PROBLEM SET 10 == ASSIGNMENT 2, 18.155
DUE DECEMBER 6, 2013

I do not mind if two, or three, people collaborate on the project if they each write their own versions of the conclusions.

Here is a preliminary list of projects. I will come up with a couple more, and am happy to try to accommodate requests.

(1) If $P(D)$ is an elliptic operator with constant coefficients and $\Omega \subset \mathbb{R}^n$ show that $P(D) : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega)$ is surjective. Involves estimates!

(2) Give a proof of Kuiper's theorem ,that the group of unitary operators on a separable but infinite-dimensional Hilbert space is 'weakly contractible' (meaning all its homotopy groups vanish) in the norm topology. Has topological consequences (e.g. that the projective unitary group is a $K(\mathbb{Z}, 2)$). Standard proofs are a bit of a slog!

(3) Show that the harmonic oscillator, $\sum\limits_j (D_j^2 + x_j^2)$ is an unbounded self-adjoint operator on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n) \cap \{u \in L^2; |x|^2 u \in L^2\}$. Show that its eigenfunctions are of the form $p_\alpha(x) \exp(-\frac{1}{2}|x|^2)$ with the p_α polynomials and that these can also be chosen to be eigenfunctions for the Fourier transform.

(4) Construct the forward fundamental solution for the wave operator $P = D_t^2 - \sum\limits_{j=1}^{n} D_j^2$ on $\mathbb{R} \times \mathbb{R}^n$ (the one with support in $t \geq 0$) and show that it is unique. Use this to prove that 'forcing problem' $Pu = f$ where $f \in \mathcal{C}^{-\infty}(\mathbb{R}^{1+n})$ has support in $t \geq 0$ has a unique solution $u \in \mathcal{C}^{-\infty}(\mathbb{R}^{1+n})$ with support in $t \geq 0$ and that this solution is smooth if f is smooth.

(5) Describe Wagner's proof of the Ehrenpreis-Malgrange theorem – that any non-trivial $P(D)$ has a fundamental solution. MR2510844.

(6) Discuss Lidskii's theorem – that the trace of a compact operator is the sum of its eigenvalues. Harder than you might think but not impossible.

(7) Radon transform. Discuss some of the properties of the Radon transform as a map $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$ given by

$$(1) \quad Rf(s, \omega) = \int_{H(s,w)} f(x) dH(s, \omega), \ H(s, \omega) = \{x \in \mathbb{R}^n ; x \cdot \omega = s\}.$$

(8) A little Fréchet analysis. Show that the operators on $L^2(\mathbb{R}^n)$ with Schwartz kernels in $\mathcal{S}(\mathbb{R}^{2n})$ form a topological algebra with the product giving a (metrically) continuous map

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^{2n}) \ni (a, b) \longmapsto a \circ b(x, y) = \int_{\mathbb{R}^n} a(x, z)b(z, y)dz. \tag{2}$$

Prove that the group of those elements $a \in \mathcal{S}(\mathbb{R}^{2n})$ such that $\mathrm{Id} + a$ (as an operator) has inverse $\mathrm{Id} + b$ for $b \in \mathcal{S}(\mathbb{R}^{2n})$ is an open and dense subset of $\mathcal{S}(\mathbb{R}^{2n})$. If you want, you can go on to show that the (unnormalized) Chern forms

$$\mathrm{Tr}\left((g^{-1}dg)^{2k+1}\right) \tag{3}$$

are well-defined closed forms on this group.

Elliptic operators1

I think you can find a proof in Hörmander's four volume treatise, probably in Volume 2. It is a special case of 'P-pseudoconvexity' in the sense that any open set is pseudoconvex with respect to an elliptic operator.

Kuiper's Theorem 2

The theorem is that a norm continuous map from a sphere of any dimension into $\mathrm{GL}(H)$ or $\mathrm{U}(H)$ can be extended to a continuous map from the ball – with the original map as the restriction to the sphere. This is 'weak contractibility' which here means that all the homotopy groups vanish.

Harmonic oscillator 3

One of the significant features of the harmonic oscillator is that one can get elliptic regularity directly, using what we already know and some commutation. The important point is that although the coefficients are not constant, they are pretty nice in that for instance $[\partial_j, x_j] = 1$.

Here is one way to proceed. Suppose you know that $u \in L^2(\mathbb{R}^n)$ and $Hu \in L^2(bbR^n)$ where $H = \sum_j(-\partial_j^2 + x_j^2)$ is the harmonic oscillator. This for instance is where you would start if you wanted to show that H is an unbounded self-adjoint operator, of course $Hu \in \mathcal{S}'(\mathbb{R}^n)$ so the assumption does make sense. Now, by *local* elliptic regularity, we can see that $u \in H^2(\mathbb{R}^n)$ locally – just write the equation $Hu = f$ as

$\Delta u = f - |x|^2 u$. Now, consider a cutoff $\psi_n = \psi(x/n)$ where $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is equal to 1 near 0. The idea is to start from the integral

$$\int (\psi_n^2 u)\overline{Hu} = \langle \phi_n^2 u, fu \rangle_{L^2}$$

and integrate by parts. In fact, take the real part of this – which cancels a term. Working on the (real part of) the left side, the piece $|x|^2$ of H contributes

$$\int |x|^2 \phi_n^2 |u|^2$$

which is positive and useful. Integration by parts from the derivative terms gives

$$\int |\nabla(\phi_n u)|^2$$

which is also good, plus some commutator terms which you should work out. These are of the form (I am being very casual, look at it carefully!)

$$\phi_n \phi_n'' |u|^2 + \phi_n \phi_n' u \overline{\partial_j u}.$$

Because we take the real part we can integrate again and get a cancelation from the second terms so that the whole commutator looks, after integration like the first term above. Really then what one gets is

$$\int |\nabla \phi_n u|^2 + |x|^2 |\phi_n u|^2 = B_n(u) + \mathrm{Re}\langle \phi_n^2 u, f \rangle.$$

where B_n is quadratic in u but involves no derivatives and in fact you can easily see that since $u \in L^2$, $B_n(u) \to 0$ as $n \to \infty$.

It follows easily from this that $\phi_n u$ converges to u in $H^1(\mathbb{R}^n)$ and also that $|x|u \in L^2(\mathbb{R}^n)$. Going back to the beginning you can now see that $H\phi_n u \to Hu$ in L^2 because the commutator terms vanish in the limit.

Now do a similar thing again, looking at $\|H\phi_n u\|^2_{L^2}$. There are two big positive terms and some cross terms. If you integrate by parts in the cross term you can get another positive term, basically the sum $\|x_j \partial_j \phi_n u\|^2_{L^2}$ plus a term which is now controlled by the fact that $\phi_n u \to u$ in H^1 etc. This now shows that $u \in H^2$ and $|x|^2 u \in L^2$.

For an eigenfunction you can easily iterate the argument to get $u \in H^\infty$, or you can use this to prove that H is self-adjoint with domain consisting of those $u \in H^2$ such that $|x|^2 u \in L^2$. In that case you can see easily that H is invertible with inverse a compact self-adjoint operator.

This still leaves the problem of showing that all eigenfunctions are as stated.

Wave equation 4

Look in Friedlander-Joshi,

Ehrenpreis Malgrange theorem 5

Lidskii's Theorem 6

Discuss the properties of theresolvent $(z-A)^{-1}$ for $A \in \mathcal{K}(H)$ (we did this) far enough to define the eigenvalues (the poles of the resolvent) and their *multiplicity*, the dimension of the associated *generalized* eigenspaces – which is to say the integer $m(z) = \lim_{k\to\infty} \dim(\mathrm{Nul}(A-z)^k)$ (the finiteness of which is the main issue). The Lidskii's theorem says that for A of trace class

$$\mathrm{Tr}(A) = \sum_{z\in\mathbb{C}} m(z). \tag{4}$$

Radon tranform 7

You can find a treatment of this in the book 'Scattering theory' by Lax and Phillips and other places too. The main thing to do is to get boundedness and invertibility properties on Sobolev spaces using the Fourier transform. More information on request.

Frechet analysis 8

To show that these 'Schwartz' operators form a topological group you only need to show that the composition and inversion maps are continuous. For the composition this is straightforward since the kernel of the composite of two operators $\mathrm{Id}+A$ and $\mathrm{Id}+B$ is $\mathrm{Id}+C$ where $C = A+B+AB$ and the kernel of AB is

$$e(x,x') = \int a(x,x'')b(x'',x')dx''. \tag{5}$$

It is easy to see that this gives a continuous bilinear map with respect to the metric. Continuity of the inverse follows by use of Neumann series and a little care about the estimates above. Density is not so hard either, I can offer help with this if anyone asks.