PROBLEM SET 7, 18.155
DUE FRIDAY 1, NOVEMBER 2013

Let H be a separable Hilbert space throughout. An *unbounded* operator on H is a linear map

(1) $$A : D \longrightarrow H$$

where $D \subset H$ is a dense subspace and

(2) $$\mathrm{Gr}(A) = \{(u, Au); u \in D\} \subset H \times H$$

is closed. Such an unbounded operator is said to be self-adjoint if

(3) $$D^* = \{g \in H; D \ni v \longrightarrow \langle g, Av\rangle \text{ extends to be continuous on } H\} = D$$

and (symmetry)

(4) $$\langle u, Av\rangle = \langle Au, v\rangle \ \forall\ u, v \in D.$$

(1) Suppose $P(D)$ is an elliptic differential operator with constant coefficients of order m. Show that $P(D) : H^m(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is an unbounded operator on $H = L^2(\mathbb{R}^n)$.

(2) Suppose that $P(D)$ is an elliptic differential operator of order m with P having real coefficients, show that $P(D) : H^m(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is self-adjoint.

(3) Suppose $A : D \longrightarrow H$ is an unbounded self-adjoint operator. Show if $z \in \mathbb{C}$ then $A + z\,\mathrm{Id} : D \longrightarrow H$ is an unbounded operator with domain D and that if $\mathrm{Im}\, z \neq 0$ then it is a bijection and that $(A + z\,\mathrm{Id})^{-1}$ is a bounded operator.

(4) Still assuming that A is unbounded self-adjoint, show that $(A + i\,\mathrm{Id})^{-1}$ and $(A - i\,\mathrm{Id})^{-1}$ commute, that their product is self-adjoint and that $E = ((A + i\,\mathrm{Id})^{-1}(A - i\,\mathrm{Id})^{-1})^{\frac{1}{2}}$ has range contained in D. Using this to see that AE is a bounded self-adjoint operator (or otherwise) define $f(A) \in \mathcal{B}(H)$ (as $g(AE)$ for an appropriate g) for each bounded continous function $f \in \mathcal{C}(\mathbb{R})$ with limits at infinity.

(5) Show that if $A = \Delta$ in the construction above, then

$$\widehat{f(A)u}(\xi) = f(|\xi|^2)\hat{u}(\xi) \ \forall\ u \in L^2(\mathbb{R}^n).$$

Notes for P4: Having shown that $(A + i\,\mathrm{Id})^{-1}(A - i\,\mathrm{Id})^{-1}$ commutes with $(A \pm i\,\mathrm{Id})^{-1}$ it follows that any polynomial in the former commutes with these two operators and hence that any continuous function on the spectrum defines an operator which commutes with them. Thus E commutes with $(A \pm i \operatorname{Id})^{-1}$ from which it follows that $E : D \longrightarrow D$. A similar argument shows that $[E, A]v = 0$ if $v \in D$. Now, compute the norm

$$\|AEv\|_H^2 = \langle EAv, EAv \rangle_H, \ v \in D.$$

The adjoint identity works here to shows this is equal to

$$\langle v, (\operatorname{Id} - E)v \rangle_H \tag{5}$$

and hence that AE extends by continuity to a bounded operator on H. The definition of self-adjointness shows that $E : H \longrightarrow D$.

Now you should define g by $f(x) = g(\frac{x}{\sqrt{x^2+1}})$ and show that g is defined and continuous on $[-1, 1]$ which contains the spectrum of AE.