PROBLEM SET 3, 18.155
DUE ON FRIDAY 27 SEPTEMBER, 2013

(1) Show that if $u \in H^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is real-valued and $f \in \mathcal{C}^\infty(\mathbb{R})$ has $f(0) = 0$ then $f \circ u \in H^1(\mathbb{R}^n)$.

(2) Under the same conditions on f and for $0 < s < 1$ show that $f \circ u \in H^s(\mathbb{R}^n)$ if $u \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is real-valued.

(3) Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a diffeomorphism of slow growth, meaning it is a smooth map with components, and the derivatives of all orders of components, having polynomial bounds and with a 2-sided inverse with bounds of the same type. Show that pull-back is defined on Schwartz functions and is an isomorphism
$$f^* : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n), \ f^*\phi(x) = \phi(f(x)).$$

(4) Let P be a differential operator with constant coefficients, so $Pu = \sum_{0 \le |\alpha| \le m} c_\alpha D^\alpha u$ for some constants c_α. Show that as a map $P : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is injective if and only if the polynomial $P(\zeta)$ has no zeros on $\mathbb{R}^n$.

(5) Explain how the Schwartz structure theorem (in L^2 form) and the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ imply the density of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$ in the weak topology.

Hints: Use the existence of strong derivatives in the first question and the 'L^2-Lipschitz' characterization of H^s in the second. For the third question, first check that $|f(x)| + 1 \ge c|x|^a$ for some positive constants c and a. In the fourth question construct a counterexample if P has a real zero and be careful about the opposite argument – if you try to divide by P directly you will need some estimates on it.

Non-problem:

Since someone asked me about it, here is a sketch of an elementary, but somewhat painful, proof that if $1 < k \in \mathbb{N}$ and one takes p real-valued elements of $H^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $f \in \mathcal{C}^\infty(\mathbb{R}^p)$ then the derivatives up to order k of $f(u_1, \ldots, u_p)$ are in L^2.

(1) Using convolution we can approximate the u_i by sequences in $\mathcal{S}(\mathbb{R}^n)$ which converge in H^k and are bounded in L^∞. This means it is okay to assume the u_k are smooth and prove L^2 estimates on the derivatives in terms of the H^k and L^∞ norms on the u_i.

(2) So we can just differentiate. In fact we know that it is enough to prove that the 'pure' derivatives are in L^2 so we just concentrate on derivatives ∂_1^j, $1 \leq j \leq k$ where we already know the result for $j = 1$ anyway

(1) $$\partial_1^j f(u_1, \ldots, u_p) = \sum \partial_1^{j_1} u_1 \ldots \partial_1^{j_p} u_p F_J(u_1, \ldots, u_p).$$

Here the sum is over partitions of j into up to p *positive* integral terms. The residual factor is in L^∞ (with a bound in terms of the bounds on the u_i) so it is actually enough to show, with bounds, that

(2) $$\prod \partial_1^{j_i} u_i \in L^2$$

where the product is over positive integers j_i such that the sum is at most $j \leq k$ – the number of factors is then at most j as well, so we can forget p now.

(3) Now, we can replace (2) by the statement that

(3) $$\int \prod \partial_1^{j_i} u_i \text{ in } L^1, \quad \text{where } j_i \leq k \text{ and } \sum_i j_i \leq 2k$$

just taking the square (everything is real) and forgetting the relationship of the factors. So in (3) we have at most $2k$ factors, all in $H^k \cap L^\infty$.

(4) Now we have (3) when there are only two factors – these are bounded by L^2 norms of derivatives which we control.

(5) We proceed by (finite) induction over the total number of factors – it cannot be larger than $2k$ since we assume everything is differentiated at least once.

(6) Look at the terms where one of the factors has the maximum number, k, of derivatives and estimate it by Cauchy-Schwarz. The L^1 norms of these terms are bounded by the product of the L^2 norm of the k-fold derivative (controlled) and the *square-root* of the L^1 norm of another product as in (3).

(7) So look at the products with l factors and no derivative of order k and proceed inductively by the size of the smallest derivative present.

(8) Single out a factor with the smallest number of derivatives. Even though there is an absolute value in the integral, we can integrate by parts once, taking a term off this smallest one. Indeed everything is real and the integral is the sum of the integrals we get by replacing every term by either its positive or negative part. All terms are continuous and have bounded derivatives and nothing goes wrong on integration by parts.

(9) The result of one integration by parts is to replace this term by a sum where the order of the smallest derivative has gone down by one. These integrals are all bounded by the L^1 integrals of products with one 'smaller' term.

(10) If the smallest order was 1 then this gives terms with one undifferentiated factor which can be removed and bounded by L^∞.

(11) After integration by parts all the terms fall into one of three classes – fewer factors which is the earlier induction, smaller minimum, which is the current induction, or a derivative of order k. The square root estimate here means we can ultimately estimate a bunch of L^1 norms by multiples of their square-roots and this means we win.

You are invited to write this out more coherently! If you are really short of things to do, try to extend this to non-integral s. The result is true, but off-hand I do not see an elementary proof like this one.