18.155 LECTURE 23: 3 DECEMBER, 2013

Last time week I talked about the Laplace-Beltrami operator on the boundary of a smoothly bounded domain and in particular proved elliptic regularity by using convolution in local coordinate patches and regularity estimates on the commutator. This was by way of a short detour before doing the same for the Dirichlet problem itself.

Recall the set up:- We have a smoothly bounded domain B and we defined

$$\dot{H}^1(B) = \{u \in H^1(\mathbb{R}^n); \operatorname{supp}(u) \subset B\} \tag{1}$$

and then showed, using Poincaré's inequality, that

$$\langle u, v\rangle = \sum_{j=1}^{n} \int_B D_j u \overline{D_j v} dx \tag{2}$$

is a Hilbert inner product on $\dot{H}^1(B)$ with norm equivalent to the usual H^1 norm.

The dual of $\dot{H}^1(B)$ with respect to the real L^2 pairing we showed to be $H^{-1}(B) = H^{-1}(\mathbb{R}^n)\big|_{B\setminus\partial B}$ the subspace of $\mathcal{C}^{-\infty}(B \setminus \partial B)$ given by the restriction of elements of $H^{-1}(\mathbb{R}^n)$. Then the Riesz' Representation Theorem gives us a linear (because we use the real pairing) isomorphism

$$A : H^{-1}(B) \longrightarrow \dot{H}^1(B) \tag{3}$$

which is a distributional right inverse of the Laplacian:

$$\Delta A f = f \text{ in } B \setminus \partial B \tag{4}$$

and which solves the Dirichlet problem in a weak sense.

We showed that A restricted to $L^2(B)$ is a self-adjoint compact operator with a complete orthonormal basis $e_j \in \dot{H}^1(B)$ (orthonormalized in $L^2(B)$) satisfying

$$\Delta e_j = \lambda_j e_j,\ \lambda_j > 0,\ \lambda_j \to \infty,\ e_j\big|_{\partial B} = 0 \tag{5}$$

where the weak form of the Dirichlet condition corresponds to the fact that there is a restriction map $H^1(\mathbb{R}^n) \longrightarrow H^{\frac{1}{2}}(\partial B)$ (by working in local flattenings of the boundary) which vanishes for on $\dot{H}^1(B)$.

Now, what we wanted to prove is that the solution to the Dirichlet problem is regular if the data is regular, the first step of which is to show

$$u \in \dot{H}^1(B),\ \Delta u \in L^2(B) \Longrightarrow u \in H^2(B) \tag{6}$$

which is also shows that Δ with domain $D = H^2(B) \cap \dot{H}^1(B)$ is an unbounded self-adjoint operator on $L^2(B)$.

The idea is the same as for the Laplacian on the boundary – differentiate the equation to get an equation satisfied by the derivative of the solution. As there we first have to smooth the solution, so that it stays in the range of the isomorphism A. Here we have an additional problem, namely the boundary. We cannot use convolution directly since this would not preserve the boundary, and hence not the boundary condition. In brief the idea is to straighten out the boundary and use convolution in the 'tangential variables' to get a partially smoothed solution which we can then differentiate in these tangential directions. We have to do a little more after that.

Proposition 1. *If $u \in \dot{H}^1(B)$ and $\Delta u\big|_{B\setminus\partial B} \in L^2(B)$ then $u \in H^2(B)$.*

Proof. Since Δ is a constant coefficient elliptic operator we do have elliptic regularity which shows that

$$\Delta u \in L^2(B \setminus \partial B) \Longrightarrow u \in H^2_{\mathrm{loc}}(B \setminus \partial B). \tag{7}$$

So it is only regularity 'up to' the boundary that we need to show.

As before, we can localize u. If we multiply by any $\rho \in \mathcal{C}^\infty(B)$ then $\rho u \in \dot{H}^1(B)$ and

$$\Delta(\rho u) = \rho\Delta u + [\Delta, \rho]u \in L^2(B) \tag{8}$$

since the commutator is a differential operator of first order.

So, take ρ to have support in the ball $B(p, \epsilon)$ around a point $p \in \partial B$ on which we have a diffeomorphism $F : B(p, \epsilon) \longrightarrow \Omega$ to an open neighbourhood of $F(p) = 0 \in \mathbb{R}^n$ which flattens ∂B to $\{x_n = 0\} \cap \Omega$. If the inverse of F is G then $v = G^*(\rho u) \in H^1(\mathbb{R}^n)$ has compact support in $\{x_n \geq 0\} \cap \Omega$ (by coordinate-invariance of Sobolev spaces) and satisfies

$$P_F v = g \in L^2 \text{ on } \Omega \cap \{x_n > 0\}. \tag{9}$$

Here P_F is the local coordinate form of the Laplacian – we know it is a second-order elliptic differential operator with smooth, but variable, coefficients but not much more. It follows that it can be written

$$P_F = a(x)Q_F,\ Q_F = D^2_{x_n} + L(x, D')D_n + R(x, D') \tag{10}$$

where L is a first order and R is a second order differential operator in the tangential variables (but with coefficients depending on x_n as well.) The fact that the smooth coefficient $a(x) \neq 0$ (so we can divide by it) follows from ellipticity, since it is $p_2(x, (0, \dots, 0, 1)$ in terms of the principal symbol p_2 of P_F.

So now, $Q_f v \in L^2(B)$ as well, since v has compact support in Ω. To proceed further we need to 'smooth' v. We cannot do this directly in all variables since the equation only holds in $x_n > 0$. So we apply a usual approximate identity but just in the x' variables and set

$$w_\epsilon = \phi_\epsilon *' v(\cdot, x_n),\ \phi_\epsilon(x') = \epsilon^{-n+1}\phi(x'/\epsilon),\ \phi \geq 0,\ \int \phi = 1. \tag{11}$$

The result is a smooth function (for ϵ sufficiently small) of x' with values in $H^1(\{x_n \geq 0\}$ in the remaining variable.

We proceed to investigate $w_j(\epsilon) = D_{x_j} w_\epsilon$, where $1 \leq j < n$ corresponds to one of the tangential variables and proceed to show that

$$P_F w_j(\epsilon) \text{ is bounded in } H^{-1}(B) \text{ as } \epsilon \to 0. \tag{12}$$

This is the same statement as the boundedness of $Q_F w_j(\epsilon)$ which we can write out as

$$Q_F w_j(\epsilon) = [Q_F, D_j]w_\epsilon + D_j[Q_F, \phi_\epsilon *']v + \phi_\epsilon *' D_j Q_F v(\cdot, x_n). \tag{13}$$

Here the first term on the right is bounded in $H^{-1}(B)$ since $[Q_F, D_j]$ is a second-order differential operator and w_ω is bounded in $\dot{H}^1(\{x_n \geq 0\})$. The reason for removing the factor a in (10) is so that term $D^2_{x_n}$ in Q_F commutes with convolution in the x' variables and so the commutator on the second term can be written

$$D_j[Q_F, \phi_\epsilon *']v = D_j[L(x, D'), \phi_\epsilon *']D_{x_n} v + D_j[R(x, D', \phi_\epsilon *']v. \tag{14}$$

The results we proved about such commutators last week show that the commutator in the first term is uniformly bounded on L^2 (in all variables) and the second term uniformly 'loses one (tangential) Sobolev derivative'. Since v is in H^1 it follows that both terms are uniformly bounded in $H^{-1}(\{x_n \ge 0\})$. The last term in (13) is also bounded in H^{-1} since $Q_F v$ is by assumption in L^2.

Now, multiplying by a again and moving back to B we conclude from the invertibility of the Dirichlet problem in the weak sense, from $\dot{H}^1(B)$ to $H^{-1}(B)$ that $w_j(\epsilon)$ is bounded in $\dot{H}^1(\{x_n \ge 0\})$ and hence that the weak limit $D_{x_j} v \in H^1$ for each $1 \le j < n$. So we have tangential regularity – all the second order derivatives $D_j D_k v \in L^2$ except if *both* $j = k = n$. However, we are saved by the equation itself which shows that

$$D^2_{x_n} v = -L(x, D')D_n v - R(x, D')v + Q_F v \in L^2 \text{ in } x_n \ge 0 \tag{15}$$

so in fact $v = \rho u \in H^2(B)$ for any cut-off supported in a coordinate patch and hence, using a partition of unity, $u \in H^2(B)$ as claimed. $\square$

So the difference here is that we can only directly recover tangential regularity. The argument above can be iterated directly. If we assume that $\Delta u \in H^k(B)$ then proceeding by induction, we can assume that in the local coordinate patches

$$D^\alpha_{x'} G^*_j(\rho_j u),\ H^1 \text{ for all } |\alpha| \le k. \tag{16}$$

Going through the same commutation arguments as above, applying another tangential derivative, we conclude that (16) holds for $|\alpha| \le k+1$ so gaining tangential derivatives. To gain the normal deriviatives just write out the same equation (15) and differentiate with respect to x_n up to k times. It follows that

$$u \in \dot{H}^1(B),\ \Delta u \in H^k(B) \Longrightarrow u \in H^{k+1}(B),\ k \in \mathbb{N}. \tag{17}$$

From this we deduce that the eigenfunctions (which we obtained from the compact operator A which solves the weak form of the Dirichlet problem) are in fact smooth. There is a lot more known about all this of course but it is still an active area of research. One thing people are currently working on is the structure of the zero sets of the Dirichlet eigenfunctions (asymptotically as the eigenvalue tends to ∞.)