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This was partly reconstructed later.

Smoothly bounded domains

(1) $B \subset \mathbb{R}^n$ is bounded
(2) $B = \overline{\operatorname{int}(B)}$ is the closure of its interior
(3) For each $p \in \partial B$ there exists $\psi \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R})$ such that $d\psi(p) \neq 0$ (so $\partial\psi/\partial x_j(p) \neq 0$ for some j) and for some $\epsilon > 0$,

$$B \cap B(p, \epsilon) = \{x \in B(p, \epsilon); f(x) \geq 0\}. \tag{1}$$

E.g. the closed ball $\{x; |x| \leq R\}$.

Lemma 1. *For each $p \in \partial B$ there exists $\delta > 0$ and a diffeomorphism $F_p : B(p, \delta) \longleftrightarrow \Omega_p$ onto an open neighbourhood of $0 \in \mathbb{R}^n$ such that*

$$F_p(B(p, \delta) = \Omega_p^+ = \Omega_p \cap \{y_n \geq 0\} \tag{2}$$

where the coordinates in the image are written $(y_1, \dots, y_n)$.

Proof. By definition there is a 'local defining function' ψ for B in an open ball around p. Choose one of the coordinates x_j such that $\partial\psi/\partial x_j \neq 0$ at p and consider the map

$$F_p : B(p, \epsilon) \ni x \longrightarrow (\tilde{x} - \tilde{x}(p), \psi) \tag{3}$$

where $\tilde{x} = (x_1, \dots, x_{j-1}, x_{j+1}, \dots x_n)$ are the coordinates other than the chosen x_j. Then the Jacobian matric of F_p is invertible at p and $F_p(p) = 0$ so by the implicit function theorem F_p is a diffeomorphism from a possibly smaller ball $B(p, \delta)$ to its image, Ω_p, an open neighbourhood of 0 in $\mathbb{R}^n$. By construction $F_p(B \cap B(p, \delta) \subset \{y_n \geq 0\}$ since $F_p^* y_n = y_n \circ F_p = \psi$. $\square$

This diffeomorphism which 'straightens out the boundary of B' (locally) is by no means unique. However if F_p and $\tilde{F}_q$ are two such diffeomorphism, where p and q might or might not be equal, then provided their domains intersect,

$$B(p, \delta) \cap B(q, \tilde{\delta}) \neq \emptyset \tag{4}$$

there is a well defined 'transition map'. Namely within the two image sets consider the image of the intersection of the balls:

$$\begin{aligned}\Omega_{pq} &= F_q(B(p,\delta)\cap B(q,\tilde{\delta}))\subset\Omega_q,\\ \Omega_{qp} &= F_p(B(p,\delta)\cap B(q,\tilde{\delta}))\subset\Omega_p.\end{aligned}\tag{5}$$

These are both open sets (not necessarily containing 0) with the composite map

$$F_pG_q:\Omega_{pq}\longrightarrow\Omega_{qp},\ G_q=F_q^{-1}\tag{6}$$

necessarily a diffeomorphism. Moreover this map has the additional property that

$$F_pG_q:\Omega^+_{pq}\longrightarrow\Omega^+_{qp},\ F_pG_q:\Omega'_{pq}\longrightarrow\Omega'_{qp}\tag{7}$$

where the first sets are the local images of B, the intersections with $y_n\ge 0$ and the second sets the images of ∂B, the intersections with $y_n=0$. As discussed further below, this means that the variables $y'=(y_1,\dots,y_{n-1})$ induced by any F_p are local coordinates on ∂B which tranform smoothly, under the transition diffeomorphism, under change to another map $\tilde{F}_q$.

Since ∂B (and of course B itself) is compact, we may cover it by a finite number of the balls $B(p_j,\frac{1}{2}\delta_j)$ where there is a diffeomorphism F_j defined on $B(p_j,\delta_j)$. Then if we take a fixed cutoff function $0\le\chi\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ with support in $|x|\le 1$ and which is strictly positive on $|x|<1$, the functions $\chi_j(x)=\chi(2(x-p)/\delta_j)\in\mathcal{C}^\infty_{\mathrm{c}}(B(p,\delta_j))$ are such that

$$\chi_M=\sum_j\chi_j(x)\text{ satisfies }\inf M\chi_M=m>0.\tag{8}$$

For each point $t\in\mathbb{R}^n\setminus\partial B$ with $|t|\le R$ where R is so large that $B\subset\{|x|<R\}$, we can a cover by balls $B(t,\delta)$ not meeting ∂B and pass to a finite collection such that the $B(t,\delta/2)$ cover the compact set

$$\{|x|\le R\}\setminus\{\chi_M>m/2\}.\tag{9}$$

Defining functions as the χ_j above, and summing separately over the points in the interior and exterior of B gives to non-negative functions

$$\chi_I,\ \chi_E+\exp(1/(|x|-R))\tag{10}$$

with supports inside and outside B such that

$$\sum_j\chi_j+\chi_I+\chi_E+\exp(1/(|x|-R))>0.\tag{11}$$

Dividing by this gives function forming a finite partition of unity

$$\eta_j,\eta_I,\eta_E\tag{12}$$

where the η_j give restricti to a partition of unity on M, η_I has compact support inside B and η_E has support disjoint from B.

This construction can be used for instance to prove

Lemma 2. *Any smoothly bounded domain B has a global defining function, $\Psi \in \mathcal{C}^\infty(\mathbb{R}^n)$, real-valued, such that*

$$\begin{aligned} \Psi(x) = 0 \Longrightarrow \partial\Psi/\partial x_j \neq 0 \text{ for some } j = j(x), \\ \text{and } B = \{\Psi \geq 0\}. \end{aligned} \tag{13}$$

Proof. Take local defining functions ψ_j which are defined in a neighbourhood of the support of each of the η_j (since they are supported in $B(p_j, \delta Ij/2)$) and set

$$\Psi = \sum_j \psi_j \eta_j + \eta_I - \eta_E. \tag{14}$$

This vanishes onlu on $M = \partial B$ and the discussion above of the transition maps shows that on any open set where both ψ_j and ψ_k are defined, each is a non-vanishing positive multiple of the other. It follows that for any j $G_j^*\Psi = a_j(y)y_n$ where G_j is the inverse of F_j with $a_j > 0$ and this is non-startionary on $y_n = 0$, so Ψ is non-stationay on M. □

To examine functions on B we start by considering the model space $\{x_n \geq 0\}$ – then subsequently do the same for functions on $M = \partial B$ which is an example of a compact manifold of dimension $n - 1$.

The Lemma of É. Borel

One of the basic results which shows how different 'smooth', i.e. $\mathcal{C}^\infty$, functions are from (real) analytic functions is that the Taylor series of such a function is completely unconstrained. That is, given an arbitrary sequence $c_j \in \mathbb{C}$ there is a function $u \in \mathcal{C}^\infty_c(\mathbb{R})$ such that

$$\frac{d^j u}{dx^j}(0) = c_j \ \forall\ j. \tag{15}$$

In fact there is a higher dimensional analogue of this.

Lemma 3. *Suppose $u_j \in \mathcal{C}^\infty(\mathbb{R}^{n-1})$ is any sequence which has* $\operatorname{supp} u_j \subset \{|x'| \leq 1\}$ *for all $j \in \mathbb{N} \cup \{0\}$ then there exists $u \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ such that*

$$\frac{\partial_n^j u}{\partial x_n^j}(x', 0) = u_j(x') \ \forall\ x' \in \mathbb{R}^{n-1} \text{ and } j \in \mathbb{N} \cup \{0\}. \tag{16}$$

The assumption on supports is just for convenience, so that we can ensure that u has compact support.

Proof. Even though there are no bounds on the u_j at all, the basic idea is still to use the Taylor series expansion that u should have around $x_n = 0$ but to cut it off in an appropriate way. So, choose a fixed cut-off function $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ where we want only that $\phi(x_n) = 0$ in $|x_n| > 1$ and $\phi(x_n) = 1$ in $|x_n| < \frac{1}{2}$. Now, for any sequence $\epsilon_j > 0$ where initially we only demand $\epsilon_j \to 0$ as $j \to \infty$ consider the formal sum

$$\sum_{j=0}^{\infty} \frac{x_n^j}{j!} u_j(x')\phi(x_n/\epsilon_j). \tag{17}$$

Observe that the jth term here is cut off to have support in $|x_n| \le \epsilon_j$, so in any region $|x_n| > \delta > 0$ only finitely many terms are non-zero (because $\epsilon_j \to 0$) so in fact the series converges to a function in $\mathcal{C}^\infty(\mathbb{R}^n \setminus \{x_n = 0\})$ – outside the hypersurface we are interested in.

So, the aim is to choose the ϵ_j (depending on the u_j) so that the series (17) actually converges in $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$, i.e. uniformly with all derivatives. To get convergence in $\mathcal{C}^k$ we need to consider derivatives up to order $k = |\alpha| + p$, $\alpha \in \mathbb{N}_0^{n-1}$. As discussed below we can consider the jth term in (16) and assume that $j \ge k$ – so the x_n derivatives do not 'exhaust' the monomial x_n^j and

$$\begin{multline} D^\alpha_{x'} D^p_n \frac{x_n^j}{j!} u_j(x')\phi(x_n/\epsilon_j) \\ = \sum_{0 \le q \le p} \frac{x_n^{j-p+q}}{(j-p+q)!} D^\alpha_{x'} u_j(x') \phi^{(q)}(x_n/\epsilon_j)\epsilon_j^{-q}. \end{multline} \tag{18}$$

For the moment we have fixed k so we are only considering a finite number of α's and $q \le p \le k$. Each term on the right side vanishes when $x_n \ge \epsilon_j$ so we can estimate the remaining powers using $x_n \le \epsilon_j$ and so estimate the derivative in (18) by

$$C(j,k)\epsilon_j^{j-p} \le C(j,k)\epsilon_j$$

absorbing the ϵ_j^{-q} and estimating the various derivatives and constants by constants depending only on k and j. Thus, for the given k we can ensure convergence in $\mathcal{C}^k$ by demanding that the ϵ_j satisfy

$$\epsilon_j \le 2^{-j}/(C(j,k)+1) \text{ for } j > k. \tag{19}$$

Note that convergence is a property of the tail – all the terms are in $\mathcal{C}^\infty$ anyway, so it is enough to estimate from some finite point onwards, which is why only $k < j$ was considered above.

The small point is that while (19) represents an infinite number of conditions on the ϵ_j for each k, it only represents a *finite* number of conditions on each ϵ_j for all k! So *all* the convergence conditions are (eventually) satisfies by demanding

$$\epsilon_j < \min_{k<j} 2^{-j}/(C(j,k)+1) \ \forall\ j \tag{20}$$

and so there is a indeed a sequence $\epsilon_j > 0$ such that (17) converges in $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$.

If u is the sum of this series then it satisfies (16). Indeed, all the terms apart from the jth have derivative D^j_n vanishing at 0 so, by the convergence in $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ the limit has $D^j_n u(x',0) = u_j(x')$ for all j. □

Extension of smooth functions

Now, we can use this to prove another natural result. How should we define $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{n-1}\times[0,\infty))$ – the space of smooth functions of compact support in an closed half-space. There are two pretty natural definitions. The one by extension and the other more 'intrinsic'. Let me adopt the first and just define

$$\begin{aligned}\mathcal{C}^\infty_{\mathrm{c}}&(\mathbb{R}^{n-1}\times[0,\infty))\\ &= \{u : \mathbb{R}^{n-1}\times[0,\infty) \longrightarrow \mathbb{C}; \exists\ \tilde u \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n),\ u = \tilde u\big|_{\mathbb{R}^{n-1}\times[0,\infty)}\}.\end{aligned} \tag{21}$$

We will also want to put a topology on this space. Observe that if we think of the restriction map to the closed half space as a linear map R then its null space consists of the functions which vanishing identically in $x_n \ge 0$. Since we are talking about smooth functions here, this is the same as having support in $\{x_n \le 0\}$. So in fact we have a short exact sequence

$$\{u \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n); \operatorname{supp} u \subset \{x_n \le 0\}\} \longrightarrow \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \xrightarrow{R} \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{n-1}\times[0,\infty)). \tag{22}$$

The space defined by supports is a closed subspace of $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ so as topology we can take the quotient topology and we will get a complete space.

Still, we look at the more intrinsic approach to regularity in a half-space. Namely consider

$$\begin{aligned}\{u \in \mathcal{C}^\infty(\mathbb{R}^{n-1}\times(0,\infty)); u(x',x_n) = 0 \text{ if } |x'| \ge R \text{ or } |x_n| \ge R,\\ D^\alpha_{x'} D^k_{x_n} u \text{ bounded for all } k,\alpha\}\end{aligned} \tag{23}$$

Lemma 4. *The space* (23) *is equal to* $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{n-1}\times[0,\infty))$ *in* (21).

Proof. Since all the derivatives are bounded it follows that they actually extend continuously down to $x_n = 0$, since they are all smooth in $x_n > 0$

and

$$D^\alpha_{x'} D^k_{x_n} u(x', x_n) = -\int_{x_n}^R D^\alpha_{x'} D^{k+1}_{x_n} u(x', x_n) dx_n \tag{24}$$

shows inductively that the limits exist. Thus we may define the sequence of limiting normal derivatives

$$u_j(x') = \lim_{x_n \to 0} \frac{\partial^j u}{\partial x_n^j}(x', x_n) \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{n-1}) \tag{25}$$

all with supports in $|x'| \le R$.

Since u is continuous on $x_n \ge 0$ it defines a distribution $u' \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n)$ by extension as zero to $x_n < 0$ and using a limiting argument from $\mathbb{R}^{n-1} \times (\delta, \infty)$

$$\begin{aligned} u'((-)^j \frac{\partial^j \phi}{\partial x_n^j}) = \int_{x_n \ge 0} -\frac{\partial^j u}{\partial x_n^j}(x', x_n)\phi(x', x_n) dx' dx_n \\ \sum_{0 \le k < j} -\int_{\mathbb{R}^{n-1}} u_k(x') \frac{\partial^{j-k-1} \phi}{\partial x_n^{j-k-1}}(x', 0) dx', \ \forall \ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n). \end{aligned} \tag{26}$$

In case all the u_j in (25) vanish, this shows that u' has continuous weak derivatives of all orders (apply the same formula to $D^\alpha_{x'} u$) and hence, by equality of weak and strong derivatives, is in $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$. In this case u is the restriction of u' which has support in $x_n \ge 0$.

Now, in the general case, where the u_j in (25) may not vanish, apply Borel's Lemma above to find $v \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ which has these as derivatives at $x_n = 0$. Then $w = u - v\big|_{x_n > 0}$ is in the space (23) and has vanishing derivatives in the sense of (25). Thus its 'zero extension' $w' \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and u itself is the restricton of $v + w'$ to $x_n > 0$, proving the result. $\square$

So the two notions of smooth function on the closed half-space – by extension or 'intrinsically' as in (23) are the same. This is a general phenomenon, reflecting the smoothness of the boundary. Before discussing the corresponding result for Sobolev spaces, let me note that there is a better result than the one proved here. It is not so hard either:-

Proposition 1 (Seeley Extension)**.** *There is a continuous linear extension map E from the space* (23) *to $\mathcal{C}^\infty_c(\mathbb{R}^n)$ with respect to the uniform norms on all derivatives.*

The point here is the linearity. The proof above constructs and extension for each element, but because it passes through Borel's Lemma it is not, and cannot be, linear. The method is like the proof of Borel's Lemma in a way, namely one can define E by 'reflection'

$$(Eu)(x', x_n) = \sum_{k=0}^{\infty} a_k u(x', -b_k x_n),\ x_n < 0. \tag{27}$$

The trick is to choose the a_k and $b_k > 0$ so that the series converges with all derivatives in $x_n \le 0$ and that the derivatives match up with those of u at $x_n = 0$. It is possible!

Supported and extendible distributions

Having discussed the space $\mathcal{C}^\infty_{\mathrm{c}}(D_+),\ D_+ = \mathbb{R}^{n-1} \times [0, \infty)$ – and hence also the corresponding space for any 'half-space', in particular $D_- = \mathbb{R}^{n-1} \times (-\infty, 0])$ by linear invariance let me now jump to the opposite extreme, of distributions associated to a close half-space. Let me introduce a notation for the space in (20):

$$\dot{\mathcal{C}}^\infty_c(D_+) = \{u \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n); \operatorname{supp}(u) \subset D_+\}. \tag{28}$$

This is a closed subspace of $\mathcal{C}^\infty_{\mathrm{c}}(D_+)$ and we can also define the corresponding spaces without restriction on supports

$$\dot{\mathcal{C}}^\infty(D_+) = \{v \in \mathcal{C}^\infty(\mathbb{R}^n); v = 0 \text{ in } x_n < 0\},\ \mathcal{C}^\infty(D_+) = \mathcal{C}^\infty(\mathbb{R}^n)\big|_{D_+} \tag{29}$$

and easily obtain similar results to the case of compact supports.

Now, we introduce four spaces of distributions associated to D_+ by duality, really two types but in each case with and without compact supports:

$$\begin{aligned}
\dot{\mathcal{C}}^{-\infty}(D_+) &= (\mathcal{C}^\infty_{\mathrm{c}}(D_+))',\\
\mathcal{C}^{-\infty}(D_+) &= (\dot{\mathcal{C}}^\infty_c(D_+))',\\
\dot{\mathcal{C}}^{-\infty}_c(D_+) &= (\mathcal{C}^\infty(D_+))',\\
\mathcal{C}^{-\infty}_{\mathrm{c}}(D_+) &= (\dot{\mathcal{C}}^\infty(D_+))'.
\end{aligned} \tag{30}$$

Note that by way of deliberate notation there are 'dots' on the left when there are none on the right and similarly there are 'c's (for compact support) on the left when there are none on the right.

We can easily identify the 'dotted' spaces:

$$\begin{aligned}
\dot{\mathcal{C}}^{-\infty}(D_+) &= \{u \in \mathcal{C}^{-\infty}(\mathbb{R}^n); \operatorname{supp}(u) \subset D_+\},\\
\dot{\mathcal{C}}^{-\infty}_c(D_+) &= \{u \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n); \operatorname{supp}(u) \subset D_+\}.
\end{aligned} \tag{31}$$

That is indeed what the 'dot' is supposed to mean, the subspace of distributions on the whole space with support in D_+, the (closed) half-space. These are the spaces of 'supported distributions on D_+ (with or without compact support)'. If you want to play along, you can do the same thing for $\mathcal{S}(\mathbb{R}^n)$.

So, how to prove (31). Of course these equalities really represent the existence of natural bijections which we can regard as identifications and ignore. Let me consider the second case, with compact supports. The definition in (30) is as the dual of the space of smooth functions on D_+, as discussed above (although there I emphasized the case of compact supports). This is a quotient

$$\mathcal{C}^\infty(D_+) = \mathcal{C}^\infty(\mathbb{R}^n)\big/\dot{\mathcal{C}}^\infty(D_-) \tag{32}$$

since we know (or defined depending) that any element can be extended to the whole space as a smooth function and different extensions differ by functions supported in D_-. Thus, we can simply extend $u \in \dot{\mathcal{C}}_c^{-\infty}(D_+)$ to a compactly supported distribution

$$\tilde{u}(\phi) = u(\phi\big|_{D_+}),\ \forall\ \phi \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{33}$$

If ϕ has compact support in $\mathbb{R}^{n-1} \times (-\infty, 0)$ then $\tilde{u}(\phi) = 0$, so indeed $\operatorname{supp}(\tilde{u}) \subset D_+$ and this provides a map from left to right in (31). This map is certainly injective, since if $\tilde{u} = 0$ then $u = 0$. Surjectivity is slightly less obvious so consider $v \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ which has support in D_+. Then it follows that $u(\phi) = 0$ if $\operatorname{supp}(\psi) \subset \mathbb{R}^{n-1} \times (-\infty, 0)$, by the definition of support. However, if $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ has support in $x_n \le 0$ then $\psi_k(x', x_n) = \phi(x', x_n - \frac{1}{k})$ has support in the open half space and converges to ϕ in $\mathcal{C}^\infty(\mathbb{R}^n)$. Thus by continuity $v(\phi) = 0$ and it follows that $v = \tilde{u}$ for $u \in \mathcal{C}_c^{-\infty}(D_+)$.

Integration by parts

References