PROBLEM SET 8, 18.155
DUE FRIDAY 8 NOVEMBER, 2013

Recall part of the spectral theorem for a bounded self-adjoint operator A that I did not prove in class. Namely the functional calculus extends from continuous functions on $[-\|A\|, \|A\|]$ so that $\chi(A) \in \mathcal{B}(H)$ is defined for the characteristic function χ of any closed interval $[a, b]$, that $\chi(A)$ is a self-adjoint projection which commutes with A, it commutes with any bounded operator that commutes with A and

$$\operatorname{spec}(\chi(A)A) \subset [a, b], \ \operatorname{spec}((\operatorname{Id} - \chi(A))A) \cap (a, b) = \emptyset \tag{1}$$

(1) Show that if $U = A + iB$ is the decomposition of a unitary operator on a separable Hilbert space into its self-adjoint and anti-self-adjoint parts then $z \in \operatorname{spec}(U)$ implies $\operatorname{Re}(z) \in \operatorname{spec}(A)$ and $\operatorname{Im}(z) \in \operatorname{spec}(B)$.

(2) Using a spectral projection as above, or otherwise (there are other methods), show that the unitary group on a separable Hilbert space is connected.

(3) Show that if L is an unbounded self-adjoint operator (with domain $D \subset H$ as defined in the preceding problem set) which is such that $(L + i)^{-1} \in \mathcal{K}(H)$ then H has an orthonormal basis consisting of eigenvectors of L, elements of D such that $Le_j = \lambda_j e_j$.

(4) If $P(D)$ is a real elliptic polynomial of positive degree on $\mathbb{R}^n$ show that $P(D) : H^m(\mathbb{T}^n) \longrightarrow L^2(\mathbb{T}^n)$ satisfies the conditions of the preceding problem. Here I am identifying the torus as the quotient $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$, so functions on the torus are functions on $\mathbb{R}^n$ which are 2π-periodic in each variable.

(5) Show that the eigenvalues appearing in the previous problem are the values $P(k)$ for $k \in \mathbb{Z}^n$.