18.155 LECTURE 24: 5 DECEMBER, 2013

Looking at this, I may be guilty of 'over explanation' here, oh well it is the end of the semester.

I wanted to finish this course with a fairly modest question, which it turns out arises in lots of places.

Question 1. Suppose $P = P(D) = \sum_{|\alpha|=m} c_\alpha D^\alpha$ is homogeneous of degree m and elliptic, on what spaces (containing $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$) is it Fredholm?

In principle of course you might say there is no reason why it needs to be Freholm at all! It is not Fredholm as a map between Sobolev spaces for instance,

$$P : H^{s+m}(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^n),\ s \in \mathbb{R} \tag{1}$$

although it is certainly bounded. Let me recall why.

We know that the null space of P on tempered distributions contains only polynomials (since $P(\xi)\hat{u} = 0$ and $P(\xi) \neq 0$ for $\xi \neq 0$ so $\operatorname{supp}(\hat{u}) \subset \{0\}$ and u is a polynomial – although not any polynomial). There are no polynomials in the H^s spaces for any s so P is injective. The problem is that its range cannot be closed. For $s = 0$ this is clear enough since to say that $f \in L^2(\mathbb{R}^n)$ is in the range is to demand that

$$\hat{f}(\xi) = P(\xi)\hat{u}(\xi),\ (1 + |\xi|)^m \hat{u} \in L^2(\mathbb{R}^n). \tag{2}$$

It follows easily that the range is dense, since the L^2 functions vanishing in some ball (depending on the function of course) around the origin are dense in L^2 and these are in the range since $c|\xi|^m \le |P(\xi)| \le C|\xi|^m$ for positive constants.

On the other hand the function with Fourier transform a multiple of the characteristic function $\hat{f} = |P(\xi)|^t \chi(|\xi| \le 1)$, for $t = -n/2 + \delta$, $\delta > 0$ small enough, cannot be in the range. Note that $\hat{f} \in L^2$ since

$$\int_{|\xi|\le 1} |P(\xi)|^{2t} d\xi < \infty \text{ if } 2t > -n \tag{3}$$

If f were in the range then $\hat{f} = P(\xi)\hat{u}$ with $\hat{u} \in L^2$ which would imply that $|P(\xi)|^{t-1}\chi(|\xi| \le 1) \in L^2(\mathbb{R}^n)$.

Thus the range of (1) is dense but not closed for $s = 0$. The same is true for all s since the multiplication by $(1 + |\xi|^2)^{s/2}$ on the Fourier transform side is an isomorphism changing the order of the Sobolev spaces and commuting with P.

Why are we interested in this question anyway? Well, the obvious examples are

(1) d/dx on $\mathbb{R}$.
(2) $\partial_x + i\partial_y$, the Cauchy-Riemann operator on $\mathbb{R}^2 = \mathbb{C}$.
(3) $\Delta = |D|^2$ on $\mathbb{R}^n$

and these are clearly important!

Although it is so simple as to be confusing, consider d/dx on the line. Notice that if we considered the non-homogeneous operator $d/dx + it$, $0 \neq t \in \mathbb{R}$ then we would indeed have an isomorphism on the standard Sobolev spaces. The aim is to get to this point, but a certain effort is required.

One thing that you might try is to work on weighted Sobolev spaces, let's think about $s = 0$ (which it turns out does not matter because the operator is elliptic) and try to find a space which makes d/dx into a Freholm operator into

$$\langle x\rangle^{-t}L^2(\mathbb{R}) = \{u \in L^2_{\text{loc}}(\mathbb{R}); u = \langle x\rangle^{-t}v,\ v \in L^2(\mathbb{R})\}, \langle x\rangle = (1+|x|^2)^{\frac{1}{2}},\ t \in \mathbb{R}. \tag{4}$$

By Cauchy-Schwartz

$$\langle x\rangle^{-t}L^2(\mathbb{R}) \subset L^1(\mathbb{R}),\ \int |u|^2 \le (\int |v|^2)^{\frac{1}{2}}(\int \langle x\rangle^{-2t})^{\frac{1}{2}} \text{ if } t > -\frac{1}{2} \tag{5}$$

since the last integral is then finite.

So, let's assume (5) and for convenience look at the subspace

$$\{f \in \langle x\rangle^{-t}L^2(\mathbb{R}) \subset L^1(\mathbb{R}); \int_{\mathbb{R}} f = 0\},\ t > \frac{1}{2} \tag{6}$$

which is clearly of codimension 1.

Then we can answer the question in a certain sense – by simply defining

$$D_t = \{u \in L^2_{\text{loc}}(\mathbb{R}); u(x) = \int_{-\infty}^x f(s)ds,\ f \text{ in } (6)\},\ t > -\frac{1}{2}. \tag{7}$$

Clearly $d/dx : D_t \longrightarrow \langle x\rangle^{-t}L^2(\mathbb{R})$ has closed range of codimension one and no null space. So it is Fredholm if we simply give it the topology from the range space using the integral to identify them – i.e. make d/dx an isomorphism onto its range.

Of course this is a bit unsatisfactory, we really should try to see what the space D_t is and what this topology on it looks like. We could do this by trying to estimate the integral in (7) to see what the range is. However, I think it is much more fruitful to try to manipulate d/dx as I indicated at the beginnning to 'turn it into $d/dx+is$ although the variable will be different.

Let's just look near $x = -\infty$. Then the solution $u \in D_t$ is the one that satisfies

$$\frac{d}{dx}u(x) = x^{-t}g(x),\ \text{on } (-\infty,-1),\ g \in L^2,\ \lim_{x\to-\infty} u(x) = 0. \tag{8}$$

Although it may seem counter-intuitive, we may multiply (8) by x and the equation becomes instead

$$x\frac{d}{dx}u(x) = x^{-t+1}g(x),\quad \text{on } (-\infty,-1),\ g \in L^2,\ \lim_{x\to-\infty} u(x) = 0. \tag{9}$$

If we commute a factor of $x^{-t+\frac{1}{2}}$ (why this one?) through the equation we are looking instead at

$$(x\frac{d}{dx} + t - \frac{1}{2})x^{-t+\frac{1}{2}}u(x) = x^{\frac{1}{2}}g(x),\quad \text{on } (-\infty,-1),\ g \in L^2,\ \lim_{x\to-\infty} u(x) = 0. \tag{10}$$

This is starting to look like what we are after! Especially if we set $v = x^{-t+\frac{1}{2}}u(x)$ and change the variable to $z = -\log(-x)$ (so it is still near $-\infty$) and the equation becomes

$$(\frac{d}{dz} + t - \frac{1}{2})v(z) = h(z) \text{ in } z < 0. \tag{11}$$

Why did I 'leave a half power behind'? Well, the Jacobian of the coordinate change is given by $\frac{dx}{x} = dz$ so if $g \in L^2$ then

$$\int |v(z)|^2 dz = \int |x^{\frac{1}{2}} g|^2 \frac{dx}{x} = \int |g|^2 dx \tag{12}$$

is 'exactly in L^2' with respect to z.

So, what we are looking for is a solution of (11) when the RHS is in L^2 (in $z < 0$ – extend it to be zero in $z > 0$). This we can do, there is a unique solution in L^2 (with respect to z of course) since

$$(\frac{d}{dz} + t - \frac{1}{2}) : H^1(\mathbb{R}_z) \longrightarrow L^2(\mathbb{R}_z) \text{ is an isomorphism for } t \neq \frac{1}{2}. \tag{13}$$

So, now we can understand what the solution to (8) actually is!

Definition 1. For any $s \in \mathbb{R}$ define

$$H^s_{\mathrm{b}}(\mathbb{R}) = \{u \in H^s_{\mathrm{loc}}(\mathbb{R}); \chi_\pm u(\pm e^z) \in H^s(\mathbb{R}_z)\tag{14}$$

where $0 \leq \chi_\pm \in \mathcal{C}^\infty(\mathbb{R})$ have support in $\pm x > 1$ and are equal to 1 near $\pm\infty$.

Notice that for $s = 0$ the condition near $x = \infty$ is that

$$\int |u(e^z)|^2 dz = \int |u(x)|^2 \frac{dx}{x} < \infty \Longleftrightarrow u = \langle x \rangle^{\frac{1}{2}} v,\ v \in L^2(\mathbb{R}_x). \tag{15}$$

Lemma 1. *The spaces $H^s_{\mathrm{b}}(\mathbb{R})$ defined by* (14) *are Hilbert spaces with*

$$H^0_{\mathrm{b}}(\mathbb{R}) = \langle x \rangle^{\frac{1}{2}} L^2(\mathbb{R}). \tag{16}$$

Proof. Take a partition of unity on $\mathbb{R}$ using $\chi_\pm$ in (14) and $0 \leq \chi_0 \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ and set

$$\langle u, v \rangle_{\mathrm{b},s} = \langle \chi_0 u(x), v(x) \rangle_{H^s} + \langle \chi_+ u(e^z), v(e^z) \rangle_{H^s} + \langle \chi_- u(-e^z), v(-e^z) \rangle_{H^s} \tag{17}$$

where the second two H^s norms are with respect to the variable z. □

The topology here is independent of the cut-offs since we can 'shift' terms supported in a compact region between the three parts.

Proposition 1. *For any $t \neq 0$*

$$d/dx : \langle x \rangle^{-t+1} H^{s+1}_{\mathrm{b}}(\mathbb{R}) \longrightarrow \langle x \rangle^{-t} H^s_{\mathrm{b}}(\mathbb{R}) \tag{18}$$

is Fredholm, with index -1 when $t > 0$ and index 1 when $t < 0$, it is not Fredholm for $t = 0$.

The space $\langle x \rangle^{-t} H^s_{\mathrm{b}}(\mathbb{R})$ consists of those $u \in H^s_{\mathrm{loc}}(\mathbb{R})$ such that $\langle x \rangle^t u \in H^s_{\mathrm{b}}(\mathbb{R})$.

Proof. This is a matter of going backwards through the preceding discussion! □

Now, what happens if we go to the general case? We need to 'guess' what the analogues of the spaces in Definition 1 should be. For $n > 1$, we think in terms of the radial action near infinity, which makes it look like $\mathbb{S}^{n-1}_\theta \times (0, \infty)_r$, $r = |x|$, $\theta = x/|x|$. I claim we can be pretty confident about how to define Sobolev spaces on a manifold such at $\mathbb{S}^{n-1}_\theta \times \mathbb{R}_y$.

Then we can try the following

Definition 2. For any $s \in \mathbb{R}$ set

$$H^s_{\mathrm{b}}(\mathbb{R}^n) = \{u \in H^s_{\mathrm{loc}}(\mathbb{R}^n); v(z, \theta) = \chi u(e^z \theta) \in H^s(\mathbb{R}_z \times \mathbb{S}^{n-1})\tag{19}$$

where $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$ has support in $|x| > 1$ and is identically equal to 1 in $|x| > 2$.

As before, observe that the case $s = 0$ is easy to analyse. Lebesgue measure in polar coordinates $r = |x|$, $\theta = x/|x|$ is

$$|dx| = r^n \frac{dr}{r} |d\theta| \tag{20}$$

were $d\theta$ is a strictly positive density on the sphere (the standard one in fact). It is written this way since then

$$|dx| = r^n |dz||d\theta|,\ z = \log r,\ r = e^z \text{ so } u \in L^2(\mathbb{R}^n) \\ \Longrightarrow \chi(|x|^{-\frac{n}{2}} u)(e^z \theta) \in L^2(|dz||d\theta|). \tag{21}$$

So in fact

$$H^0_{\mathrm{b}}(\mathbb{R}^n) = \langle x \rangle^{\frac{n}{2}} L^2(\mathbb{R}^n) \tag{22}$$

generalizing (16).

These spaces are sometimes called 'homogeneous Sobolev spaces'. On reason for this is that if $u \in \mathcal{C}^\infty(\mathbb{R}^n)$ is homogeneous of some real degree m in $|x| > R$ then

$$u \in \langle x \rangle^{m+\epsilon} H^s_{\mathrm{b}}(\mathbb{R}^n)\ \forall\ s \text{ and }\ \forall\ \epsilon > 0. \tag{23}$$

The following result is a better reason!

Theorem 1. *Let $P(D)$ be a homogeneous, elliptic constant coefficient differential operator of order m on $\mathbb{R}^n$, then*

$$P : \langle x \rangle^{-t+m} H^{s+m}_{\mathrm{b}}(\mathbb{R}^n) \longrightarrow \langle x \rangle^{-t} H^s_{\mathrm{b}}(\mathbb{R}^n) \tag{24}$$

is Fredholm when $t \neq m - k$, $k = 0, 1, \ldots,$ and $t \neq n + j$, $j = 0, 1, \ldots.$

Notice that these two ranges of integers overlap if $m \geq n$ but there is a 'gap' if $m < n$ in the range $t \in (m, n)$ in which it turns out that P is an isomorphism. The null space is non-trivial only in the ranges $(m - j, m - j - 1)$ for $j = 0, 1, \ldots$ where it consists precisely of the polynomials in the null space of $P(D)$ of degree at most j – for $t > m$ P in (24) is injective. Similarly for $t \in (n + j, n + j + 1)$, $j = 0, 1, \ldots,$ the range of P has complement which is the null space of the corresponding space of polynomial solutions for the adjoint. So in principle one also has a formula for the index – which decreases as one crosses each of the integral thresholds.