18.155 LECTURE 20: 19 NOVEMBER, 2013

These note were reconstructed some weeks after the lecture.

At this point I had decided to detour before proving boundary regularity for the Dirichlet problem by examining the Laplacian, either for the induced or some other metric, on the boundary $M = \partial B$ of a smoothly bounded domain. This is a very similar, but slightly simpler question. On the other hand setting up the (global) definition of the Laplacian and distributional spaces on M on which it acts takes some time. Perhaps this was not such a good idea!

Let me start by going through the 'trivial' differential geometry of Euclidean space, $\mathbb{R}^n$, as preparation for doing the same on M – which is an example of a compact (oriented) manifold.

The basic object we have studied on $\mathbb{R}^n$ is the space $\mathcal{C}^\infty(\mathbb{R}^n)$ of smooth, for the moment real-evalued, functions. The tangent space $T_p\mathbb{R}^n$ of $\mathbb{R}^n$ at p is then defined to be the space of *derivations* at p, meaning

$$
\begin{aligned}
(1)\quad T_p\mathbb{R}^n = \{\delta : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R};\ &\text{linear and satisfying} \\
&\delta(fg) = f(p)\delta(g) + g(p)\delta(f)\ \forall\ f, g \in \mathcal{C}^\infty(\mathbb{R}^n)\}.
\end{aligned}
$$

Clearly $T_p\mathbb{R}^n$ is a linear space over $\mathbb{R}$.

The obvious examples of such maps are given by the partial derivatives at p :

$$
\partial_i : \mathcal{C}^\infty(\mathbb{R}^n) \ni f \longmapsto \frac{\partial f}{\partial x_i}(p) \in \mathbb{R}. \tag{2}
$$

By the global form of Taylor's theorem

$$
f(x) = f(p) + \sum_{i=1}^n \partial_i f(p)(x_i - p_i) + \sum_{i,j=1}^n (x_i - p_i)(x_f - p_j) f_{ij} \tag{3}
$$

with $f_{ij} \in \mathcal{C}^\infty(\mathbb{R}^n)$. From the distribution law in (1) $\delta c = 0$ for any derivation and constant function. Similarly, any derivation at p must annihilate each term in the second sum in (3) so for any $\delta \in T_p\mathbb{R}^n$,

$$
\delta f = \sum_{i=1}^n \partial_i f(p)\delta(x_i - p_i) = \sum_{i=1}^n c_i \partial_i f(p),\ c_i = \delta(x_i - p_i). \tag{4}
$$

Thus in fact, $T_p\mathbb{R}^n$ has dimension n with the ∂_i being a 'canonical' basis.

So, one can just identify $T_p\mathbb{R}^n = \mathbb{R}^n$ using this natural basis. The reason for *not* doing so is the behaviour under diffeomorphism.

Exercise 1. If we replace $\mathcal{C}^\infty(\mathbb{R}^n)$ by $\mathcal{C}^\infty(O)$ where $p \in O \subset \mathbb{R}^n$ is open, then the definition (1) makes sense and so defines T_pO. Show that under the restriction map (not surjective) $\mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(O)$ derivations on $\mathcal{C}^\infty(O)$ define derivations on $\mathcal{C}^\infty(\mathbb{R}^n)$ and the resulting map is in fact a linear isomorphism $T_p\mathbb{R}^n \longrightarrow T_pO$.

In view of this we really do identify T_pO and T_pO' for any two open sets containing p; this is not like the coordinate identification with $\mathbb{R}^n$.

So suppose F is a smooth map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$, or in view of Exercise 1, just defined on some open set containing p. Then there is a natural map

$$F_* T_p\mathbb{R}^n \longrightarrow T_{F(p)}\mathbb{R}^n, \qquad F_*(\delta)g = \delta(g \circ F),\ g \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{5}$$

Exercise 2. Using Taylor series as above, show that if $F(x) = (F_1(x), \dots F_n(x))$ then

$$F_*(\partial_i(p)) = \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j}(p)\partial_j(F(p)). \tag{6}$$

In particular the linear map in (5) is an isomorphism if and only if the Jacobian determinant at p, $\det(\frac{\partial F_i}{\partial x_j}(p)) \neq 0$ and conversely, if this is true, then by the Implicit Function Theorem, $F : B(p,\epsilon) = O_1 \longrightarrow O_2 = F(O_1)$ is a diffeomorphism from a small ball around p to its image.

Dually there is a similar construction of the cotangent space at $p \in \mathbb{R}^n$. From an algebraic point of view, p can be identified with the prime ideal $\mathcal{I}_p \subset \mathcal{C}^\infty(\mathbb{R}^n)$ of functions which vanish at p.

Exercise 3. Show that the only prime ideals (ideals without any proper subideals) $\mathcal{I} \subset \mathcal{C}^\infty(O)$ for $O \subset \mathbb{R}^n$ open are the $\mathcal{I}_p$ for $p \in O$.

The 'square' $\mathcal{I}^2$ of an ideal is the finite linear span of the products of elements of $\mathcal{I}$. Then we may *define*

$$T^*_p\mathbb{R}^n = \mathcal{I}_p/\mathcal{I}^2_p. \tag{7}$$

Here the $*$ does not in principal mean dual, however what the notation suggests is of course true:-

Exercise 4. The bilinear map

$$\mathcal{I}_p \times T_p\mathbb{R}^n \ni (f, \delta) \longmapsto \delta(f) \in \mathbb{R} \tag{8}$$

descends to a non-degenerate pairing which identifies $T^*_p\mathbb{R}^n$ with the dual of $T_p\mathbb{R}^n$.

You should also check that if $F : O \longrightarrow \mathbb{R}^n$ is a smooth map then $F^* : T_F(p)^*\mathbb{R}^n \longrightarrow T^*_p bbR^n$ is well defined from $\mathcal{I}_{F(p)} \ni g \longrightarrow g \circ F = F^*g \in \mathcal{I}_p$ and that under the duality identification F^* is the dual map to F_*.

Of course you know all this, but it is worth going through it carefully at least once!

Why did I go through all this nonsense? The main point is that it extends directly to $M = \partial B$ for a smoothly bounded domain B, and indeed to a general manifold. First we need to define the smooth functions on M. The obvious definition is

$$\mathcal{C}^\infty(M) = \mathcal{C}^\infty(\mathbb{R}^n)\big|_M, \tag{9}$$

so a function on M is smooth if and only if it can be extended to a smooth function on $\mathbb{R}^n$. Recall that we have defined 'adapted coordinates' near a point $p \in M$ in terms of a diffeomorphism which strainghtens M to $y_n = 0$.

Lemma 1. *A function $f : M \longrightarrow \mathbb{C}$ (or $\mathbb{R}$) is in $\mathcal{C}^\infty(M)$ if and only if for each $p \in M$ and 'adapted coordinate system' $F : B(p,\epsilon) \longrightarrow F(B(p,\epsilon)) = \Omega \subset \mathbb{R}^n$ (mapping p to 0 and $B(p,\epsilon) \cap B$ to $\{y_n > 0\}$) with inverse G, $G \circ f \in \mathcal{C}^\infty(\{x_n = 0\} \cap$. The smoothness of f follows from this regularity for and collection of such map F_p for which the $B(p, \epsilon_p)$ cover M.*

Proof. If $f \in \mathcal{C}^\infty(M)$ and $\tilde f \in \mathcal{C}^\infty(\mathbb{R}^n)$ is such that $f = \tilde F\big|_M$ then the fact that F is a diffeomorhism implies that $\tilde g = G^*\tilde f = \tilde f \circ G$ is smooth on Ω. Since $G \circ f = \tilde g\big|_{y_n=0}$ the smoothness of $G \circ f$ follows.

Conversely, if this smoothness holds for f for a covering of M by such 'adapted coordinates' F_j then we can choose a finite partition of unity $\rho_j \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ with ρ_j supported in the ball $B(p_j, \epsilon_j)$ and $\sum\limits_j \rho_j = 1$ in a neighbourhood of $M = \partial B$. If $\rho'_j = G_j^*\rho_j$ it follows from the hypothesis that $G_j^* f \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega_j \cap \{y_n = 0\})$ which can then be extended to $g'_j \in \mathcal{C}^\infty(\Omega_j)$ and hence $\rho'_j g'_j \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega_j)$ so $\tilde f = \sum\limits_j F_j^*(\rho'_j g'_j) \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$. From the properties of the partition of unity $\tilde f\big|_M = f \in \mathcal{C}^\infty(M)$. □

This is the general pattern with spaces of functions, and indeed of distributions, on M we can define them either 'extrinsically' by some sort of restriction process from $\mathbb{R}^n$ or 'intrinsically' in terms of the adapted coordinate maps F. Generally speaking the latter approach is to be preferred since this extends directly to the case of an arbitrary compact manifold.

Now we go back to basics. The tangent space to M at $p \in M$ is defined by

$$T_pM = \{\delta : \mathcal{C}^\infty(M) \longrightarrow \mathbb{R};\ \text{derivations at } p\}. \tag{10}$$

That is, $\delta \in T_pM$ is a linear map satisfying $\delta(fg) = f(p)\delta(g) + g(p)\delta(f)$ for all f, $g \in \mathcal{C}^\infty(M)$. Since we have defined $\mathcal{C}^\infty(M)$ by restriction, these actually define derivations on $\mathcal{C}^\infty(\mathbb{R}^n)$,

$$T_pM \longrightarrow T_p\mathbb{R}^n,\ \tilde\delta \tilde f = \delta(\tilde f\big|M). \tag{11}$$

Lemma 2. *The extension map* (11) *identifies T_pM with the subspace of $T_p\mathbb{R}^n$ consisting of the derivations at p 'tangent to M', namely this satisfying $\tilde\delta(f) = 0$ if $f\big|_M = 0$.*

Proof. Recall that local, and indeed global, defining functions exist for B and hence $M = \partial B$. Using Taylor's theorem it follows that any $h \in \mathcal{C}^\infty(\mathbb{R}^n)$ which vanishes on M is of the form $h'\Psi$ where $\Psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ is a global defining function and $h' \in \mathcal{C}^\infty(\mathbb{R}^n)$ is a (determined) element of $\mathcal{C}^\infty(\mathbb{R}^n)$. This can be visualized as a short exact sequence

$$\mathcal{C}^\infty(\mathbb{R}^n) \cdot \Psi \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(M) \tag{12}$$

from which the result follows (if necessary using the natural basis of $T_p\mathbb{R}^n$. □

The cotangent space T_p^*M may be defined in terms of the ideal $\mathcal{I}_p \subset \mathcal{C}^\infty(M)$ and the arguments above show that there is a linear map $T_p^*\mathbb{R}^n \longrightarrow T_p^*M$ which is surjective with one-dimensional null space. Again the argument above showing that T_p^*M is naturally the dual space of T_pM carries over and this may be seen more explicitly as follows:

Exercise 5. Show that under an 'adapted coordinate' diffeomorphism T_qM for $q \in M \cap B(p, \epsilon)$ is mapped isomorphically to $T_{F(q)}\mathbb{R}^{n-1}$ which is spanned by the ∂_{y_1}, $\dots$, $\partial_{y_{n-1}}$.

Both on $\mathbb{R}^n$ and on M we can define the differential of a (smooth) function at a point. Namely, if $f \in \mathcal{C}^\infty(M)$ then $f - f(p) \in \mathcal{I}_p \subset \mathcal{C}^\infty(M)$ since it vanishes at p. The deRham differential of f is then the element this defines in the quotient

$$df(p) = [f - f(p)] \in T_p^*M = \mathcal{I}_p/\mathcal{I}_p^2. \tag{13}$$

This applies equally well on $\mathbb{R}^n$ where the linear functions x_i define a basis at each point

$$dx_i(p) = [x_i - p_i] \in T_p^*\mathbb{R}^n. \tag{14}$$

Exercise 6. Check that this is the dual basis to the basis ∂_i of $T_p\mathbb{R}^n$ and for any $g \in \mathcal{C}^\infty(\mathbb{R}^n)$,

$$dg(p) = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j}(p) dx_j. \tag{15}$$