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TRACE CLASS OPERATORS CONT

Last time I define and operator to be of 'trace class' if it is a finite sum of products of Hilbert-Schmidt operators

$$(1) \qquad T = \sum_{i=1}^{N} A_i B_i, \ A_i, \ B_i \in \mathrm{HS}(H).$$

from which it follows that

$$(2) \qquad \|T\|_{\mathrm{TC}} = \sup \sum_{j} |\langle T e_j, f_j \rangle| < \infty$$

where the supremum is over pairs of orthonormal bases. From this it follows that

$$(3) \quad \|T\| = \sup_{\|u\|=\|v\|=1} |\langle T u, v \rangle| \le \|T\|_{\mathrm{TC}} \text{ and}$$

$$\|BT\| \le \|B\| \|T\|, \ B \in \mathcal{B}(H), \ T \in \mathrm{TC}(H).$$

Conversely, it follow that if $\|T\|_{\mathrm{TC}} < \infty$ then T has polar decomposition

$$(4) \qquad T = VA, \ A = (T^*T)^{\frac{1}{2}}.$$

with $A^{\frac{1}{2}}$ Hilbert Schmidt, so $T \in \mathrm{TC}(H)$. If T_j is Cauchy with respect to $\|\cdot\|_{\mathrm{TC}}$ then it is Cauchy, and hence convergent, in $\mathcal{B}(H)$. The boundedness of the sequence in $\mathrm{TC}(H)$ implies that the limit is in $\mathrm{TC}(H)$ and convergence with respect to the trace class norm follows. Thus $\mathrm{TC}(H)$ is a Banach space with respect to the trace norm.

Furthermore the trace functional

$$(5) \qquad \mathrm{Tr}(T) = \sum_{i} \langle T e_i, e_i \rangle$$

is continuous on $\mathrm{TC}(H)$ and is independent of the orthonormal basis used to define it since if $T = AB$ is a product of Hilbert-Schmidt operators then

$$(6) \qquad \mathrm{Tr}(T) = \langle B, A^* \rangle.$$

Moreover, the fundamental trace identity

$$(7) \qquad \mathrm{Tr}([T, B]) = 0 \text{ if } T \in \mathrm{TC}(H), \ B \in \mathcal{B}(H)$$

holds,

To prove (7) it suffices to prove it for finite rank operators and then use continuity in the trace norm on T. Since we know that Hilbert-Schmidt operators are the closure of the finite rank operators with respect to the Hilbert-Schmidt nor we can just use the observation that

$$\|AB\|_{\text{TC}} \leq \|A\|_{\text{HS}}\|B\|_{\text{HS}}$$

to see that finite rank operatos are dense in the trace class operators with respect to the trace norm.

Since the identity is clearly linear in T it is enough to assume tht T has rank one and then use linearity. So consider a non-zero operator of rank one $Tu = \langle u, v \rangle w$. Its trace can be computed by taking an orthonormal basis with first element $\hat{v} = v/\|v\|$ then

$$\text{Tr}(T) = \langle \hat{v}, v \rangle \langle w, \hat{v} \rangle = \langle w, v \rangle$$

since all other terms in the sum (5) vanish. So

$$TBu = \langle u, B^*v \rangle w, \ BTu = \langle u, v \rangle Bw \Longrightarrow$$
$$\text{Tr}(TB) = \langle w, B^*v \rangle = \text{Tr}(BT) = \langle Bw, v \rangle.$$

Notice that (**??**) means that if $A \in \text{GL}(H)$ is invertible then

(8) $\qquad \text{Tr}(ATA^{-1}) = \text{Tr}(T)$, since $ATA^{-1} - T = [A, TA^{-1}]$.

This means that in a certain sense, which I am not making precise, (5) remains true if e_i is replaced by a 'non-orthogonal basis'. In fact there is a not-so-simple result in this direction

Theorem 1. *[Lidskii – without proof] If $T \in \text{TC}(H)$ and its non-zero eigenvalues λ_j are repeated in a sequence with* algebraic multiplicity *(meaning according to the dimension of the generalized eigenspace, see the discussion of the spectrum of a compact operator) say with $|\lambda_j|$ non-increasing then*

(9) $$\text{Tr}(T) = \sum_j \lambda_j.$$

I will not use Lidskii's theorem – see for instance [1] for a proof.

Now, back to the result about Fredholm operators. We defined the index of a Fredholm operator in terms of its generalized inverse G satisfying (**??**). Note that finite rank operators are of trace class and for a self-adjoint operator we know that the trace is the sum of the

eigenvalues. Thus

$$(10) \quad \operatorname{ind}(B) = \dim \operatorname{Nul}(B) - \dim \operatorname{Nul}(B^*)$$
$$= \operatorname{Tr}(\Pi_{\operatorname{Nul}(B)}) - \operatorname{Tr}(\Pi_{\operatorname{Nul}(B^*)} = \operatorname{Tr}(\operatorname{Id} - GB) - \operatorname{Tr}(\operatorname{Id} - BG).$$

Note the apparent conundrum here that the last two operators are of trace class, in view of (**??**) and hence we may write

$$(11) \qquad\qquad \operatorname{ind}(B) = \operatorname{Tr}([B, G])$$

so you might think the index always vanishes (which we know it doesn't). Of course the point here is that neither of the operators in the commutator is of trace class, so (7) does not apply.

The formula (11) generalizes as follows

Lemma 1 (Calderón?). *If B is Fredholm and $L \in \mathcal{B}(H)$ is any generalized inverse modulo trace class operators, so $LB - \operatorname{Id}$ and $BL - \operatorname{Id}$ are both of trace classe, then*

$$(12) \qquad\qquad \operatorname{ind}(B) = \operatorname{Tr}([B, L]).$$

Proof. Let G be the generalized inverse of B and consider the segment $L_s = (1 - s)G + sL$ in $\mathcal{B}(H)$. Computing both composites

$$L_S B = (1 - s)GB + sLB$$
$$= (1 - s)\operatorname{Id} - (1 - s)\Pi_{\operatorname{Nul}(B)} + s\operatorname{Id} + sT_1 = \operatorname{Id} + (sT_1 - (1 - s)\Pi_{\operatorname{Nul}(B)},$$
$$BL_S = (1 - s)BG + sBL$$
$$= (1 - s)\operatorname{Id} - (1 - s)\Pi_{\operatorname{Nul}(B^*)} + s\operatorname{Id} + sT_2 = \operatorname{Id} + (sT_2 - (1 - s)\Pi_{\operatorname{Nul}(B^*)}$$

shows that L_s consists of two-sided inverses modulo trace class operators. Now consider the commutator $[B, L_s]$ – this is a linear map into trace class operators so its trace is a linear function with derivative

$$(13) \qquad\qquad \frac{d}{ds} \operatorname{Tr}([B, L_s]) = \operatorname{Tr}([B, L - G]) = 0$$

Here we use the identity (**??**) which applied in the case of the trace ideal shows that the two inverses modulo trace class operators must differ by a trace class operator, then (7) does apply and we conclude that $\operatorname{Tr}([B, L_s])$ is constant in s and so (12) follows from (**??**) in the form of (11). ☐

We wish to show that the index is constant on each component of the Fredholm operators. Since this is a metric space, it is enough to show that it is locally constant near 0 on any (continuous) curve $B(t); (-1, 1) \longrightarrow \mathcal{F}(H)$. Start with the generalized inverse $G(0)$ of $B(0)$,

$$(14) \quad G(0)B(t) = \operatorname{Id} - \Pi_{\operatorname{Nul}(0)} + E_+(t), \ B(t)G(0) = \operatorname{Id} - \Pi_{\operatorname{Nul}^*(0)} + E_-(t)$$

where $E_\pm(t)$ are continous with small norm, $\|E_\pm(t)\| < 1$ in $|t| < \epsilon$.

Thus $\mathrm{Id} + E_+(t)$ is invertible and

(15)
$$(\mathrm{Id} + E_+(t))^{-1}G(0)B(t) = \mathrm{Id} - F_+(t), \ B(t)G(0)(\mathrm{Id} + E_-(t))^{-1} = \mathrm{Id} - F_-(t)$$

where $F_\pm(t)$ are continuous in t with values in $\mathrm{TC}(H)$. As above, the difference between these left and righ inverses modulo trace class operators is

(16)
$$D(t) = (\mathrm{Id} + E_+(t))^{-1}G(0) - G(0)(\mathrm{Id} + E_-(t))^{-1} = (\mathrm{Id} + E_+(t))^{-1}F_-(t) - F_+(t)(\mathrm{Id} + E_-(t))^{-1}$$

which is also continous as a map into $\mathrm{TC}(H)$. Thus in fact $G(t) = (\mathrm{Id} + E_+(t))^{-1}G(0)$ is a curve of 2-sided inverses modulo a continuous error with values in trace class. It follows from the discussion of the index above that

(17) $\mathrm{ind}(B(t)) = \mathrm{Tr}([B(t), G(t)])$ is continous near 0.

However, we know this takes values in the integers so it must be locally, and hence globally, constant.

Thus the index is constant on (path) components of $\mathcal{F}(H)$.

Conversely, we wish to show that if two elements of $\mathcal{F}(H)$ have the same index then they can be connected by a path in $\mathcal{F}(H)$. First observe that it is enough to treat the operators with $\mathrm{ind} = 0$ and show they are connected to Id.

If $\mathrm{ind}(B) < 0$ consider B^* which has index $k = -\mathrm{ind}(B) > 0$ and if we connect B^* to T^k, the k-fold downward shift, the B is connected to $S^k = (T^k)^*$, the k-fold upward shift. Similarly, if $k = \mathrm{ind}(B) > 0$ it suffices to connect $S^k B$ to Id since then $T^k S^k B = B$ is connected to T^k as desired.

Thus we may suppose that $\mathrm{ind}(B) = 0$. Then $\mathrm{Nul}(B)$ and $\mathrm{Nul}(B^*)$ have the same dimensions, so there is an isomorphism $E : \mathrm{Nul}(B) \longrightarrow \mathrm{Nul}(B^*)$. The operator $B + sE$ is invertible for small $s \neq 0$ since we know it is Fredholm, of index 0 and has no null space.

Finally then, it is enough to show that $\mathrm{GL}(H)$, the group of invertible operators on H is connected, since we have shown that every element of $\mathcal{F}(H)$ of index 0 can be connected to point in $\mathrm{GL}(H)$ withing $\mathcal{F}(H)$.

I will see who comes up with the best proof that $\mathrm{GL}(H)$ is connected from Problems8!

Note that although I have been talking about Fredholm operators on a fixed separable Hilbert space, this discussion immediately carries over to the Fredholm operators between two separable Hilbert spaces, both infinite-dimensional to avoid triviality. Namely invertible operators are Fredholm and products of Fredholm operators is Fredholm, so to say

$B \in \mathcal{F}(H_1, H_2)$ is just to say $UB \in \mathcal{F}(H_1)$ for $U : H_2 \longrightarrow H_1$ an isomorphism, since the conclusion does not depend on the choice of U.

Another thing I wanted to mention is the space if semi-Fredholm operators. These are the bounded operators $B \in \mathcal{B}(H)$ which have closed range and *either* $\mathrm{Nul}(B)$ *or* $\mathrm{Nul}(B^*)$ finite dimensional. So of course Fredholm operators are semi-Fredholm.

Lemma 2. *The semi-Fredholm operators form and open dense subset of $\mathcal{B}(H)$.*

Proof. If B is semi-Fredholm then
(18)
$$\dim \mathrm{Nul}(B) < \infty \Longrightarrow B : H \longrightarrow \mathrm{Ran}(B) \text{ is Fredholm and surjective}$$
$$\dim \mathrm{Nul}(B^*) < \infty \Longrightarrow B : \mathrm{Nul}(B)^\perp \longrightarrow H \text{ is Fredholm and injective.}$$

Now, consider $B + A$ where $A \in \mathcal{B}(H)$ has $\|A\| < \epsilon$ to be chosen. In the first case above, $B + \Pi_{\mathrm{Ran}(B)} A$ is Fredholm from H to $\mathrm{Ran}(B)$ provided $\|A\|$ is small enough. In particular the null space of $B + A$ is finite-dimensional and of the same dimension. Moreover the range of $B + A$ is of the form $w = (B + \Pi_{\mathrm{Ran}(B)} A)u + (\mathrm{Id} - \Pi_{\mathrm{Ran}(B)})u$ for $u \in H$. If G_A is the generalized inverse of $(B + \Pi_{\mathrm{Ran}(B)} A)$ then $u = G_A(B + \Pi_{\mathrm{Ran}(B)} A)u + u'$, $(B + \Pi_{\mathrm{Ran}(B)} A)u' = 0$ and hence

$$w = f + (Id - \Pi_{\mathrm{Ran}(B)} A)G_A f + (\mathrm{Id} - \Pi_{\mathrm{Ran}(B)} A)\mathrm{Nul}(B + \Pi_{\mathrm{Ran}(B)} A).$$

The last, finite-dimensional, part is closed as is the first part – since $f = \Pi_{\mathrm{Ran}(B)} w$. Thus in the first case in (18) the range remains closed under perturbation by a small bounded operator.

The second case is similar but perhaps simpler. Namely, $A = A\Pi_{\mathrm{Nul}(B)^\perp} + A(\mathrm{Id} - \Pi_{\mathrm{Nul}(B)^\perp})$. The first part perturbs the Fredholm operator which therefore remains Fredholm and injective, provided $\|A\|$ is small. Thus the range of the whole operator is closed and of finite codimension, since it must contain the range of the restriction to $\mathrm{Nul}(B)^\perp$.

To see that semi-Fredholm operators are dense, observe that any $B \in \mathcal{B}(H)$ projects to a bounded operator with dense range from $\mathrm{Nul}(B)^\perp$ to $\overline{\mathrm{Ran}(B)}$, so these two Hilbert spaces are either of the same finite dimension or both infinite dimensional. Then B can be perturbed to be invertible as an operator between these two spaces, by perturbing the self-adjoint part of its polar decomposition to be invertible on $\mathrm{Nul}(B)^\perp$. If one of $\mathrm{Nul}(B)$ or $\mathrm{Ran}(B)^\perp$ is finite dimensional, then this gives a semi-Fredholm perturbation. If not, both are infinite dimensional and hence there is an isomorphism between them. Adding a small multiple of this perturbs the original operator to be invertible.

Thus the semi-Fredholm operators are dense in the bounded operators. □

References

[1] J.R. Ringrose, *Compact non-self-adjoint operators*, Van Nostrand Reinhold Company, 1971.