LECTURE 13, 18.155, 22 OCTOBER 2013

Operators on Hilbert space.

In the next three lectures I will go through the basic results on operators on Hilbert space before returning to the behaviour of particular differential operators. I am assuming that everyone has some familiarity with the basic properties of Hilbert space itself, so I shall just describe these very quickly.

(1) Hermitian (sesquilinear) form on a vector space V over $\mathbb{C}$ is a map

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{C} \text{ s.t.}$$
$$\langle av_1 + bv_2, w \rangle = a\langle v_1, w\rangle + b\langle v_2, w\rangle \ \forall\ a, b \in \mathbb{C},\ v_1, v_2, w \in V \text{ and}$$
$$\langle v, w \rangle = \overline{\langle w, v \rangle}\ \forall\ v, w \in V.$$

Thus the form is linear in the first variable, and using the second 'Hermitian symmetry' identity, it is 'antilinear' in the second variable. We are interested in positive definite forms of this type, i.e. such that

$$\langle v, v \rangle \geq 0 \text{ and } \langle v, v \rangle = 0 \Longrightarrow v = 0. \tag{1}$$

From this Cauchy's inequality follows

$$\begin{gathered} \|v\| = |\langle v, v\rangle|^{\frac{1}{2}} \text{ is a norm and} \\ |\langle v, w\rangle| \leq \|v\|\|w\|\ \forall\ v, w \in V. \end{gathered} \tag{2}$$

Really the first part follows from the second (which you can prove by expanding out $0 \leq \langle v + sw, v + sw\rangle$.

So, a vector space with such a Hermitian inner product is a pre-Hilbert space and a Hilbert space if it is complete with respect to the norm.

(2) A norm comes from an inner product if and only if the parallelogram law holds

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2). \tag{3}$$

(3) As for any Banach space a linear map $A : H \longrightarrow N$ for any normed space is continuous if and only if it is bounded in the sense that

$$\|Au\| \leq \|A\|\|u\|,\ \|A\| = \inf_{\|u\|=1} \|Au\|_N < \infty.$$

This turns the linear space of such 'operators' into a normed space which is a Banach space if N is Banach.

(4) An orthonormal set – in which each element has norm 1 and district elements are orthogonal in the sense that $\langle v, w\rangle = 0$ – is complete if the only element of H orthogonal to all elements of the set is 0. A separable Hilbert space (having a countable dense subset) has a (n at most) countable orthonormal set, called an orthonomal basis (Gram-Schmidt).

(5) Bessel's inequality – if $\{e_i\}$ is a finite orthonormal set then

$$\sum_i |\langle u, e_i\rangle|^2 \le \|u\|^2 \;\forall\; u \in H. \tag{4}$$

(6) If H is a separable Hilbert space and $\{e_i\}$ is an orthonormal basis then

$$\begin{gathered} \|u\|^2 = \sum_i |\langle u, e_i\rangle|^2 \text{ and} \\ u = \sum_i \langle u, e_i\rangle e_i \text{ converges in } H \;\forall\; u \in H. \end{gathered} \tag{5}$$

This is the Fourier-Bessel series.

(7) If $S \subset H$ is a closed, non-empty set in a Hilbert space which is convex in the sense that if $u, v \in S$ then $\frac{1}{2}(u+v) \in S$ then there is a unique element $s \in S$ of minimal length. The proof uses the parallelogram law and this result is not true in a general Banach space.

(8) Riesz Representation: The dual space of a Hilbert space (the linear space of bounded linear functionals) is isometrically isomorphic to $\bar{H}$ (which is H with the complex structure and inner product reversed). The second dual is naturally isomorphic to H.

(9) Any bounded operator $A : H \longrightarrow H$ on a Hilbert space has a unique adjoint operator satisfying

$$\langle Au, v\rangle = \langle u, A^*v\rangle \;\forall\; u, v \in H \text{ and } \|A^*\| = \|A\|.$$

(10) For any $W \subset H$ which is a closed subspace there is a well-defined projection operator P_W with $P_W H = W$ and $P_W^2 = P_W = P_W^*$. Then $\mathrm{Id} - P_W$ is the orthogonal projection onto $W^\perp$,

$$H = W \oplus W^\perp.$$

(11) A sequence u_j in H converges weakly if $\langle u_j, v\rangle$ converges for each $v \in H$. It follows that u_j is bounded and that there exists a unique weak limit $u \in H$ such that $\langle u_j, v\rangle \to \langle u, v\rangle$ for all $v \in H$. This is written $u_j \rightharpoonup u$. The weak limit satisfies $\|u\| \le$

$\liminf \|u_j\|$ and if $u_j \rightharpoonup u$ then $u \to u_j$ if and only if $\|u_j\| \to \|u\|$.

(12) Bounded sets in a separable Hilbert space are weakly precompact, i.e. any bounded sequence has a weakly convergent subsequence.

(13) Heine-Borel replacement. A subset U of a Hilbert space is compact if and only if it is closed and bounded and for each $\epsilon > 0$ there is a finite-dimensional subspace $F \subset H$ such that

$$\sup_{u \in U} \inf_{f \in F} \|u - f\| < \epsilon.$$

Proof: If U is compact then the open cover by the ϵ-balls around each point of U have a finite subcover. The linear span of the centres is a finite dimensional subspace F and by the covering property (13) holds.

Conversely if U is closed and bounded then the projection $P_F U$ onto F is bounded and hence any sequence in U has a subsequence which projects to a convergent sequence in F, since finite-dimensional Hilbert spaces do satisfy Heine-Borel. Starting with a sequence u_k in U, take $\epsilon = 1/n$ for which there is a finite dimensional space F_n and extract successive subsequences so that $P_{F_n} u_{n,p}$ converges in F_n. Here $u_{n,p}$ is a subsequence of $u_{n-1,p}$ for each n with $u_{0,p} = u_p$. Now, take the diagonal sequence $v_k = u_{k,k}$. It is a subsequence of u_k and by construction $P_{F_n} v_k$ converges in F_n for each n – since convergence is a property of the tail and v_k is 'eventually' a subsequence of $u_{n,p}$. Since every element of v_k is distant at most $1/n$ from F_n it follows from the triangle inequalty

$$\|v_k - v_j\| \le \|v_k - P_n v_k\| + \|P_n v_k - P_n v_j\| + \|P_n v_j - v_j\|$$

that v_k is Cauchy so converges by the completeness of H.

(14) The bounded operators $\mathcal{B}(H)$, on H form a Banach-$\star$-algebra, so if A, $B \in \mathcal{B}(H)$ then

$$A^* \in \mathcal{B}(H),\ AB \in \mathcal{B}(H),\ \|AB\| \le \|A\|\|B\|.$$

(15) Norm convergence of operators is defined with respect to the Banach space structure on $\mathcal{B}(H)$ and *strong* convergence is defined for a sequence $A_n \in \mathcal{B}(H)$ by the condition that $A_n u \to Au$ (in the norm topology on H) for each $u \in H$. In fact it is enought just to require the convergence of $A_n u$ for each u since it follows that Au is well-defined, that it is linear and its boundedness follows from the uniform boundedness principle (Banach-Steinhaus).

(16) Open Mapping, Closed Graph theorems. A linear map $A : H \longrightarrow H$ is bounded if and only if its graph

$$\mathrm{Gr}(A) = \{(u, Au); u \in H\} \subset H \times H$$

is closed. If $A \in \mathcal{B}(H)$ is surjective then it is *open* in the sense that $A(O) \subset H$ is open for each $O \subset H$ open. In particular if $A \in \mathcal{B}(H)$ is a bijection then its inverse is continuous.

(17) For $A \in \mathcal{B}(H)$ the resolvent set is

$$\mathrm{res}(A) = \{z \in \mathbb{C}; A - z\,\mathrm{Id} \text{ is a bijection}\}.$$

By the open mapping theorem the resolvent $(A - z\,\mathrm{Id})^{-1} \in \mathcal{B}(H)$ is defined for $z \in \mathrm{res}(A)$ which, using a Neumann series argument, is open and contains the region $|z| > \|A\|$. The complement, the spectrum $\mathrm{spec}(A) \subset \mathbb{C}$ is therefore compact and if z is an eigenvalue, i.e. $A - z\,\mathrm{Id}$ is not injective, then $z \in \mathrm{spec}(A)$, in general the spectrum is larger than the set of eigenvalues. The resolvent is a holomorphic map from $\mathrm{res}(A)$ to $\mathcal{B}(H)$.

(18) By definition an operator is of finite rank if its range is finite dimensional. It can then be written in the form

$$Au = \sum_{i=1}^{N} \langle u, v_i \rangle w_i \tag{6}$$

for some elements u_i, v_i of H. The operators of finite rank for a two-sided $\star$-closed ideal in $\mathcal{B}(H)$.

(19) An operator $A \in \mathcal{B}(H)$ is compact if the closure of $A(B(0,1))$, the image of the unit ball, is compact. These operators form a two-sided $\star$-closed ideal $\mathcal{K}(H) \subset \mathcal{B}(H)$ which is the closure of the ideal of finite-rank operators.

(20) For a compact operator the spectrum is of the form $D \cup \{0\}$ where $D \subset \mathbb{C} \setminus \{0\}$ is a discrete set (possibly empty) consisting of eigenvalues; $\{0\}$ may or may not be an eigenvalue. In particular there are quasi-nilpotent compact operators which have spectrum just consisting of $\{0\}$ but have no eigenvalues. The generalized eigenspace for $A \in \mathcal{K}(H)$ and $z \neq 0$,

$$\{u \in H; (A - z\,\mathrm{Id})^N u = 0 \text{ for some } N\} \text{ is finite-dimensional.} \tag{7}$$