LECTURE NOTES FOR 18.157, FALL 2009

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Abstract.

Contents

Introduction

1. Lecture 4: 22 September, 2009

What I did today.

(1) Recalled the basic properties of pseudodifferential operators that I have shown so far.

First I showed that if $a \in S^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$ then the convergent integral

$$I(a)u(z) = (2\pi)^{-n}\int e^{i(z-z')\cdot\zeta}a(z,z',\zeta)u(z')dz'd\zeta,\ u \in \mathcal{S}(\mathbb{R}^n) \tag{1}$$

defines a continuous linear map $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{S}(\mathbb{R}^n)$ and for each $m \in \mathbb{R}$ there exists L such that

$$\|I(a)u\|_p \le C_{m,L}\|a\|_{p+L,m}\|u\|_{p+L}. \tag{2}$$

Here $\|\cdot\|_p$ are the norms on $\mathcal{S}(\mathbb{R}^n)$ and $\|a\|_{q,m}$ the norms on $S^m(\mathbb{R}^{2n};\mathbb{R}^n)$.

Then I defined $\Psi^m_\infty(\mathbb{R}^n)$ by continuity using these estimates so by definition it is the range of I on $S^m(\mathbb{R}^{2n};\mathbb{R}^n)$ as a space of continuous linear operators on $\mathcal{S}(\mathbb{R}^n)$.

The main result is left-reduction, that restricted to symbols which are independent of the z' variable this I gives an isomorphism – which I am calling left quantization

$$q_L : S^m(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow \Psi^m_\infty(\mathbb{R}^n). \tag{3}$$

I showed that this is a bijection and declared it to be a topological isomorphism by using it to transfer the topology from S^m to Ψ^m_∞. The inverse of q_L is the left symbol map σ_L.

(2) Formal adjoint defines an isomorphism of $\Psi^m_\infty(\mathbb{R}^n)$.

(3) Composition as operators on $\mathcal{S}(\mathbb{R}^n)$ gives a non-commutative product

$$S^m(\mathbb{R}^n;\mathbb{R}^n) \times S^{m'}(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow S^{m+m'}(\mathbb{R}^n;\mathbb{R}^n),\ (a,b) \longmapsto \sigma_L(q_L(a) \circ q_L(b)) \tag{4}$$

and that this product has an asymptotic expansion in terms of the commutative product

$$\sigma_L(AB) = \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} D_z^\alpha \sigma_L(A) \cdot D_\zeta^\alpha \sigma_L(B). \tag{5}$$

Here the terms in the sum are of order $m + m' - |\alpha|$.

(4) Today I showed the uniform ellipticity of $\sigma_L(A)$ is equivalent to the existence of a two-sided inverse modulo errors of order $-\infty$.

A symbol $a \in S^m(\mathbb{R}^{2n}; \mathbb{R}^n)$ is uniformly elliptic of order m if there exists $c > 0$ such that

$$|a(t,\zeta)| \ge c|\zeta|^m \text{ in } |\zeta| > 1/c \ \forall\ t \in \mathbb{R}^p. \tag{6}$$

So, if $A \in \Psi^m(\mathbb{R}^n)$ then (6) for $\sigma_L(A)$ (and this is equivalent to the the same result for $\sigma_R(A)$) is equivalent to the existence of $B \in \Psi_\infty^{-m}(\mathbb{R}^n)$ such that

$$AB - \mathrm{Id},\ BA - \mathrm{Id} \in \Psi_\infty^{-\infty}(\mathbb{R}^n). \tag{7}$$

(5) Next I defined the essential support of a symbol and the local elliptic set of a symbol.

(8)

$$\text{ess-supp}(a) = \{(\bar z, \bar\zeta) \in \mathbb{R}^n \times \mathbb{S}^{n-1}; \exists\ \epsilon > 0 \text{ s.t.} \sup_{|z-\bar z| \le \epsilon, |\frac{\zeta}{|\zeta|} - \bar\zeta| \le \epsilon} |a(z,\zeta)(1+|\zeta|)^{-N} < \infty\ \forall\ N\}^{\complement}$$

$$\mathrm{Ell}_m(a) = \{(\bar z, \bar\zeta) \in \mathbb{R}^n \times \mathbb{S}^{n-1}; \exists\ \epsilon > 0 \text{ s.t.} \inf_{|z-\bar z| \le \epsilon, |\frac{\zeta}{|\zeta|} - \bar\zeta| \le \epsilon, |\zeta| \ge 1/\epsilon} |a(z,\zeta)||\zeta|^{-m} > 0.$$

Certainly $\mathrm{Ell}(a) \subset \text{ess-supp}(a)$. For pseudodifferential operators we define

$$\begin{aligned} \mathrm{WF}'(A) &= \text{ess-supp}(\sigma_L(A)) = \text{ess-supp}(\sigma_R(A)), \\ \mathrm{Ell}_m(A) &= \mathrm{Ell}_m(\sigma_L(A)) = \mathrm{Ell}_m(\sigma_R(A)),\ \Sigma_m(A) = \mathrm{Ell}_m(A)^{\complement}. \end{aligned} \tag{9}$$

Lemma 1. *For any* $A \in \Psi_\infty^m(\mathbb{R}^n)$, $B \in \Psi_\infty^{m'}(\mathbb{R}^n)$

$$\begin{aligned} \mathrm{WF}'(AB) &\subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B), \\ \mathrm{Ell}_{m+m'}(AB) &= \mathrm{Ell}_m(A) \cap \mathrm{Ell}_{m'}(B), \\ \Sigma_{m+m'}(AB) &= \Sigma_m(A) \cup \Sigma_{m'}(B). \end{aligned} \tag{10}$$

(6) Then, after correction, I defined for any $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$

$$\mathrm{WF}(u) = \bigcap_{A \in \Psi_\infty^0(\mathbb{R}^n); Au \in \mathcal{C}^\infty(\mathbb{R}^n)} \Sigma(A). \tag{11}$$

This is the same as saying $(\bar z, \bar\zeta) \notin \mathrm{WF}(u)$ iff there exists $A \in \Psi_\infty^m(\mathbb{R}^n)$ such that $Au \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $(\bar z, \bar\zeta) \in \mathrm{Ell}(A)$.

(7) Finally I proved the easy half of

Theorem 1. *If* $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ *then* $(\bar z, \bar\zeta) \notin \mathrm{WF}(u)$ *iff there exists* $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ *with* $\phi(\bar z) \ne 0$ *and* $\epsilon > 0$ *such that*

$$\sup_{|\frac{\zeta}{|\zeta|} - \bar\zeta| \le \epsilon, |\zeta| \ge 1} |\zeta|^N |\widehat{\phi u}(\zeta)| < \infty\ \forall\ N. \tag{12}$$

Proof. That (12) implies that $(\bar{z},\bar{\zeta}) \notin \mathrm{WF}(u)$ according to the definition (11) requires us to construct an operator $A \in \Psi^0_\infty(\mathbb{R}^n)$ such that $(\bar{z},\bar{\zeta}) \notin \Sigma_0(A)$, meaning that $(\bar{z},\bar{\zeta}) \in \mathrm{Ell}_0(A)$ but $Au \in \mathcal{C}^\infty(\mathbb{R}^n)$. So, simply choose a cutoff $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with support in $|\zeta| \le 1$ and which is equal to 1 in $|\zeta| \le \frac{1}{2}$. Then consider

$$a = \phi(z')\psi(\epsilon^{-1}(\frac{\zeta}{|\zeta|} - \bar{\zeta}))(1 - \psi(\zeta) \in S^0(\mathbb{R}^n;\mathbb{R}^n). \tag{13}$$

This is smooth since the last term keep the support away from $\zeta = 0$ where the middle term is singular. The symbol estimates follow easily from the fact that in $|\zeta| > 1$ it is homogeneous of degree 0 in ζ. Moreoverl $(\bar{z},\bar{\zeta}) \in \mathrm{Ell}_0(a)$. So if we take $A \in \Psi^0_\infty(\mathbb{R}^n)$ to have *right* symbol a then $(\bar{z},\bar{\zeta}) \in \mathrm{Ell}_0(A)$ and directly from (12)

$$\widehat{Au}(\zeta) = \psi(\epsilon^{-1}(\frac{\zeta}{|\zeta|} - \bar{\zeta}))(1 - \psi(\zeta)\widehat{\phi u}(\zeta) \longrightarrow Au \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{14}$$

□

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