad. Sci. Paris Sér. I Math. **300** (1985), no. 17, 605–608.

[3] Armand Borel and Lizhen Ji, *Compactifications of symmetric and locally symmetric spaces*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 2006.

[4] Jean-Luc Brylinski, *Cyclic homology and equivariant theories*, Ann. Inst. Fourier (Grenoble) **37** (1987), no. 4, 15–28.

[5] J. J. Duistermaat and J. A. C. Kolk, *Lie groups*, Universitext, Springer-Verlag, Berlin, 2000.

[6] Victor W. Guillemin and Shlomo Sternberg, *Supersymmetry and equivariant de Rham theory*, Mathematics Past and Present, Springer-Verlag, Berlin, 1999, With an appendix containing two reprints by Henri Cartan.

[7] Andrew Hassell, Rafe Mazzeo, and Richard B. Melrose, *Analytic surgery and the accumulation of eigenvalues*, Comm. Anal. Geom. **3** (1995), no. 1-2, 115–222.

[8] Wu-yi Hsiang, *On the classification of differentiable* $\mathrm{SO}(n)$ *actions on simply connected* π-*manifolds*, Amer. J. Math. **88** (1966), 137–153.

[9] Sören Illman, *Equivariant algebraic topology*, Annales de l'institut Fourier **23** (1973), no. 2, 87–91.

[10] Klaus Jänich, *On the classification of $O(n)$-manifolds*, Math. Ann. **176** (1968), 53–76.

[11] Katsuo Kawakubo, *The theory of transformation groups*, The Clarendon Press Oxford University Press, New York, 1991.

[12] Richard B. Melrose, *Differential Analysis on Manifolds with Corners*, In Preparation.

[13] Ioanid Rosu, *Equivariant K-theory and equivariant cohomology*, Math. Z. **243** (2003), no. 3, 423–448, With an appendix by Allen Knutson and Rosu.

[14] Graeme Segal, *Equivariant K-theory*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 129–151.

[15] Arthur G. Wasserman, *Simplifying group actions*, Topology Appl. **75** (1997), no. 1, 13–31.

Department of Mathematics, Massachusetts Institute of Technology

E-mail address: `pierre@math.mit.edu`

E-mail address: `rbm@math.mit.edu`