18.155 LECTURE 4: 17 SEPTEMBER, 2013

Last week I went through the proof that the Fourier transform

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto \hat{\phi} = \int e^{i\bullet\cdot x}\phi(x)dx \in \mathcal{S}(\mathbb{R}^n) \tag{1}$$

is an isomorphism and its basic properties and extension to a bijection on $\mathcal{S}'(\mathbb{R}^n)$ and on $L^2(\mathbb{R}^n)$. The latter was based in part at least on a plausibility argument for the injectivity of the map $I : L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$. This in turn allows one to define the L^2-based Sobolev spaces

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); (1 + |\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)\},\ s \in \mathbb{R}. \tag{2}$$

Written out more carefully this last statement says we consider those $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $(1 + |\xi|^2)^{s/2}\hat{u} = I(v)$ for some $v \in L^2(\mathbb{R}^n)$ which makes sense since you showed in the homework that $(1 + |\xi|^2)^{s/2}$ is a multiplier on $\mathcal{S}'(\mathbb{R}^n)$.

The most basic properties of $H^s(\mathbb{R}^n)$ are that these spaces *decrease* as s increases – since for any $s' \le s$, $(1+|\xi|^2)^{\frac{1}{2}(s'-s)}$ is a multiplier on $L^2(\mathbb{R}^n)$. They are all Hilbert spaces where the inner product can be taken to be

$$\langle u, v\rangle_s = (2\pi)^{-n}\int (1 + |\xi|^2)^s\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi. \tag{3}$$

By Parseval's identity the constant is chosen so this is consistent with the inner product on L^2 for $s = 0$.

Towards the end I discussed weak and strong derivatives and the Sobolev embedding theorem. Since I went rather quickly let me recall these.

First

Lemma 1. *If $s = k \in \mathbb{N}$ then*

$$H^k(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); u \text{ has strong derivatives in } L^2 \text{ to order } k\}. \tag{4}$$

Here a strong partial derivative in $L^2(\mathbb{R}^n)$ means that the limit

$$\lim_{t\to 0}\frac{u(x + te_j) - u(x)}{t} = v_j \text{ exists in } L^2(\mathbb{R}^n). \tag{5}$$

As usual we define the notion of having strong derivatives up to order k iteratively – by demanding that the strong partial derivatives exist and have strong L^2 partial derivatives up to order $k - 1$.

Proof. To prove the forward result it suffices to prove that if $u \in H^k(\mathbb{R}^n)$ with $k \in \mathbb{N}$ then it has strong partial derivatives and these lie in $H^{k-1}(\mathbb{R}^n)$. The existence of strong partial derivatives follows from first from the existence of weak partial derivatives. If $u \in H^k(\mathbb{R}^n)$ then $u \in H^1(\mathbb{R}^n)$ and $(1 + |\xi|)\hat{u} = w \in L^2(\mathbb{R}^n)$. This implies that $\hat{v}_j = \xi_j\hat{u} = \frac{\xi_j}{(1+|\xi|)}w \in L^2$ defines $v_j \in L^2(\mathbb{R}^n)$ and this satisfies

$$(D_jI(u))(\overline{\phi}) = I(u)(\overline{D_j\phi}) = -\int uD_j\phi = (2\pi)^{-n}\int \hat{u}\overline{\xi_j\hat{\phi}} = \int v_j\overline{\phi}\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{6}$$

I put the bars in here to make the computation easier. So v_j is a weak derivative in $L^2(\mathbb{R}^n)$. As I showed last time this implies that it is a strong derivative as well. Indeed computing the Fourier transform of the difference quotient gives

$$\widehat{\frac{u(x+te_j)-u(x)}{t}} = \frac{e^{it\xi_j}-1}{t}\hat{u}(\xi) \tag{7}$$

and using Taylors formula with remainder we showed last time that

$$\frac{e^{it\xi_j}-1}{t}(1+|\xi|)^{-1} \longrightarrow i\xi_j(1+|\xi|)^{-1} \tag{8}$$

pointwise in ξ and with a uniform bound. In (7) this means by Lebesgues' dominated convergence that the right side converges to $i\xi_j\hat{v}_j$ in L^2 so in fact (5) holds. If $u \in H^k(\mathbb{R}^n)$ then $v_j \in H^{k-1}(\mathbb{R}^n)$ and the result this way holds.

The statement the other way, that if u has strong partial derivatives in L^2 up to order k then it is in $H^k(\mathbb{R}^n)$ is a bit easier. Again induction in k reduces the problem to showing that if u has strong first derivatives and they are in $H^{k-1}(\mathbb{R}^n)$ then $u \in H^k(\mathbb{R}^n)$. Change of variable of integration shows that for $u \in L^2(\mathbb{R}^n)$

$$\int u(x)\frac{\phi(x-t)-\phi(x)}{t} = \int \frac{u(x-t)-u(x)}{t}\phi(x)dx,\ \ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{9}$$

The difference quotient for ϕ does converge in $L^2(\mathbb{R}^n)$ – $\mathcal{S}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$ for instance – so both sides converge and this shows that a strong derivative is a weak derivative. The inductive hypothesis means that

$$(1+|\xi|^2)^{\frac{k-1}{2}}\xi_j\hat{u} \in L^2(\mathbb{R}^n) \tag{10}$$

for all j and this implies that $(1+|\xi|^2)^{\frac{k}{2}}\hat{u} \in L^2(\mathbb{R}^n)$, i.e. $u \in H^k(\mathbb{R}^n)$. □

So, now we have some idea of what the Sobolev spaces of positive integral order are like. What about the intermediate ones?

Proposition 1. *For $0 < s < 1$, $u \in L^2(\mathbb{R}^n)$ is in the space $H^s(\mathbb{R}^n)$ if and only if the integral*

$$\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy < \infty. \tag{11}$$

Proof. The integrand here is a non-negative measureable function so this makes good sense. So, start with $u \in H^s(\mathbb{R}^n)$. This means that $u \in L^2$ and $(1+|xi|^2)^{s/2}\hat{u} \in L^2$. So, we can use the fact that $(1+|\xi|^2)^{s/2}$ and $1+|\xi|^s$ are bounded by multiples of each other, so given that $u \in L^2$ and hence $\hat{u} \in L^2$ this just means that

$$|\xi|^s\hat{u} \in L^2(\mathbb{R}^n). \tag{12}$$

Notice that the integral in (11) is like and integrated difference quotient, it can be rewritten as

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x+t)-u(x)|^2}{|t|^{n+2s}}dxdt.$$

With this in mind we proceed to compute the L^2 integral using the Fourier transform

$$\int_{\mathbb{R}^n} |u(x+t)-u(x)|^2dx = (2\pi)^{-n}\int_{\mathbb{R}^n} |e^{it\xi}-1|^2|\hat{u}(\xi)|^2d\xi. \tag{13}$$

So the t integral we want to evaluate is

$$F(\xi) = \int |e^{it\cdot\xi} - 1|^2 |t|^{-n-2s} dt. \tag{14}$$

This is finite for each ξ since the singularity at the origin is integrable – the first term vanishes at $t = 0$ so contributes a $|t|^2$ and the integral is bounded by $C|t|^{-n-2s+2}$ which is integrable for $s < 1$. Near infinity the first term is bounded so it is integrable there too.

We can also make a rotation in the integral to check that $F(\xi) = f(|\xi|)$ is constant on spheres. We can also scale the variables, sending $\xi \longmapsto r\xi$ and $t \to t/r$ for r positive shows that

$$F(r\xi) = r^{2s} F(\xi) \Longrightarrow F(\xi) = C|\xi|^{2s},\ C > 0. \tag{15}$$

From this it follows that the finiteness of (11) for $u \in L^2(\mathbb{R}^n)$ is equivalent to $u \in H^s(\mathbb{R}^n)$. □

Next we can relate the two spaces $H^{\pm s}(\mathbb{R}^n)$.

Lemma 2. *For any $s \in \mathbb{R}$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ and if $u \in \mathcal{S}'(\mathbb{R}^n)$ then $u \in H^s(\mathbb{R}^n)$ if and only if*

$$|u(\phi)| \le C\|\phi\|_{H^{-s}}\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{16}$$

Proof. The density follows from the density of $\mathcal{S}(\mathbb{R}^n)$ in L^2. If $u \in H^s(\mathbb{R}^n)$ then $(1+|\xi|^2)^{s/2}\hat{u} = v \in L^2(\mathbb{R}^n)$ so there exists $v_n \in \mathcal{S}(\mathbb{R}^n)$ such that $v_n \to v$ in $L^2(\mathbb{R}^n)$. Then defining $w_n \in \mathcal{S}(\mathbb{R}^n)$ by $\hat{w}_n = (1 + |\xi|^2)^{-s/2} v_n(\xi)$ we see that

$$\|u - w_n\|^2_{H^s} = (2\pi)^{-n} \int (1 + |\xi|^2)^s |\hat{u} - \hat{w}_n|^2 = (2\pi)^{-n} \int |v - v_n|^2 \to 0. \tag{17}$$

Setting $\psi = \overline{\phi}$

$$u(\overline{\psi}) = (2\pi)^{-n} \hat{u}(\overline{\hat{\psi}}) = (2\pi)^{-n} \int (1 + |\xi|^2)^{s/2} \hat{u} \cdot (1 + |\xi|^2)^{-s/2} \overline{\hat{\psi}} \tag{18}$$

Here the last integral should really be interpreted as the distributional pairing. Then the inequality (16) (and Riesz' Representation Theorem) tells us that $v = (1 + |\xi|^2)^{s/2}\hat{u}$ does exist in L^2 so that (18) holds and hence $u \in H^s(\mathbb{R}^n)$. The converse is easier. □

For Sobolev spaces of negative integral order we get a corresponding 'dual' characterization to that above.

Lemma 3. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ then $u \in H^{-k}(\mathbb{R}^n)$, with $k \in \mathbb{N}$, if and only if there exist $u_\alpha \in L^2(\mathbb{R}^n)$ for all $0 \le |\alpha| \le k$ such that*

$$u = \sum_{0 \le |\alpha| \le k} D^\alpha u_\alpha. \tag{19}$$

I.e. it is a sum of up to k fold derivatives of L^2 functions.

Proof. We know that $u \in H^{-k}(\mathbb{R}^n)$ if and only if $\hat{u} = (1 + |\xi|^2)^{k/2} v$ with $v \in L^2$. This can also be written

$$\hat{u} = (1 + |\xi|^2)^k v',\ v' = (1 + |\xi\|^2)^{-k/2} v,\ v \in L^2(\mathbb{R}^n). \tag{20}$$

This however means precisely that $v' = \hat{w}$ where $w \in H^k(\mathbb{R}^n)$. Multiplying out the factor in (20) we see that for some constants c_α,

$$u = \sum_{0\le|\alpha|\le k} D^\alpha u_\alpha,\ u_\alpha = (c_\alpha D^\alpha w) \in L^2(\mathbb{R}^n). \tag{21}$$

□

This can be refined in various. Using the fact that the three functions

$$(1+|\xi|^2)^k,\ 1 + \sum_{|\alpha|=k} |\xi^\alpha|^2 \text{ and } 1 + \sum_{j=1}^n |\xi_j|^{2k} \tag{22}$$

are bounded above and below by constant multiples of each other, the second two can be inserted in place of the first in (20). This allows the condition $u \in H^{-k}(\mathbb{R}^n)$ to be shown to be equivalent to either of

$$\begin{gathered} u = u_0 + \sum_{|\alpha|=k} D^\alpha u_\alpha,\ u_\bullet \in L^2(\mathbb{R}^n) \text{ or} \\ u = v_0 + \sum_{j=1}^n D_j^k v_j,\ v_\bullet \in L^2(\mathbb{R}^n). \end{gathered} \tag{23}$$

As an exercise towards 'elliptic regularity' which we discuss later you should check that similarly for the positive integral case

(24)

$$\text{For } u \in \mathcal{S}'(\mathbb{R}^n),\ u \in H^k(\mathbb{R}^n) \Longleftrightarrow u \in L^2(\mathbb{R}^n) \text{ and } D_j^k u \in L^2(\mathbb{R}^n)\ j = 1, \dots, n.$$

Now we can prove Schwartz' structure theorem.

Theorem 1. *Any $u \in \mathcal{S}'(\mathbb{R}^n)$ can be written in the form of a finite sum*

$$u = \sum_{|\alpha|+|\beta|\le N} x^\alpha D^\beta u_{\alpha,\beta},\ u_{\alpha,\beta} \in L^2(\mathbb{R}^n) \tag{25}$$

or, for a possibly different N, with the $u_{\alpha,\beta}$ bounded continuous functions.

Detour on isotropic Sobolev spaces – I will add some notes a bit later.