PROBLEM SET 6, 18.155
DUE FRIDAY 25 OCTOBER, 2013

Let H be a separable Hilbert space throughout. Recall that the bounded operators on H form a Banach *-algebra: if $A, B \in \mathcal{B}(H)$ then $\|A^*\| = \|A\|$, $AB \in \mathcal{B}(H)$ and $\|AB\| \leq \|A\|\|B\|$. Let $\mathcal{K}(H) \subset \mathcal{B}(H)$ be the closed subspace of compact operators and $\mathrm{GL}(H) \subset \mathcal{B}(H)$ be the subset of invertible operators, those with two-sided bounded inverses.

You may to use the Uniform Boundedness Principle!

(1) Show that the open ball $B(\mathrm{Id}, 1) = \{B \in \mathcal{B}(H); \|B - \mathrm{Id}\| < 1\} \subset \mathrm{GL}(H)$.
(2) Show that in the unitary group, those $U \in \mathrm{GL}(H)$ with two-sided inverse U^*, multiplication is continuous in both the norm and the strong topologies, i.e. if $U_n \to U$ and $V_n \to V$ in $\mathrm{U}(H)$ then $U_nV_n \to UV$ in the same sense.
(3) Show that the map $\{A \in \mathcal{B}(H); A^* = A, \|A\| < 1\} \longrightarrow \exp(iA)$ may be defined by convergence of the Taylor series of the exponential function around 0 and that its image contains a neighbourhood of $\mathrm{Id} \in \mathrm{U}(H)$ in the norm topology. [You could try the Taylor series for log].
(4) Show that $T \in \mathcal{B}(H)$ has rank at most one (range of dimension at most one) if and only if $TKT = cT$ for some $c \in \mathbb{C}$ for each $K \in \mathcal{K}(H)$.
(5) Suppose that $V : \mathcal{K}(H) \longrightarrow \mathcal{K}(H)$ is an isomorphism (of Banach spaces) such that $V(K_1K_2) = V(K_1)V(K_2)$ and $V(K^*) = V(K)^*$. Show that there exists $U \in \mathrm{U}(H)$ such that $V(K) = UKU^*$.

Hint: For the last problem you might want to show that a rank one operator which is self-adjoint and satisfies $S^2 = S$ is of the form $v \longmapsto \langle v, u\rangle u$ for some element $u \in H$ of norm one (and conversely). The image under V is another such. Fixing this, look at the rank one operators which satisfy $TS = T$ – show that these form a linear space which can be identified with H. The image of this linear space is another version of the same thing. This gives a linear map from H to H and you can use the $*$ property to show it is norm-preserving and hence continuous. Then extend to finite rank operators and by density to $\mathcal{K}(H)$. You have shown that the 'automorphism group of $\mathcal{K}$ is the projective unitary group $\mathrm{U}(H)/\,\mathrm{U}(1)\,\mathrm{Id}$' (since if you check you can see that the multiples of the identity act trivially).