PROBLEM SET 4, 18.155
DUE FRIDAY 4 OCTOBER, 2013

(1) Recall that for an open set $\Omega \subset \mathbb{R}^n$, $\mathcal{C}^\infty_{\mathrm{c}}(\Omega) = \{\phi \in \mathcal{S}(\mathbb{R}^n); \operatorname{supp}(\phi) \Subset \Omega\}$. For an exhaustion $K_p \Subset \Omega$, $K_p \subset \overset{\circ}{K}_{p+1}$, $\Omega = \bigcup_p K_p$, we fix a topology on $\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ by declaring a set to be open if it meets each $\mathcal{C}^\infty_{\mathrm{c}}(K_p) = \{\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega); \operatorname{supp}(\phi) \subset K_p\}$ in an open set with respect to the induced (metric) topology from $\mathcal{S}(\mathbb{R}^n)$.

Show that this topology is independent of the exhaustion used to define it.

Show that the supremum norm is continous on $\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$.

Show that multiplication by a smooth function $\psi \in \mathcal{C}^\infty(\Omega)$ is continous as a map from $\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ to itself.

Show that if a sequence $\phi_j \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ converges to $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$, in the sense that $\phi_j \in \mathcal{O}$ for $j > J(\mathcal{O})$ if $\phi \in \mathcal{O}$ is open, then then $\phi_j \to \phi$ in some $\mathcal{C}^\infty_{\mathrm{c}}(K_p)$.

(2) Suppose we have shown that $c/|x|$ is a fundamental solution for $\varDelta = D_1^2 + D_2^2 + D_3^2$ on $\mathbb{R}^3$ for some constant c. Find all other tempered fundamental solutions.

(3) Define the 'local' Sobolev spaces of non-negative integral order on an open set $\Omega \in \mathbb{R}^n$ by

$$H^s_{\mathrm{loc}}(\Omega) = \{u \in \mathcal{C}^{-\infty}(\Omega); \phi u \in H^s(\mathbb{R}^n)\ \forall\ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)\}.$$

Show that a differential operator with smooth coefficients

$$P(x, D) = \sum_{|\alpha| \le m} p_\alpha(x) D^\alpha,\ p_\alpha \in \mathcal{C}^\infty(\Omega)$$

defines a map $H^{s+m}_{\mathrm{loc}}(\Omega) \longrightarrow H^s_{\mathrm{loc}}(\Omega)$ for any non-negative integer s.

(4) Show that if $u \in H^s_{\mathrm{loc}}(\Omega)$ for all s then $u \in \mathcal{C}^\infty(\Omega)$.

(5) Suppose that $P(x, D) = \sum\limits_{k=0}^{m} p_k(x) D_x^k$ is an *ordinary differential operator* with smooth coefficients $p_k(x) \in \mathcal{C}^\infty(\mathbb{R})$. Show that if $p_m(x)$, has no zeros then $u \in \mathcal{C}^{-\infty}(\mathbb{R})$ and $P(x, D)u \in \mathcal{C}^\infty(\mathbb{R})$ implies $u \in \mathcal{C}^\infty(\mathbb{R})$.

Hint:

P1 (There are probably easier ways) For the last part, show that if not, then we can assume (by passing to a subsequence and relabelling) that there a sequenc $x_j \notin K_j$ without repeats and with $\phi_j(x_j) \neq 0$. Show that there exists $\psi \in \mathcal{C}^\infty(\Omega)$ such that $\psi(x_j)\phi_j(x_j) = j$ and deduce a contradiction.

P2 It is enough to relate a general tempered fundamental solution to a harmonic polynomial – a polynomial in the null space of the operator, you don't have to describe all these (although that is possible).

P5 Aim to get to a point where you can apply Problem 4. First show that if u is any distribution on $\mathbb{R}$ then restricted to $(-N, N)$ it is in some $H^s_{\text{loc}}(-N, N)$ (compare to ϕu where ϕ is a cutoff which is 1 on a bigger interval) where s may depend on N. Use the equation to show that $D^m_x u \in H^{s-m+1}(-N, N)$. Now cut off again and show that this implies $u \in H^{s+1}(-N, N)$. One way to do this is to choose two cutoffs, ϕ and ψ both supported in the interval with $\phi = 1$ in a neighbourhood of the support of ψ. Check that $\psi D^m(\phi u) = \psi D^m u$. After that, use induction on s.