

Introduction to Microlocal Analysis

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Contents

Preface	9
Introduction	11
Chapter 1. Tempered distributions and the Fourier transform	13
1.1. Schwartz test functions	13
1.2. Linear transformations	15
1.3. Tempered distributions	16
1.4. Two big theorems	17
1.5. Examples	18
1.6. Two little lemmas	19
1.7. Fourier transform	21
1.8. Differential operators	24
1.9. Radial compactification	25
1.10. Problems	26
Chapter 2. Pseudodifferential operators on Euclidean space	29
2.1. Symbols	29
2.2. Pseudodifferential operators	33
2.3. Composition	35
2.4. Reduction	36
2.5. Asymptotic summation	37
2.6. Residual terms	39
2.7. Proof of Composition Theorem	40
2.8. Quantization and symbols	41
2.9. Principal symbol	42
2.10. Ellipticity	44
2.11. Elliptic regularity and the Laplacian	46
2.12. L^2 boundedness	47
2.13. Square root and boundedness	48
2.14. Sobolev boundedness	50
2.15. Polyhomogeneity	53
2.16. Topologies and continuity of the product	55
2.17. Linear invariance	57
2.18. Local coordinate invariance	58
2.19. Semiclassical limit	59
2.20. Smooth and holomorphic families	63
2.21. Problems	64
Chapter 3. Residual, or Schwartz, algebra	69

3.1.	The residual algebra	69
3.2.	The augmented residual algebra	70
3.3.	Exponential and logarithm	73
3.4.	The residual group	73
3.5.	Traces on the residual algebra	74
3.6.	Fredholm determinant	76
3.7.	Fredholm alternative	79
Chapter 4. Isotropic calculus		81
4.1.	Isotropic operators	81
4.2.	Fredholm property	84
4.3.	The harmonic oscillator	86
4.4.	L^2 boundedness and compactness	89
4.5.	Sobolev spaces	90
4.6.	Representations	92
4.7.	Symplectic invariance of the isotropic product	93
4.8.	Metaplectic group	95
4.9.	Complex order	102
4.10.	Resolvent and spectrum	102
4.11.	Residue trace	103
4.12.	Exterior derivation	106
4.13.	Regularized trace	107
4.14.	Projections	108
4.15.	Complex powers	108
4.16.	Index and invertibility	108
4.17.	Variation 1-form	110
4.18.	Determinant bundle	112
4.19.	Index bundle	113
4.20.	Index formulæ	113
4.21.	Isotropic essential support	113
4.22.	Isotropic wavefront set	113
4.23.	Isotropic FBI transform	113
4.24.	Problems	113
Chapter 5. Microlocalization		117
5.1.	Calculus of supports	117
5.2.	Singular supports	118
5.3.	Pseudolocality	118
5.4.	Coordinate invariance	119
5.5.	Problems	120
5.6.	Characteristic variety	121
5.7.	Wavefront set	122
5.8.	Essential support	123
5.9.	Microlocal parametrices	124
5.10.	Microlocality	125
5.11.	Explicit formulations	126
5.12.	Wavefront set of K_A	127
5.13.	Hypersurfaces and Hamilton vector fields	127
5.14.	Relative wavefront set	129

5.15.	Proof of Proposition 5.9	133
5.16.	Hörmander's propagation theorem	136
5.17.	Elementary calculus of wavefront sets	137
5.18.	Pairing	138
5.19.	Multiplication of distributions	140
5.20.	Projection	140
5.21.	Restriction	142
5.22.	Exterior product	143
5.23.	Diffeomorphisms	144
5.24.	Products	146
5.25.	Pull-back	147
5.26.	The operation F_*	149
5.27.	Wavefront relation	151
5.28.	Applications	152
5.29.	Problems	153
Chapter 6.	Pseudodifferential operators on manifolds	155
6.1.	C^∞ structures	155
6.2.	Form bundles	156
6.3.	Pseudodifferential operators	157
6.4.	The symbol calculus	162
6.5.	Pseudodifferential operators on vector bundles	164
6.6.	Hodge theorem	165
6.7.	Sobolev spaces and boundedness	169
6.8.	Pseudodifferential projections	171
6.9.	Semiclassical algebra	173
6.10.	Heat kernel	175
6.11.	Resolvent	175
6.12.	Complex powers	175
6.13.	Problems	175
Chapter 7.	Scattering calculus	177
7.1.	Scattering pseudodifferential operators	177
Chapter 8.	Elliptic boundary problems	179
	Summary	179
	Introduction	179
	Status as of 4 August, 1998	179
8.1.	Manifolds with boundary	179
8.2.	Smooth functions	180
8.3.	Distributions	183
8.4.	Boundary Terms	184
8.5.	Sobolev spaces	187
8.6.	Dividing hypersurfaces	188
8.7.	Rational symbols	190
8.8.	Proofs of Proposition 8.7 and Theorem 8.1	191
8.9.	Inverses	191
8.10.	Smoothing operators	192
8.11.	Left and right parametrices	194

8.12.	Right inverse	195
8.13.	Boundary map	196
8.14.	Calderón projector	197
8.15.	Poisson operator	198
8.16.	Unique continuation	198
8.17.	Boundary regularity	198
8.18.	Pseudodifferential boundary conditions	198
8.19.	Gluing	200
8.20.	Local boundary conditions	200
8.21.	Absolute and relative Hodge cohomology	200
8.22.	Transmission condition	200
Chapter 9. The wave kernel		201
9.1.	Hamilton-Jacobi theory	213
9.2.	Riemann metrics and quantization	217
9.3.	Transport equation	218
9.4.	Problems	222
9.5.	The wave equation	222
9.6.	Forward fundamental solution	227
9.7.	Operations on conormal distributions	230
9.8.	Weyl asymptotics	232
9.9.	Problems	236
Chapter 10. K-theory		237
10.1.	What do I need for the index theorem?	237
10.2.	Odd K-theory	237
10.3.	Computations	241
10.4.	Vector bundles	242
10.5.	Isotropic index map	244
10.6.	Bott periodicity	247
10.7.	Semiclassical quantization	249
10.8.	Symplectic bundles	249
10.9.	Thom isomorphism	249
10.10.	Chern-Weil theory and the Chern character	249
10.11.	Todd class	253
10.12.	Stabilization	253
10.13.	Delooping sequence	253
10.14.	Looping sequence	253
10.15.	\mathcal{C}^* algebras	253
10.16.	K-theory of an algebra	253
10.17.	The norm closure of $\Psi^0(X)$	253
10.18.	The index map	253
10.19.	Problems	253
Chapter 11. Hochschild homology		255
11.1.	Formal Hochschild homology	255
11.2.	Hochschild homology of polynomial algebras	256
11.3.	Hochschild homology of $\mathcal{C}^\infty(X)$	261
11.4.	Commutative formal symbol algebra	264

11.5. Hochschild chains	265
11.6. Semi-classical limit and spectral sequence	265
11.7. The E_2 term	267
11.8. Degeneration and convergence	271
11.9. Explicit cohomology maps	272
11.10. Hochschild homology of $\Psi^{-\infty}(X)$	272
11.11. Hochschild homology of $\Psi^{\mathbb{Z}}(X)$	272
11.12. Morita equivalence	272
Chapter 12. The index theorem and formula	273
12.1. Outline	273
12.2. Fibrations	273
12.3. Smoothing families	275
12.4. Elliptic families	277
12.5. Spectral sections	277
12.6. Analytic index	277
12.7. Topological index	277
12.8. Proof of index theorem	277
12.9. Chern character of the index bundle	277
Problems	277
Appendix A. Bounded operators on Hilbert space	279
Index of Mathematicians	280
Appendix. Bibliography	281

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Introduction

I shall assume some familiarity with distribution theory, with basic analysis and functional analysis and with (local) differential geometry. A passing knowledge of the theory of manifolds would also be useful. Any one or two of these prerequisites can be easily picked up along the way, but the prospective student with none of them should perhaps do some preliminary reading:

Distributions: A good introduction is Friedlander's book [5]. For a more exhaustive treatment see Volume I of Hörmander's treatise [9].

Analysis on manifolds: Most of what we need can be picked up from Munkres' book [10] or Spivak's little book [13].

