# LECTURE 15, 18.155, 29 OCTOBER 2013

Fredholm and Trace class operators

## PARIAL ISOMETRIES

A unitary operator is an inner-product preserving bijection – possibly between two different Hilbert spaces. More generally a *partial isometry* is a bounded operator $V : H_1 \longrightarrow H_2$ which satisfies

$$(1) \qquad \|Vu\|_{H_2} = \|u\|_{H_1} \ \forall \ u \in \mathrm{Nul}(V)^\perp.$$

It follows directly that the range is closed and the resulting map $V' : \mathrm{Nul}(V)^\perp \longrightarrow \mathrm{Ran}(V) \subset H_2$, is a norm-preserving bijection and hence is a unitary isomorphism, since

$$(2) \quad \langle V'u, V'u \rangle = \langle u, u \rangle \Longrightarrow (V')^*V' = \mathrm{Id}_{\mathrm{Nul}(V)^\perp}, \ V'(V')^* = \mathrm{Id}_{\mathrm{Ran}(V)}.$$

One of the important examples of partial isometry is the shift operator. If $e_i$, $i \geq 0$, is an orthonormal basis of a separable Hilbert space then there are uniquely defined bounded operators $S$ and $T$ determined by

$$(3) \qquad Se_i = e_{i+1} \ \forall \ i \geq 0, \ Te_j = e_{j-1} \ \forall \ j \geq 1, \ Te_0 = 0.$$

## POLAR DECOMPOSITION

If $B \in \mathcal{B}(H)$ then we can think of $B$ as a $2 \times 2$ matrix between the two decompositions $H = (\mathrm{Nul}\,B)^\perp \oplus (\mathrm{Nul}\,B)$ with $\pi_N$ projection off the null space and $\overline{\mathrm{Ran}(B)} \oplus \mathrm{Nul}(B^*)$ with $\pi_R$ projection onto the closure of the range:

$$(4) \qquad B = \begin{pmatrix} \pi_R B \pi_N & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $B^*B$ is self-adjoint and can be considered as an operator on $\mathrm{Nul}(B)^\perp$. Since it is non-negative, its square-root $A = (B^*B)^{\frac{1}{2}} \in \mathcal{B}(H)$ is also well-defined as an operator on $H$ which maps $\mathrm{Nul}(B)^\perp$ to itself and is otherwise zero. Then we can define a linear map

$$V : \mathrm{Ran}(B) \longrightarrow \mathrm{Nul}(B)^\perp, \ V(Bu) = Au.$$

In fact $V$ extends by continuity to a bounded operator on the closure since

$$\|V(Bu)\|^2 = \|Au\|^2 = \langle Au, Au \rangle = \langle u, B^*Bu \rangle = \langle Bu, Bu \rangle = \|Bu\|^2.$$

So in fact $V : \overline{\operatorname{Ran}(B)} \longrightarrow \operatorname{Nul}(B)^{\perp}$ is an isometry. If we extend it as zero to $\operatorname{Ran}(B)^{\perp}$ and take $W = V^*$ we get the polar decomposition of $B$ :

(5)

$B = WA$, $A$ non-negative and $W$ a partial isometry with $\operatorname{Ran}(W) = \overline{\operatorname{Ran}(B)}$.

## FREDHOLM OPERATORS

An operator, $B \in \mathcal{B}(H)$, is said to be Fredholm if

(1) It has a finite dimensional null space $\operatorname{Nul}(B) \subset H$.
(2) It has closed range, $\overline{\operatorname{Ran}(B)} = \operatorname{Ran}(B)$
(3) It adjoint has a finite dimensional null space $\operatorname{Nul}(B^*)$.

Of course we know that $H = \overline{\operatorname{Ran}(B)} \oplus \operatorname{Nul}(B^*)$ so the second two conditions can be combined by saying that the range has a finite-dimensional complement. It is just that this is a little too easy to interpret as saying that the closure of the range has a finite dimensional complement, which is by no means enough to guarantee that the operator is Fredholm.

The operators of the form $\operatorname{Id} + K$, $K \in \mathcal{K}(B)$ compact are Fredholm. Indeed, the null space of $\operatorname{Id} + K$ consists of vectors $v \in H$ such that $v + Kv = 0$ which means that the unit ball in $\operatorname{Nul}(\operatorname{Id} + K)$ is mapped onto itself by $K$, and hence is precompact – so the null space is finite dimensional.

Since $K$ is compact there is a finite rank operator $F$ such that $\|F - K\| < \frac{1}{2}$. Then

$$(6) \quad (\operatorname{Id} + K) = \operatorname{Id} + (K - F) + F = (\operatorname{Id} + F(\operatorname{Id} + K - F)^{-1})(\operatorname{Id} + K - F).$$

So $\operatorname{Id} + K$ has the same range as $\operatorname{Id} + F'$ where $F' = F(\operatorname{Id} + K - F)^{-1}$ is finite rank. So the range contains the null space of $F'$ which is a closed subspace of finite codimension, so itself is closed and of finite codimension.

These examples proved a useful restatement of the Fredholm property.

**Lemma 1.** *An operator $B \in \mathcal{B}(H)$ is Fredholm if and only if it has a right and a left inverse modulo compact operators, i.e. there exist $L$ and $R \in \mathcal{B}(H)$ such that*

$$(7) \qquad LB = \operatorname{Id} + K_L, \;\; BR = \operatorname{Id} + K_R, \;\; K_L, \;\; K_R \in \mathcal{K}(H).$$

Note that as is the case for left and right inverses in a ring, in this case if $B$ has both a left and a right inverse modulo compacts (an ideal)

then either of them is both a left and a right inverse modulo that ideal. Namely, they differ by a compact operator:

$$(8) \quad L = L(BR) - LK_R = (LB)R - LK_R = R + K_L R - LK_R$$
$$\implies L - R \in \mathcal{K}(H).$$

It follows from this criterion that the adjoint of a Fredholm operator is Fredholm and that the product of two Fredholm operators is Fredholm. It also follows directly that if $A$, $B \in \mathcal{B}(H)$ and the product $AB \in \mathcal{F}(H)$ then both $A$ and $B \in \mathcal{F}(H)$. This is all saying that the property of being Fredholm is a sort of invertibility, namely invertibility modulo compact operators.

*Proof.* The existence of a right inverse modulo compacts shows that

$$(9) \quad \mathrm{Ran}(B) \supset \mathrm{Ran}(BR) = \mathrm{Ran}(\mathrm{Id} + K)$$
$$\text{is closed with finite-dimensional complement}$$

and hence so is $\mathrm{Ran}(B)$. Similarly, the existence of a left inverse modulo compact implies that

$$(10) \quad \mathrm{Nul}(B) \subset \mathrm{Nul}(LB) = \mathrm{Nul}(\mathrm{Id} + KL) \text{ is finite dimensional.}$$

Conversely, if $B$ is Fredholm then it defines a continuous bijection between the Hilbert spaces

$$(11) \quad B : (\mathrm{Nul}(B))^{\perp} \longrightarrow \mathrm{Ran}(B)$$

which therefore has a continuous inverse $G : \mathrm{Ran}(B) \longrightarrow (\mathrm{Nul}(B))^{\perp} \subset H$. Extending $G$ as zero to $(\mathrm{Ran}(B))^{\perp}$ therefore gives a bounded operator and

$$(12) \quad GB = \mathrm{Id} - \Pi_{\mathrm{Nul}(B)}, \ BG = \mathrm{Id} - \Pi_{\mathrm{Nul}(B^*)}.$$

This is the *generalized inverse* of $B$ and is certainly both a left and right inverse modulo compacts. It also follows that $B$ is Fredholm, since $B$ is its generalized inverse. $\qquad\square$

**Proposition 1.** *The Fredholm operators $\mathcal{F}(H) \subset \mathcal{B}(H)$ form an open set with components labelled by the integer*

$$(13) \quad \mathrm{ind}(B) = \dim \mathrm{Nul}(B) - \dim \mathrm{Nul}(B^*)$$

*and which is stable under the addition of compact operators,*

$$(14) \quad \mathcal{F}(H) + \mathcal{K}(H) = \mathcal{F}(H).$$

*Partial proof, not including index.* The fact that the Fredholm operators form an open set follows directly from Lemma 1 above. Namel, if $G$ is the generalized inverse of a Fredholm operator $B$ and $A \in \mathcal{B}(H)$ has $\|A\| < 1/\|G\|$ then

(15)
$$G(B + A) = \mathrm{Id} + GA - \Pi_{\mathrm{Nul}(B)} \Longrightarrow$$
$$((\mathrm{Id} + GA)^{-1}G)(B + A) = \mathrm{Id} - (\mathrm{Id} + GA)^{-1}\Pi_{\mathrm{Nul}(B)},$$
$$(B + A)G = \mathrm{Id} + AG - \Pi_{\mathrm{Nul}(B^*)} \Longrightarrow$$
$$(B + A)(G(\mathrm{Id} + AG)^{-1}) = \mathrm{Id} - \Pi_{\mathrm{Nul}(B^*)}(\mathrm{Id} + AG)^{-1}$$

gives left and right inverses for $B + A$ modulo compact (in fact finite rank) operators.

Similarly (14) holds since an inverse modulo compact operators for $B$ is also an inverse modulo compact operators for $B + K$ for any $K \in \mathcal{K}(H)$.                                                              □

It is certainly possible to go through the remainder of the proof, which is to show that two Fredholm operators $B_0$ and $B_1$ have the same index (13) if and only if they are in the same component of $\mathcal{F}(H)$, i.e. can be joined by a curve, a continuous map $B : [0,1] \longrightarrow \mathcal{F}(H)$, with $B(0) = B_0$ and $B(1) = B_1$. However, this is a bit painful without using the trace functional, so I will complete the proof of Proposition 1 after discussing the trace and trace-class operators.

## HILBERT-SCHMIDT OPERATORS

If $H$ is a separable Hilbert space then an operator $A \in \mathcal{B}(H)$ is *Hilbert-Schmidt* if for (any) one orthonormal basis $\{e_i\}$,

(16)
$$\|A\|_{\mathrm{HS}}^2 = \sum_i \|Ae_i\|^2 < \infty.$$

Using Bessel's identity and the fact that the (double) series are absolutely convergent, if $f_j$ is another orthonormal basis then

(17)
$$\sum_i \|Ae_i\|^2 < \infty = \sum_i \sum_j |\langle Ae_i, f_j \rangle|^2$$
$$= \sum_i \sum_j |\langle e_i, A^* f_j \rangle|^2 = \sum_j \|A^* f_j\|^2.$$

Applying this twice shows that the sum defining (16) is indeed independent of the orthonormal basis used and also that $A^*$ is Hilbert-Schmidt. It follows that the Hilbert-Schmidt operators form a 2-sided

$\star$-ideal HS($H$) since if $B \in \mathcal{B}(H)$, $\|BAe_i\| \leq \|B\|\|Ae_i\|$ so

(18) $$\|BA\|_{\mathrm{HS}} \leq \|B\|\|A\|_{\mathrm{HS}}, \ AB = (B^*A^*)^*.$$

In fact the Hilbert-Schmidt operators form a Hilbert space, with the inner product given in terms of any choice of orthonormal basis by

(19) $$\langle A_1, A_2 \rangle_{\mathrm{HS}} = \sum_i \langle A_1 e_i, A_2 e_i \rangle.$$

The completeness follows from the fact that the finite-rank operators are dense in HS($H$), since directly from (16) $A\Pi_N \to A$, in HS($H$) where $\Pi_N$ is projection onto the span of the first $N$ elements of the orthonormal basis $\{e_i\}$ and that

$$\|A\| \leq \|A\|_{\mathrm{HS}}.$$

Indeed, there is a sequence $u_i \in H$ with $\|u_i\| = 1$ such that $\|A\| = \lim_i \|Au_i\|$ and each $u_i$ can be extended to an orthonormal basis, so $\|Au_i\| \leq \|A\|_{\mathrm{HS}}$. Thus of $A_n$ is Cauchy in HS($H$) it is Cauchy, and hence convergent, in norm and the limit, from the boundedness of $\|A_n\|_{\mathrm{HS}}$ is in HS($H$) and the sequence converges to it in HS($H$). Of course it also follows from this, and the density of finite rank operators in HS($H$) that HS($H$) $\subset \mathcal{K}(H)$.

*Exercise* 1. Show that for the case of $H = L^2(\mathbb{R}^n)$ that HS($H$) may be identified with $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ with the function on the product giving the operator as an integral operatorL

(20) $$Au(x) = \int_{\mathbb{R}^n} A(x,y)u(y)dy.$$

### Trace class operators

Now the 'trace class' operators are by definition those compact operators which can be written in the form of a finite sum of products of Hilbert-Schmidt operators

(21) $$T = \sum_{i=1}^N A_i B_i, \ A_i, \ B_i \in \mathrm{HS}(H).$$

This is a more complicated definition since the presentation (21) is clearly not unique. On the other hand it follows that TC($H$) is a 2-sided $\star$-ideal contained in the compact operators.

Now, notice that using a presentation (21) it follows that for *any* two orthonormal bases $e_j$ and $f_j$,

(22) $$\sum_j |\langle Te_j, f_j \rangle| \leq \sum_j \sum_i |\langle B_i e_j, A_i^* f_j \rangle| \leq \sum_i \|B_i\|_{\mathrm{HS}} \|A_i\|_{\mathrm{HS}}.$$

Thus for any $T \in \mathrm{TC}(H)$,

$$(23) \qquad \|T\|_{\mathrm{TC}} = \sup \sum_j |\langle Te_j, f_j \rangle| < \infty.$$

Here the supremum is over all pairs of orthonormal bases. That this is indeed a norm follows directly – with the triangle inequality following from the fact that for any pair of orthonormal bases

$$(24) \qquad \sum_j |\langle (T_1 + T_2)e_j, f_j \rangle| \leq \|T_1\|_{\mathrm{TC}} + \|T_2\|_{\mathrm{TC}}.$$

Note that as for the Hilbert-Schmidt norm

$$(25) \qquad \|T\| = \sup_{\|u\|=\|v\|=1} |\langle Tu, v \rangle| \leq \|T\|_{\mathrm{TC}}.$$

Again by looking at $TP_N$ it follows that $\mathrm{TC}(H) \subset \mathcal{K}(H)$.

Now, in fact $\mathrm{TC}(H)$ is a Banach space with respect to the trace norm (23). To see this, apply the polar decomposition to some $T$ for which the norm (23) is finite:

$$(26) \qquad T = VA, \; A = (T^*T)^{\frac{1}{2}}.$$

Then let $f_j$ be an orthonormal basis containing the eigenvectors for $A$ (which is compact) and $e_j$ one that includes the $Ve_j$. It follows that if $\lambda_j \geq 0$ are the eigenvalues of $A$ repeated with multiplicity then

$$(27) \qquad \sum_j \lambda_j \leq \|T\|_{\mathrm{TC}} < \infty.$$

Then if we define $E = A^{\frac{1}{2}}$, with eigenvalues $\lambda_j^{\frac{1}{2}}$ and use an orthonormal basis including an eigenbasis, $E \in \mathrm{HS}(H)$ since (27) is just the Hilbert-Schmidt norm of $E$. Thus in fact

$$(28) \qquad T = (VE)E \in \mathrm{TC}(H).$$

Summing all this up we have shown most of:

**Proposition 2.** *The operators for which the Trace Class norm* (25) *is finite form the Banach space of Trace class operators, each of which can be written in the form* (21) *or* (28) *and satisfy* $\|BT\|_{\mathrm{TC}} \leq \|B\| \|T\|_{\mathrm{TC}}$; *the trace functional*

$$(29) \qquad \mathrm{Tr}(T) = \sum_i \langle Te_i, e_i \Longrightarrow$$

*is continuous on* $\mathrm{TC}(H)$, *independent of the orthonormal basis used to define it and satisfies*

$$(30) \qquad \mathrm{Tr}([T, B]) = 0 \; \textit{if } T \in \mathrm{TC}(H), \; B \in \mathcal{B}(H).$$