18.155 LECTURE 10, 3 OCTOBER, 2013

This week I will talk about hypoellipticity and ellipticity.

First let me recall properties of convolution which we will use below. We started by investigating the integration involved in

$$u * v(x) = \int_{\mathbb{R}^n} u(x-y)v(y)dy. \tag{1}$$

Some decay or support conditions is needed to make the integrals converge, and for today the case of interest is where one has compact support.

$$* : \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \times \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n),\ * : \mathcal{C}^\infty(\mathbb{R}^n) \times \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n) \tag{2}$$

are continouous bilinear maps with the additional properties

$$\begin{gathered} u * v = v * u \\ P(D)(u * v) = (P(D)u) * v = u * (P(D)v), \\ \operatorname{supp}(u * v) \subset \operatorname{supp}(u) + \operatorname{supp}(v), \\ u * v(\phi) = u(\wedge v * \phi) \text{ two with compact supports.} \end{gathered} \tag{3}$$

The continuity and the last, weak, reformulation allows us to extend convolution directly and see that

$$\begin{gathered} * : \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \times \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n),\ * : \mathcal{C}^\infty(\mathbb{R}^n) \times \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n), \\ * : \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \times \mathcal{C}^{-\infty}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n),\ * : \mathcal{C}^{-\infty}(\mathbb{R}^n) \times \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n), \\ * : \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \times \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n),\ * : \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \times \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n), \end{gathered} \tag{4}$$

with (3) still holding. Again continuity allows us to extend this to distributions:-

(5)

$$\begin{gathered} * : \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \times \mathcal{C}^{-\infty}(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R}^n),\ * : \mathcal{C}^{-\infty}(\mathbb{R}^n) \times \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R}^n), \\ * : \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \times \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \end{gathered}$$

still with (3) holding.

Proofs:

Last week I showed that a homogeneous, constant coefficient, differential operator has a fundamental solution which is smooth outside the orgin. Since compactness is important in the convolution identities above we will cut this off near the origin.

Definition 1. A differential operator $P(D)$ is said to be *hypoelliptic* if for each $\epsilon > 0$ there exists a *parameterix* $U_\epsilon \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n)$ with support in $\{|x| \le \epsilon\}$, meaning it satisfies

$$P(D)U_\epsilon = \delta_0 + F,\ F \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{6}$$

Thus (6) is the definition of a (convolution) parameterix. In this case, $F \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ since we are assuming that U_ϵ has compact support.

Lemma 1. *Homogeneous elliptic operators are hypoelliptic.*

Proposition 1. *If $P(D)$ is hypoelliptic and $u \in \mathcal{C}^{-\infty}(\Omega)$, for $\Omega \subset \mathbb{R}^n$ open satisfies $P(D)u \in \mathcal{C}^{\infty}(\Omega)$ then $u \in \mathcal{C}^{\infty}(\Omega)$.*

This is the real definition of hypoellipticity. It is actually equivalent to the existence of a fundamental solution which is smooth outside the origin. Indeed, this is clear if one knows the existence of a fundamental solution since, given the Proposition, any fundamental solution of a hypoelliptic operator must have this property. Then the property in the definition follows by cutting this fundamental solution off near 0.

So, at this point it is natural to define the singular support of a distribution.