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Operators on Hilbert space, continued.

(1) For a compact operator the spectrum is of the form $D \cup \{0\}$ where $D \subset \mathbb{C} \setminus \{0\}$ is a discrete set (possibly empty) consisting of eigenvalues; $\{0\}$ may or may not be an eigenvalue. In particular there are quasi-nilpotent compact operators which have spectrum just consisting of $\{0\}$ but have no eigenvalues. The generalized eigenspace for $A \in \mathcal{K}(H)$ and $z \neq 0$,

$$\{u \in H; (A - z\operatorname{Id})^N u = 0 \text{ for some } N\} \text{ is finite-dimensional.} \tag{1}$$

Proof: This follows from the approximation by finite rank operators. It suffices to consider the case that H is separable, since the closure of the range and the orthocomplement of the null space are both separable. Given $\epsilon > 0$, choose a projection onto a finite dimensional space P such that $\|(\operatorname{Id} - P)A(\operatorname{Id} - P)\| < \epsilon$ and consider the decomposition of A into a '2×2 matrix'

$$A = \begin{pmatrix} PAP & PA(\operatorname{Id} - P) \\ (\operatorname{Id} - P)AP & (\operatorname{Id} - P)A(\operatorname{Id} - P) \end{pmatrix} \tag{2}$$

which acts on the 'vectors' in $PH \oplus (\operatorname{Id} - P)H$. If $|z| > \epsilon$ then $T(z) = ((\operatorname{Id} - P)A(\operatorname{Id} - P) - zP)^{-1}$ is a holomorphic function of z as a bounded operator on the Hilbert space $(\operatorname{Id} - P)H$. This allows one to write

$$\begin{aligned} &\begin{pmatrix} P & -PA(\operatorname{Id} - P)T(z) \\ 0 & T(z) \end{pmatrix} (A - z\operatorname{Id}) \\ &\qquad = \begin{pmatrix} PAP - zP - PA(\operatorname{Id} - P)T(z)(\operatorname{Id} - P)AP & 0 \\ M(z) & \operatorname{Id} - P \end{pmatrix} \end{aligned} \tag{3}$$

where $M(z)$ is holomorphic in $|z| > \epsilon$. The matrix on the left is invertible in $|z| > \epsilon$ and the matrix on the right is invertible, since it is lower semi-triangular, if $PAP - zP - PA(\operatorname{Id} - P)T(z)(\operatorname{Id} - P)AP$ is invertible. This is a finite dimensional matrix depending holomorphically on z. It is invertible if and only if its determinant (with respect to some chosen basis) is non-vanishing. The determinant is a holomorphic function in $|z| > \epsilon$ which is non-vanishing when $|z| > \|A\|$, since both terms on the left are invertible then. Thus it, and hence $A - z\operatorname{Id}$, is invertible outside the zeros of the determinant which form a discrete set in $|z| > \epsilon$. Since this works for each $\epsilon > 0$ the discreteness of the spectrum follows. You can readily check from (3) that near a point $\zeta \in \operatorname{spec}(A) \setminus \{0\}$ the inverse must look like

$$(A - z\,\mathrm{Id})^{-1} = \sum_{j=1}^{N} (z-\zeta)^{-j} B_j + R(z),\ B_N \neq 0, \tag{4}$$

where the matrices B_j are finite rank and $R(z)$ is holomorphic near $z = \zeta$.

In fact using the 'resolvent identity' $(A - z\,\mathrm{Id})(A - z\,\mathrm{Id})^{-1} = \mathrm{Id}$ it follows from (4) that

$$\begin{aligned}(A-\zeta)B_N &= 0 = B_N(A-\zeta),\\ (A-\zeta)B_j &= B_{j+1} = B_j(A-\zeta),\ 1 \le j < N,\\ (A-\zeta)R(\zeta) &= \mathrm{Id} + B_1 = R(\zeta)(A-\zeta).\end{aligned} \tag{5}$$

Thus the range of B_N consists of eigenvectors with eigenvalue ζ and the range of B_j for $1 \le j < N$ consists of generalized eigenvectors, satisfying $(A-\zeta)^{N-j+1}B_j = 0$. Indeed, the null spaces of the B_j increase with increasing j and their ranges decrease. The range of $A - \zeta$ has a finite dimensional complement, namely the range of B_1. We will come back to these properties later!

(2) For any $B \in \mathcal{B}(H)$ the *closure* of the range of B is the orthocomplement of $\operatorname{Nul}(B^*)$:

$$H = \overline{B(H)} \oplus \operatorname{Nul}(B^*).$$

Proof: Certainly if $v \in \operatorname{Nul}(B^*)$, i.e. $B^*v = 0$, then $\langle Bu, v\rangle = \langle u, B^*v\rangle = 0$ so $\operatorname{Nul}(B^*)$ is orthogonal to the range of B, and hence to its closure. Conversely, if $\langle Bu, v\rangle = 0$ for all $u \in H$, i.e. $v \perp BH$, then $B^*v = 0$ so (2) holds.

(3) If $A \in \mathcal{B}(H)$ is self-adjoint then $\operatorname{spec}(A) \subset [-\|A\|, \|A\|]$ is contained in the real axis and one of $\pm\|A\| \in \operatorname{spec}(A)$.

Proof: Certainly if $A^* = A$ and $z \in \mathbb{C} \setminus \mathbb{R}$ then $A - z$ is injective, since

$$\operatorname{Im}\langle A - zu, u\rangle = -\operatorname{Im} z\|u\|^2.$$

The adjoint is $A - \bar{z}$ so it follows that the range $(A - z)H$ is dense. If $f_j = (A - z)u_j$ is in the range then from (3)

$$|\operatorname{Im} z|\|u_j\|^2 \le \|f_j\|\|u_j\|.$$

Applying this inequality to $(A-z)(u_j-u_k)=f_j-f_k$ it follows that if $f_j\to f$ then $u_j\to u$ (using completeness) and hence the range is closed. Thus it follows that $\operatorname{spec}(A)\subset[-\|A\|,\|A\|]\subset\mathbb{R}$.

Now, to see that at least one of $\pm\|A\|$ is in the spectrum note that for a self-adjoint operator

$$\|A\|=\alpha,\ \alpha=\sup_{\|u\|=1}|\langle Au,u\rangle|. \tag{6}$$

Certainly $\alpha\le\|A\|$. It $u,\ v\in H$ then

$$\begin{aligned}4\langle Au,v\rangle=&\langle A(u+v),u+v\rangle-\langle A(u-v),u-v\rangle\\&+i\langle A(u+iv),u+iv\rangle-i\langle A(u-iv),u-iv\rangle.\end{aligned} \tag{7}$$

If $\langle Au,v\rangle=Re^{i\theta}$ with $R\ge 0$ then

$$|\langle Au,v\rangle|=R=\langle Au',v\rangle,\ u'=e^{-i\theta}u.$$

Thus it suffices to suppose that the left side in (7) is positive, in which case the second two terms on the right can be dropped, since they are not real, and

$$\begin{aligned}4|\langle Au,v\rangle|&=\langle A(u'+v),u'+v\rangle-\langle A(u'-v),u'-v\rangle\\&\le\|A\|(\|u'+v\|^2+\|u'-v\|^2)=2\|A\|(\|u\|^2+\|v\|^2)\end{aligned}$$

from which the opposite inequality follows and (6) is proved.

Thus there is a sequence u_n, which we can assume converges weakly, $u_n\rightharpoonup u$, with $\|u_n\|=1$ with one of $\langle Au_n,u_n\rangle\to\pm\|A\|$; suppose $\langle Au_n,u_n\rangle\to\|A\|$. Then

$$\|(A-\|A\|)u_n\|^2=\|Au_n\|^2-2\|A\|\langle Au_n,u_n\rangle+\|A\|^2.$$

The middle term converges to $-2\|A\|^2$ so the limit supremum of the right side is 0 and hence the left side converges to 0 from which it follows that $A-\|A\|$ is not invertible. A similar argument applies for the opposite sign, so indeed one of $A\pm\|A\|$ is not invertible so one of $\pm\|A\|$ is in $\operatorname{spec}(A)$.

(4) This is the key to the functional caculus for self-adjoint operators and hence the spectral theorem. We are still supposing that $A^*=A\in\mathcal{B}(H)$. Consider a polynomial $p(z)$ with real coefficients. Then we can define, unambigously

$$p(A)=\sum_{j=0}^{n}c_jA^j,\ p(z)=\sum_{j=0}^{n}c_jz^j$$

where n is the degree, so $c_n \neq 0$. Since the coefficients are real, $p(A) \in \mathcal{B}(H)$. What can we say about the spectrum of $p(A)$? For each $\lambda \in \mathbb{C}$ we can factorize the polynomial $p(z) - \lambda$:

$$p(A) - \lambda = c_n \prod_{j=1}^{n} (A - \lambda_j),\ p(z) - \lambda = c_n \prod_{j=1}^{n} (z - \lambda_j)$$

where the λ_j are the roots of $p(z) - \lambda$ (repeated with multiplicity of course). Since $\operatorname{spec}(A) \subset [-\|A\|, \|A\|]$ it follows that if no λ_j is in this interval then $p(A) - \lambda$ is invertible. That is

$$\operatorname{res}(p(A)) \supset \{\lambda; p(z) - \lambda \text{ does not vanish on } [-\|A\|, \|A\|\}.$$

Reversing this and using the fact that the spectrum is in the reals

$$\operatorname{spec}(p(A)) \subset p([-\|A\|, \|A\|)$$

– only points in the range of p on this interval can be in the spectrum.

Now, this actually gives an estimate on the norm of $p(A)$, since we know that the spectum containes one of $\pm\|p(A)\|$:

$$\|p(A)\| \leq \sup_{[-\|A\|, \|A\|]} |p|.$$

The RHS is the supremum norm in p as an element of $\mathcal{C}([-\|A\|, \|A\|])$ so this map from polynomials with real coefficients in $\mathcal{B}(H)$ is continous.

(5) Recall the Stone-Weierstrass Theorem, that polynomials restricted to a closed interval are dense in continous functions in the supremum norm. From this it follows that the map extends by continuiy to

$$\mathcal{C}([-\|A\|, \|A\|]) \ni f \longmapsto f(A) \in \mathcal{B}(H). \tag{8}$$

We can extend to complex-valued functions by separately defining the real and imaginary parts. Observe it follows by continuity from the explict case of polynomials that

$$f(A)g(A) = g(A)f(A) = fg(A), \tag{9}$$

So the map (8) is actually a map of algebras – the image consists of a commutative subalgebra of $\mathcal{B}(H)$ associated to the given self-adjoint operator.

(6) Going back to (7) we see that we can recover the whole operator A from its 'diagonal' part $\langle Au, u\rangle$. So it is reasonable to consider, for f real-valued, the function

$$H \times \mathcal{C}([-\|A\|, \|A\|]; \mathbb{R}) \ni (u, f) \longmapsto \langle f(A)u, u\rangle \in \mathbb{R}. \tag{10}$$

For each fixed u this is a continuous linear functional on continuous functions – which is to say it is given by a measure.

As long as I can quite Riesz' Representation theory for measures in this sense – which says that these really are given by integrals against a Borel measure (with some more properties) then if you fix a set $(-\infty, t]$ the map (10) can be extended to the characteristic function and defines a bounded, self-adjoint operator

$$P_t(A) \in \mathcal{B}(H),\ P_t(A)^* = P_t(A). \tag{11}$$

This is the spectral projection onto the interval and

$$\operatorname{spec}(AP_t(A)) \subset [-\|A\|, t]. \tag{12}$$

I will not go into detail about the properties of these spectral projections, but it is certainly important to understand. In particular one can interpret P_t, which is a function defined on intervals with values in projections, as a measure and then write the functional calculus as

$$f(A) = \int_{\mathbb{R}} f(t) dP_t \tag{13}$$

since continous functions are integrable in this sense.

(7) Of course you should think about the relationship between this form of the spectral theorem and the discussion above of the spectrum of compact operators. For a self-adjoint compact operator the spectrum outside 0 must consist of real points of course, and the generalized eigenspace (1) is the eigenspace – so $N = 1$. To see this just apply (3) to (4) for $z = \zeta + it$ where $\zeta \neq 0$ is an eigenvalue and $t > 0$ is small, this shows that all the B_j for $j > 1$ must vanish. Then $-B_1$ is just the orthogonal projection onto the null space. In this case the 'spectral measure' in (13) becomes a countable (possibly finite) sum involving a real sequence $\lambda_j \to 0$ and a sequence of mutually commuting finite rank projections P_j such that

$$f(A) = \sum_j f(\lambda_j) P_j,\ dP_t = \sum_{\lambda_j \leq t} \delta(t - \lambda_j) P_j. \tag{14}$$

The series for $f(A)$ here converges in norm.

(8) Although I will not discuss the full measure-theoretic spectral theorem, you should not think that there is anything misterious here. In general the spectral measure can behave rather badly – see the book(s) of Reed and Simon for instance. For us the two most important examples are the discrete case – basically

(14) – and in the opposite extreme, the 'continuous' (or even smooth) case.

Consider for instance $H = L^2([0,1])$ and a continuous real-valued function $g(t) \in \mathcal{C}([0,1])$. Multiplication by g defines a bounded self-adjoint operator and its spectral decomposition is easy to arrive at. Namely the spectrum occupies the interval $[\inf g, \sup g]$ and the spectral projection is

$$P_t u = \chi(\{g(t) \leq t\})u. \tag{15}$$

Namely it is just the function which is u where $g(t) \leq t$ and 0 elsewhere. In this case there are no eigenfunctions at all, although there are 'generalized eigenfunctions' – in this case $\delta(x - t)$ is a generalized eigenfunction.