QUESTIONS 18.102 FINAL, SPRING 2013

I expect to choose 6 of these questions on the final. Note that some are rather straightforward and some are less so, I will choose some of each. Before the day of the exam, you can ask me for hints.

All Hilbert spaces should be taken to be separable and non-trivial, i.e. containing an element other than 0.

Problem 1

Let H_i, $i = 1, 2$ be two Hilbert spaces with inner products $(\cdot, \cdot)_i$ and suppose that $I : H_1 \longrightarrow H_2$ is a continuous linear map between them. Suppose that

- the range of I is dense in H_2
- I is injective.
- if $v_n \rightharpoonup v$ (converges weakly) in H_1 then $I(v_n) \to I(v)$ in H_2.

Show that

(1) there is a continuous linear map $Q : H_2 \longrightarrow H_1$ such that $(u, I(f))_2 = (Qu, f)_1 \ \forall \ f \in H_1$.
(2) as a map from H_1 to itself, $Q \circ I$ is bounded and self-adjoint.
(3) as a map on H_1, $Q \circ I$ is compact.
(4) as a map on H_2, $I \circ Q$ has the same eigenvalues as $Q \circ I$ on H_1.

Problem 2

Let $A_j \subset \mathbb{R}$ be a sequence of subsets with the property that the characteristic function, χ_j of A_j, is integrable for each j. Show that the characteristic function of $\mathbb{R} \setminus A$, where $A = \bigcup_j A_j$ is locally integrable.

Problem 3

If H is a Hilbert space let $l^2(H)$ be the space of sequences $h : \mathbb{N} \longrightarrow H$ such that $\sum_j \|u(j)\|_H^2 < \infty$. Show that this is a Hilbert space and that there is a bounded linear bijection $l^2(H) \longrightarrow H$ if and only if H is not finite dimensional.

Problem 4

Let A be a Hilbert-Schmidt operator on a separable Hilbert space H, which means that for some orthonormal basis $\{e_i\}$

$$\sum_i \|Ae_i\|^2 < \infty. \tag{1}$$

Using Bessel's identity to expand $\|Ae_i\|^2$ with respect to another orthonormal basis $\{f_j\}$ show that $\sum_j \|A^* f_j\|^2 = \sum_i \|Ae_i\|^2$. Conclude that the sum in (1) is independent of the othornormal basis used to define it and that the Hilbert-Schmidt operators form a Hilbert space.

Problem 5

Let $a : [0, 2\pi] \longrightarrow L^2(0, 2\pi)$ be a continuous map. Show that

$$Af(x) = \int_0^{2\pi} a(x)f, \;\; f \in L^2(0, 2\pi)$$

defines a compact operator from $L^2(0, 2\pi)$ to $\mathcal{C}([0, 2\pi])$ (i.e. the image of a bounded set is equicontinuous). Conversely show that if $A : L^2(0, 2\pi) \longrightarrow \mathcal{C}([0, 2\pi])$ is given as a compact (in particular bounded) operator for the supremum norm on $\mathcal{C}([0, 2\pi])$ then there exists such a map a.

Problem 6

Let $u_n : [0, 2\pi] \longrightarrow \mathbb{C}$ be a sequence of continuously differentiable functions which is uniformly bounded, with bounded derivatives i.e. $\sup_n \sup_{x\in[0,2\pi]} |u_n(x)| < \infty$ and $\sup_n \sup_{x\in[0,2\pi]} |u_n'(x)| < \infty$. Show that u_n has a subsequence which converges in $L^2([0, 2\pi])$.

Problem 7

Suppose that $f \in \mathcal{L}^1(0, 2\pi)$ is such that the constants $c_k = \int_{(0,2\pi)} f(x)e^{-ikx}, \; k \in \mathbb{Z}$, satisfy $\sum\limits_{k\in\mathbb{Z}} |c_k|^2 < \infty$. Show that $f \in L^2(0, 2\pi)$.

Problem 8

Carefully justify each step in the following proof of the uniform boundedness principle (see Alan D. Sokal, 'A really simple elementary proof of the uniform boundedness theorem', Arxiv math:1005.1585).

Theorem: If a sequence, or more generally a collection $\mathcal{T}$, of bounded operators $T : V \longrightarrow W$, where V is a Banach space and W is a *normed* space, is such that for each $v \in V$, $\sup_{T\in\mathcal{T}} \|T(v)\|_W < \infty$ then $\sup_{T\in\mathcal{T}} \|T\| < \infty$.

Proof:

(1) Suppose to the contrary that $\sup_{T\in\mathcal{T}} \|T\| = \infty$.
(2) Choose a sequence T_k in $\mathcal{T}$ such that $\|T_k\| \ge 4^k$.
(3) Observe that for a bounded operator, S, between two normed spaces

$$\max\big[\|S(v + v')\|, \, \|S(v - v')\|\big] \ge \frac{1}{2}\big[\|S(v + v')\| + \|S(v - v')\|\big] \ge \|Sv'\|$$

(4) Deduce that

$$\sup_{u\in B(v,r)} \|Su\| \ge \|S\|r \;\forall\; v \in V \text{ and } r > 0.$$

(5) Set $v_0 = 0$ and note that for $k \ge 1$, proceeding inductively, points $v_k \in V$ can be chosen such that $\|v_k - v_{k-1}\| \le 3^{-k}$ and $\|T_k v_k\| \ge \frac{2}{3}3^{-k}\|T_k\|$.
(6) The sequence $\{v_k\}_{k=1}^{\infty}$ is Cauchy, hence converges to some $v \in V$.
(7) Since $\|v - v_k\| \le \frac{1}{2}3^{-k}$, $\|T_k v\| \ge \frac{1}{6}3^{-k}\|T_k\| \ge \frac{1}{6}(4/3)^k \to \infty$.
(8) Hence $\sup_{T\in\mathcal{T}} \|T\| < \infty$.

Problem 9

Let B_n be a sequence of bounded linear operators on a Hilbert space H such that for each u and $v \in H$ the sequence $(B_n u, v)$ converges in $\mathbb{C}$. Show that there is a uniquely defined bounded operator B on H such that

$$(Bu, v) = \lim_{n\to\infty}(B_n u, v)\ \forall\ u, v \in H.$$

Problem 10

Let $T : H_1 \longrightarrow H_2$ be a continuous linear map between two Hilbert spaces and suppose that T is both surjective and injective.

(1) Let $A_2 \in \mathcal{K}(H_2)$ be a compact linear operator on H_2, show that there is a compact linear operator $A_1 \in \mathcal{K}(H_1)$ such that

$$A_2 T = T A_1.$$

(2) If A_2 is self-adjoint (as well as being compact) and H_1 is infinite dimensional, show that A_1 has an infinite number of linearly independent eigenvectors.

Problem 11

Suppose $P \subset H$ is a closed linear subspace of a Hilbert space, with $\pi_P : H \longrightarrow P$ the orthogonal projection onto P. If H is separable and A is a compact self-adjoint operator on H, show that there is a complete orthonormal basis of H each element of which satisfies $\pi_P A \pi_P e_i = t_i e_i$ for some $t_i \in \mathbb{R}$.

Problem 12

Let $e_j = c_j C^j e^{-x^2/2}$, $c_j > 0$, where $j = 1, 2, \dots,$ and $C = -\frac{d}{dx} + x$ is the creation operator, be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. You may assume completeness in $L^2(\mathbb{R})$ and use the facts established in class that $-\frac{d^2 e_j}{dx^2} + x^2 e_j = (2j+1)e_j$, that $c_j = 2^{-j/2}(j!)^{-\frac{1}{2}}\pi^{-\frac{1}{4}}$ and that $e_j = p_j(x)e_0$ for a polynomial of degree j. Compute Ce_j and Ae_j in terms of the basis and hence arrive at a formula for de_j/dx. Use this to show that the sequence $j^{-\frac{1}{2}}\frac{de_j}{dx}$ is bounded in $L^2(\mathbb{R})$. Conclude that if

$$H^1_{\mathrm{iso}} = \{u \in L^2(\mathbb{R}); \sum_{j\ge1} j|(u, e_j)|^2 < \infty\} \tag{2}$$

then there is a uniquely defined operator $D : H^1_{\mathrm{iso}} \longrightarrow L^2(\mathbb{R})$ such that $De_j = \frac{de_j}{dx}$ for each j.

Problem 13

Let A be a compact self-adjoint operator on a separable Hilbert space and suppose that for *any* orthonormal basis

$$\sum_i |(Ae_i, e_i)| < \infty.$$

Show that the eigenvalues of A, if infinite in number, form a sequence in l^1.

Problem 14

Consider the subspace $H \subset \mathcal{C}[0, 2\pi]$ consisting of those continuous functions on $[0, 2\pi]$ which satisfy

$$u(x) = \int_0^x U(x), \ \forall\, x \in [0, 2\pi] \tag{3}$$

for some $U \in L^2(0, 2\pi)$ (depending on u of course). Show that the function U is determined by u (given that it exists), that

$$\|u\|_H^2 = \int_{(0,2\pi)} |U|^2 \tag{4}$$

turns H into a Hilbert space.

Problem 15

Let $e_j = c_j C^j e^{-x^2/2}$, $c_j > 0$, where $C = -\frac{d}{dx} + x$ is the creation operator, be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. Define an operator on $L^2(\mathbb{R})$ by

$$Au = \sum_{j=0}^{\infty} (2j+1)^{-\frac{1}{2}} (u, e_j)_{L^2} e_j.$$

(1) Show that A is compact as an operator on $L^2(\mathbb{R})$.
(2) Suppose that $V \in \mathcal{C}^0_\infty(\mathbb{R})$ is a bounded, real-valued, continuous function on $\mathbb{R}$. Show that $L^2(\mathbb{R})$ has an orthonormal basis consisting of eigenfunctions of $K = AVA$, where V is acting by multiplication on $L^2(\mathbb{R})$.

Problem 16

Suppose that $f \in \mathcal{L}^1(0, 2\pi)$ is such that the constants

$$c_k = \int_{(0,2\pi)} f(x) e^{-ikx}, \ k \in \mathbb{Z},$$

satisfy

$$\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty.$$

Show that $f \in \mathcal{L}^2(0, 2\pi)$.

Problem 17

Consider the space of those complex-valued functions on $[0, 1]$ for which there is a constant C (depending on the function) such that

$$|u(x) - u(y)| \le C|x - y|^{\frac{1}{2}} \ \forall\, x, y \in [0, 1]. \tag{5}$$

Show that this is a Banach space with norm

$$\|u\|_{\frac{1}{2}} = \sup_{[0,1]} |u(x)| + \inf_{(5) \text{ holds}} C. \tag{6}$$

Problem 18

Let $B : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \longrightarrow \mathbb{C}$ be a bilinear form (meaning it is linear in each factor when the other is held fixed) such that there is a constant $C > 0$ and

$$|B(u,v)| \le C\|u\|_{L^2}\|v\|_{L^2} \ \forall\ u, v \in L^2(\mathbb{R}).$$

Show that there is a bounded linear operator $T : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} T(u)(x)v(x) = B(u,v) \ \forall\ u, v \in L^2(\mathbb{R}).$$